Topology (TOP) / Alex Küronya / Fall 2005

Homework 5

The problems with an asterisk are the ones that you are supposed to submit. The ones with two asterisks are meant as more challenging, and by solving them you can earn extra credit.

1. Find an example of a topological space which is not T_0, an example which is not T_1 but T_0, and one which is not T_2 but T_1.

2. * Show that if X is a Hausdorff topological space, then a convergent sequence of points of X can have at most one limit point.

3. Prove that a subspace of a Hausdorff space is itself Hausdorff with respect to the subspace topology.

4. Verify that every connected open set in a locally path connected space is path connected.

5. Prove that if $f : X \to Y$ is a closed continuous map of topological spaces, X compact, Y Hausdorff, then f maps closed sets to closed sets (such functions are called *closed*).

6. Consider a continuous map $f : X \to Y$ for which the inverse image of every point ot y is compact. Show that if Y is compact, then so is X.

7. ** Let (X, d) be a metric space, $f : X \to X$ a contraction, that is, a continuous function for which there exists a positive real number $c < 1$ such that for every pair of point $x, y \in X$,

$$d(f(x), f(y)) \leq c \cdot d(x, y) \ .$$

Show that if X is compact, then f has a unique fixed point (i.e. an element $x \in X$, for which $f(x) = x$).

8. ** Let R be a commutative ring (with 1), and let

$$\operatorname{Spec} R \stackrel{\text{def}}{=} \{ \mathfrak{p} \subseteq R \,|\, \mathfrak{p} \text{ is a prime ideal of } R \} \ .$$

For an ideal $I \subseteq R$, consider the set

$$V(I) \stackrel{\text{def}}{=} \{ \mathfrak{p} \in \operatorname{Spec} R \,|\, \mathfrak{p} \supseteq I \} \ .$$

Show that the sets of the form $V(I)$ (where I runs through all ideals of R) satisfy the properties of closed sets of a topology. The set $\operatorname{Spec} R$ is called the *spectrum* of the ring R, and the topology just defined is the *Zariski topology* on $\operatorname{Spec} R$. How could you describe the Zariski topology on $R = \mathbb{C}[x]$? And on $\mathbb{Z}$?

9. Let X be a compact topological space, $\{ A_\alpha \,|\, \alpha \in I \}$ an arbitrary collection of closed sets, which is closed with respect to finite intersections. If for an open set $U \in X$ one has $\bigcap_\alpha A_\alpha \subseteq U$, then there exists $\alpha \in I$ for which $A_\alpha \subseteq U$.

10. * A map $f : X \to Y$ between topological spaces is said to be *proper*, if the inverse image of every compact subset in Y is compact in X. Prove the following: if $f : X \to Y$ is a closed (continuous) map, and for every $y \in X$ the inverse image $f^{-1}(y) \subseteq X$ is compact, then f is proper.

11. ** Show that for any commutative ring R (with unit), $\operatorname{Spec} R$ is compact in the Zariski topology.