Topology (TOP) / Alex Küronya / Fall 2006

Homework 1

The problems with an asterisk are the ones that you are supposed to submit. The ones with two asterisks are meant as more challenging, and by solving them you can earn extra credit.

1. Come up with a definition for convergence and Cauchy sequences in metric spaces.

2. (i) Show that the functions $s, p : \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by $s(x, y) = x+y$, $p(x, y) = xy$ are continuous.
(ii) Let $f, g : X \longrightarrow \mathbb{R}$ continuous functions. Then all of $f \pm g, f \cdot g$ are continuous; if $g(x) \neq 0$ for all $x \in X$, then $\frac{f}{g}$ is continuous as well.

3. (i) Is the function $f : \mathbb{R}^2 - \{(0, 0)\} \longrightarrow \mathbb{R}^2$

$$f(x, y) = \left(\frac{x}{x^2 + y^2}, -\frac{y}{x^2 + y^2}\right)$$

continuous on $\mathbb{R}^2 - \{(0, 0)\}$?
(ii) Is there a continuous function $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ for which

$$g|_{\mathbb{R}^2 - \{(0,0)\}} = f \ ?$$

4. Let $f : X \longrightarrow Y$ be a homeomorphism, x_k a sequence in X. Then x_k is convergent in X if and only if $f(x_k)$ is convergent in Y.

5. * Let α, β, γ be arbitrary real numbers. Then the so-called *open half-space*

$$H = \left\{(x, y, z) \in \mathbb{R}^3 \mid \alpha x + \beta y + \gamma z > 0\right\}$$

is indeed an open subset of $\mathbb{R}^3$ with respect to the Euclidean topology.

6. (i) Let $X \subseteq \mathbb{R}^n$, $p \in X$, $\delta > 0$. Then $\mathbb{B}_X(p, \delta)$ is open in X.
(ii) Show that the closed ball

$$D(x.\delta) = \{y \in \mathbb{R}^n \mid |x - y| \leq \delta\}$$

is indeed a closed subset of $\mathbb{R}^n$.
(iii) Find an example of a metric space X, a point $x \in X$, and a nonnegative real number ϵ such that the closure of $\mathbb{B}(x, \epsilon)$ is different from $\mathbb{D}(x, \epsilon)$.

7. Prove that the set

$$\left\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \geq 10\right\}$$

is closed.

8. Is the set consisting of all point of the form $\frac{1}{n}$, n a natural number, open/closed in $\mathbb{R}$?

9. Give examples of infinitely many open sets in $\mathbb{R}$, the intersection of which is (i) open (ii) closed (iii) neither open nor closed.

10. * Prove that

$$d_1(f, g) = \int_{[a,b]} |f - g| dx$$

is a metric on $\mathcal{C}[a, b]$. Is this still true if we replace continuous functions by Riemann integrable ones?

11. ** (Kuratowski complement and closure problem) Let X be an arbitrary topological space. Show that for any subspace $A \subseteq X$ the process of repeated applications of taking closure/taking complement generates at most 14 distinct subsets. Give an example of a subset $T \subseteq \mathbb{R}$ (with the Euclidean topology) which realizes this maximum.