Topology (TOP) / Alex Küronya / Fall 2005

Homework 3

The problems with an asterisk are the ones that you are supposed to submit. The ones with two asterisks are meant as more challenging, and by solving them you can earn extra credit.

1. Let X be an arbitrary set, $\mathcal{S}$ an arbitrary collection of subsets of X. Then there exists precisely one topology on X for which $\mathcal{S}$ is a subbasis. This is called the *topology generated by* $\mathcal{S}$.

2. Let X be a topological space, Y a metric space, $f_n : X \to Y$ a sequence of functions. We say that f_n *converges uniformly* to a function $f : X \to Y$, if for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that if $n \geq N$, then $d(f_n(x), f(x)) < \epsilon$ for every $x \in X$.

Prove that the limit function of a uniformly convergent sequence of continuous functions is also continuous.

3. Consider the topology on $\mathbb{R}$ generated by the sets $[x, y)$ and $(x, y]$ for all $x, y \in \mathbb{R}$. Verify that it coincides with the discrete topology.

4. Let $f : X \to Y$ be a function between topological spaces, $\mathcal{S}$ a subbasis for Y. Then f is continuous if and only if for every $U \in \mathcal{S}$ the set $f^{-1}(U)$ is open in X.

5. * Let X be a metric space. Then $\overline{A} =$ set of limits of sequences of points in A.

6. Let X be a topological space, $A, B \subseteq X$ arbitrary.
(*i*) Prove that $\operatorname{int}(A) = \{\, x \in X \mid \exists U \text{ open for which } a \in U \subseteq A \,\}$.
(*ii*) Show that $\overline{A} = \{\, x \in X \mid \forall U \text{ open with } x \in U, U \cap A \neq \emptyset \,\}$.
(*iii*) Show that A is open iff $A = \operatorname{int}(A)$ and A is closed iff $A = \overline{A}$. Verify furthermore that $X - \operatorname{int}(A) = \overline{X - A}$ and $X - \overline{A} = \operatorname{int}(X - A)$.
(*iv*) Prove the following identities: $\operatorname{int}(A) \cap \operatorname{int}(B) = \operatorname{int}(A \cap B)$, $\overline{A \cup B} = \overline{A} \cup \overline{B}$. Verify also that

$$\bigcap_\alpha \operatorname{int}(A_\alpha) \supseteq \operatorname{int}\left(\bigcap_\alpha A_\alpha\right) = \operatorname{int}\left(\bigcap_\alpha \operatorname{int}(A_\alpha)\right) \; , \; \bigcup_\alpha \operatorname{int}(A_\alpha) \subseteq \operatorname{int}\left(\bigcup_\alpha A_\alpha\right) \; .$$

Give examples showing that the inclusions need not necessarily be equalities.
(*v*) Prove that $A \subseteq B$ implies $\operatorname{int}(A) \subseteq \operatorname{int}(B)$, and $\overline{A} \subseteq \overline{B}$.

7. For X a topological space, $A \subseteq X$, $X = \operatorname{int}(A) \coprod \partial A \coprod X - \overline{A}$.

8. A finite union of nowhere dense sets is again nowhere dense.

9. Let X be a metric space. Then X is second countable iff it has a countable dense set.

10. * (*i*) A topological space X is said to be *irreducible* if whenever $X = F \cup G$, where F, G are closed, then either $X = F$ or $X = G$. A subspace of X is irreducible if it is so in the subspace topology. Prove that if X is irreducible and $U \subseteq X$ open, then U is irreducible.
(*ii*) A topological space is called *noetherian* if every descending chain of closed subsets is eventually constant. Show that a noetherian topological space can be expressed as a finite union

$$X = X_1 \cup \cdots \cup X_r \; ,$$

where the X_i are closed, irreducible, and none of them contains any other. Verify that this decomposition is unique up to the reordering of the terms.
(*iii*) Show that $\mathbb{R}$ equipped with the finite complement topology is an irreducible noetherian space.