Topology (TOP) / Alex Küronya / Fall 2006

Homework 3

The problems with an asterisk are the ones that you are supposed to submit. The ones with two asterisks are meant as more challenging, and by solving them you can earn extra credit.

1. * (i) A topological space X is said to be *irreducible* if whenever $X = F \cup G$, where F, G are closed, then either $X = F$ or $X = G$. A subspace of X is irreducible if it is so in the subspace topology. Prove that if X is irreducible and $U \subseteq X$ open, then U is irreducible.
(ii) A topological space is called *noetherian* if every descending chain of closed subsets is eventually constant. Show that a noetherian topological space can be expressed as a finite union

$$X = X_1 \cup \cdots \cup X_r ,$$

where the X_i are closed, irreducible, and none of them contains any other. Verify that this decomposition is unique up to the reordering of the terms.
(iii) Show that $\mathbb{R}$ equipped with the finite complement topology is an irreducible noetherian space.

2. The relation "$p \ominus q$ if for every discrete valued map d on X, $d(p) = d(q)$" is an equivalence relation, the equivalence classes of which are called *quasi-components.*
(i) Show that quasi-components are either equal or disjoint, and fill out X.
(ii) The quasi-components of a topoligical space are closed; each connected set is contained in a quasi-component.
(iii) Let $X \overset{\text{def}}{=} \{ \{ (0,0) \} , \{ (0,1) \} \} \cup \bigcup_{n=1}^{\infty} \{ \frac{1}{n} \} \times [0,1] \subseteq \mathbb{R}^2$. Then the points $(0,0)$ and $(0,1)$ are components, but not quasi-components.

3. Let $(X,\tau),(X,\sigma)$ two topologies on the same set, assume that $\sigma \subseteq \tau$. Does connectivity of a subset with respect to one topology imply anything for connectivity in the other?

4. Let C_n be an infinite sequence of connected subspaces of a topological space X, such that for every n, one has $C_n \cap C_{n+1} \neq \emptyset$. Show that $\bigcup_{n=1}^{\infty} C_n$ is connected as well.

5. Verify that an infinite set is always connected in the finite complement topology.

6. * A topological space X is called *totally disconnected*, if the only connected sets are those consisting of one element only. Show that a discrete topological space is totally disconnected. Is the converse true?

7. Let $A \subseteq X$ be an arbitrary subspace, $C \subseteq X$ connected. Prove that $A \cap C \neq \emptyset$ and $(X - A) \cap C \neq \emptyset$ together imply $\partial A \neq \emptyset$.

8. Show that if $X \subseteq \mathbb{R}^n$ is a convex subset, then X is connected.

9. (i) The spaces $[0,1]$, $[0,1)$, and $(0,1)$ are pairwise non-homeomorphic.
(ii) $\mathbb{R} \not\approx \mathbb{R}^n$ for $n \geq 2$.

10. Prove that a continuous map $f : [0,1] \to [0,1]$ always has a fixed point. Give an example to show that the same statement is false for $[0,1)$.

11. Prove that if X and Y are topological spaces with X non-empty and connected, and all points of Y closed, then every locally constant map from X to Y is constant.

12. (i) Let $\Sigma \subseteq \mathbb{R}^2$ be a finite set of points. Show that $\mathbb{R}^2 - \Sigma$ is connected.
(ii) ** Now let $\Gamma \subseteq \mathbb{R}^3$ be a finite set of lines and prove that $\mathbb{R}^3 - \Gamma$ is connected.