Topology (TOP) / Alex Küronya / Fall 2006

Homework 5

The problems with an asterisk are the ones that you are supposed to submit. The ones with two asterisks are meant as more challenging, and by solving them you can earn extra credit.

1. Let (X, d) be a metric space, $A \subseteq X$ an arbitrary fixed subset. For $x \in X$ define

$$d(x, A) \stackrel{\text{def}}{=} \inf_{y \in A} d(x, y) \ .$$

Show that the function $x \mapsto d(x, A)$ is continuous.

2. * (Lebesgue number lemma) Let (X, d) be a compact metric space, $\mathfrak{U}$ an open covering of X. Prove that there is a positive real number δ (depending on $\mathfrak{U}$) such that for every $A \subseteq X$ with diameter less than δ, there exists an element $U \in \mathfrak{U}$ for which $A \subseteq U$. (Note: the diameter of the subset A is is defined as $\sup \{ d(x, y) \mid x, y \in A \}$.)

3. Let (X, d_X),(Y, d_Y) be metric spaces with X compact, and let $f : X \to Y$ be a continuous function. Then f is uniformly continuous, that is, for every $\epsilon > 0$ there exists $\delta > 0$ such that for every pair of points $x_1, x_2 \in X$

$$d_X(x_1, x_2) < \delta \Rightarrow d_Y(f(x_1), f(x_2)) < \epsilon \ .$$

4. * Prove that if X, Y are connected topological spaces then so is $X \times Y$.

5. Let X, Y, Z be topological spaces, $f : X \times Y \to Z$ an arbitrary function. Show that f is continuous if and only if it is continuous in both variables separately.

Definition. A *topological group* G is a Hausdorff topological space equipped with a group structure in such a way that

(1) the multiplication map $G \times G \to G$, $(g, h) \mapsto gh$ is continuous,
(2) taking inverse images $G \to G$, $g \mapsto g^{-1}$ is continuous.

A *subgroup* of a topological group is a subspace $H \subseteq G$ which is an abstract subgroup as well. If G, G' are topological groups then a function $f : G \to G'$ is a *homomorphism*, if it is a homomorphism of abstract groups which is continuous.

6. Let G be a group, which is a Hausdorff topological space as well. Show that G is a topological group iff the function $G \times G \to G$ sending (x, y) to xy^{-1} is continuous.

7. Verify that the following groups (with their usual topology) are topological groups: $(\mathbb{Z}, +)$,$(\mathbb{Q}, +)$, $(\mathbb{R}^+, \cdot)$, (complex numbers of absolute value $1, \cdot$).

8. ** Prove that the general linear group $GL(n, \mathbb{R})$ together with the usual matrix multiplication is a topological group (here $GL(n, \mathbb{R})$ denotes the set of $n \times n$ invertible matrices; we give this group the topology inherited from $\mathbb{R}^{n^2}$).

9. Let G be a topological group, and denote the connected component of G containing the identity by G°. Show that G° is a closed normal subgroup of G.