Topology (TOP) / Alex Küronya / Spring 2006

Homework 2

The problems with an asterisk are the ones that you are supposed to submit. The ones with two asterisks are meant as more challenging, and by solving them you can earn extra credit.

1. Is the set $\left\{ \frac{1}{n} \,|\, n \in \mathbb{N} \right\} \subseteq \mathbb{R}$ open/closed?

2. Let X be a topological space, Y a metric space, $f_n : X \to Y$ a sequence of functions. We say that f_n *converges uniformly* to a function $f : X \to Y$, if for every $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that if $n \geq N_\varepsilon$, then $d(f_n(x), f(x)) < \varepsilon$ for every $x \in X$.

Prove that the limit function of a uniformly convergent sequence of continuous functions is also continuous.

3. Let $f : X \to Y$ be a function between topological spaces, $\mathcal{S}$ a subbasis for Y. Show that f is continuous if and only if for every $U \in \mathcal{S}$ the set $f^{-1}(U)$ is open in X.

4. * Let X be a metric space, $A \subseteq X$ an arbitrary subset. Then $\overline{A}$ = the set of limit points of convergent sequences of points in A.

5. Let X be an arbitrary topological space, $A, B \subseteq X$ subsets.
(i) Show that $\operatorname{int}(A) = \{ x \in X \,|\, \exists U \subseteq X \text{ open for which } x \in U \subseteq A \}$.
(ii) Prove that $\overline{A} = \{ x \in X \,|\, \forall U \text{ open in } X, U \cap A \neq \emptyset \}$.
(iii) Show that A is open if and only if $A = \operatorname{int}(A)$, and A is closed if and only $A = \overline{A}$. Verify furthermore that $X - \operatorname{int}(A) = \overline{X - A}$, and $X - \overline{A} = \operatorname{int}(X - A)$.
(iv) Prove the following identities: $\operatorname{int}(A) \cap \operatorname{int}(B) = \operatorname{int}(A \cap B)$, $\overline{A} \cup \overline{B} = \overline{A \cup B}$. Verify also that

$$\bigcap_\alpha \operatorname{int}(A_\alpha) \supseteq \operatorname{int}\left(\bigcap_\alpha A_\alpha\right) = \operatorname{int}\left(\bigcap_\alpha \operatorname{int}(A_\alpha)\right), \bigcup_\alpha \operatorname{int}(A_\alpha) \subseteq \operatorname{int}\left(\bigcup_\alpha A_\alpha\right) .$$

(v) Prove that $A \subseteq B$ implies $\operatorname{int}(A) \subseteq \operatorname{int}(B)$, and $\overline{A} \subseteq \overline{B}$.

6. Consider the topology on $\mathbb{R}$ generated by the sets $[x, y)$ and $(x, y]$ for all $x, y, \in \mathbb{R}$. Show that it coincides with the discrete topology.

7. For a topological space X and an arbitrary subset $A \subseteq X$,

$$X = \operatorname{int}(A) \coprod \partial A \coprod (X - \operatorname{int}(X - \overline{A}) .$$

8. * A finite union of nowhere dense sets is again nowhere dense.

9. Let X be a metric space. Then X is second countable iff it has a countable dense set.

10. ** How many non-homeomorphic topologies are there on a three-element set?

Definition. A function $f : X \to Y$ between topological spaces is called *locally constant* if for every $x \in X$ there exists an open neighbourhood $U \subseteq X$ of x such that $f(U)$ consists of one point.

11. Show that every locally constant function is continuous.