Topology (TOP) / Alex Küronya / Fall 2005

Homework 9

The problems with an asterisk are the ones that you are supposed to submit. The ones with two asterisks are meant as more challenging, and by solving them you can earn extra credit.

1. * Let G be a topological group, $H \subseteq G$ a closed normal subgroup. Show that the quotient group G/H, with the quotient topology, is a topological group.

2. Let G be a topological group, G^0 the connected component of G containing the identity element. Prove the G^0 is a closed normal subgroup of G.

3. ** Let $\phi : G \to H$ be a map of topological groups (that is, a continuous homomorphism).
(i) Verify that $\ker\phi \subseteq G$ is a closed normal subgroup of G.
(ii) Show that if G is compact, then

$$G/\ker\phi \approx H$$

as topological groups (in other words: show that the map $\tilde{\phi} : G/\ker\phi \to H$ induced by ϕ is a homeomorphism of topological spaces and an isomorphism of groups).

4. Let $f, g : X \to Y$ be homotopic maps with f being a homemorphism. Does it follow that g is a homeomorphism as well? Prove your answer.

Definition. For a topological space X, the *suspension of* X is defined to be the quotient of $X \times I$ obtained by collapsing the subspace $X \times \{0\}$ to one point, and $X \times \{1\}$ to another point. The suspension of X is denoted by $S(X)$. Let $f : X \to Y$ be a map of topological spaces. Then the *suspension of* f is the map $S(f) : S(X) \to S(Y)$ induced by

$$f \times \mathbf{1}_I : X \times I \to Y \times I \ .$$

5. Prove that $S(\mathbb{S}^n) \approx \mathbb{S}^{n+1}$.

6. Decide if the following is true: if $f, g : X \to Y$ are two homotopic maps, then $S(f)$ and $S(g)$ are homotopic as well.

7. * Show that if a topological space X deformation retracts to a point $x \in X$, then for each neighbourhood U of x there exists a neighbourhood $x \in V \subseteq U$ such that the inclusion map $V \hookrightarrow U$ is nullhomotopic.