Topology (TOP) / Alex Küronya / Fall 2005

Homework 4

The problems with an asterisk are the ones that you are supposed to submit. The ones with two asterisks are meant as more challenging, and by solving them you can earn extra credit.

1. * The relation "$p \ominus q$ if for every discrete valued map d on X, $d(p) = d(q)$" is an equivalence relation, the equivalence classes of which are called *quasi-components*.
(i) Show that quasi-components are either equal or disjoint, and fill out X.
(ii) The quasi-components of a topoligical space are closed; each connected set is contained in a quasi-component.
(iii) Let $X \stackrel{\text{def}}{=} \{\{(0,0\},\{(0,1)\}\} \cup \bigcup_{n=1}^{\infty} \{\frac{1}{n}\} \times [0,1] \subseteq \mathbb{R}^2$. Then the points $(0,0)$ and $(0,1)$ are components, but not quasi-components.

2. Let $(X,\tau),(X,\sigma)$ two topologies on the same set, assume that $\sigma \subsetneq \tau$. Does connectivity of a subset with respect to one topology imply anything for connectivity in the other?

3. Let C_n be an infinite sequence of connected subspaces of a topological space X, such that for every n, one has $C_n \cap C_{n+1} \neq \emptyset$. Show that $\bigcup_{n=1}^{\infty} C_n$ is connected as well.

4. An infinite set is always connected in the finite complement topology.

5. * A topological space X is called *totally disconnected*, if the only connected sets are those consisting of one element only. Show that a discrete topological space is totally disconnected. Is the converse true?

6. Let $A \subseteq X$ be an arbitrary subspace, $C \subseteq X$ connected. Prove that $A \cap C \neq \emptyset$ and $(X-A) \cap C \neq \emptyset$ together imply $\partial A \neq \emptyset$.

7. If $X \subseteq \mathbb{R}^n$ is a convex subset, then X is connected.

8. (i) The spaces $[0,1]$, $[0,1)$, and $(0,1)$ are pairwise non-homeomorphic.
(ii) $\mathbb{R} \not\approx \mathbb{R}^n$ for $n \geq 2$.

9. A continuous map $f : [0,1] \to [0,1]$ always has a fixed point; the same statement is false for $[0,1)$.