Topology (TOP) / Alex Küronya / Fall 2005

Homework 2

The problems with an asterisk are the ones that you are supposed to submit. The ones with two asterisks are meant as more challenging, and by solving them you can earn extra credit.

1. Prove that the set

$$\left\{(x,y) \in \mathbb{R}^2 \,|\, x^2+y^2 \geq 10\right\}$$

is closed.

2. Is the set consisting of all point of the form $\frac{1}{n}$, n a natural number, open/closed in $\mathbb{R}$?

3. Give examples of infinitely many open sets in $\mathbb{R}$, the intersection of which is (i) open (ii) closed (iii) neither open nor closed.

4. Show that the closed ball

$$D(x.\delta) = \{y \in \mathbb{R}^n \,|\, |x-y| \leq \delta\}$$

is indeed a closed subset of $\mathbb{R}^n$.

5. * Prove that

$$d_1(f,g) \;=\; \int_{[a,b]} |f-g| dx$$

is a metric on $\mathcal{C}[a,b]$. Is this still true if we replace continuous functions by Riemann integrable ones?

6. Show that for every $x, y \in \mathbb{Q}$
   (1) $\mathrm{ord}_p(xy) = \mathrm{ord}_p(x) + \mathrm{ord}_p(y)$
   (2) $\mathrm{ord}_p(x+y) \geq \min\{\mathrm{ord}_p(x), \mathrm{ord}_p(y)\}$ with equality if $\mathrm{ord}_p(x) \neq \mathrm{ord}_p(y)$.
7. Compute the p-adic order of $5, 100, 24, -\frac{1}{48}, -\frac{12}{28}$ for $p = 2, 3, 5$.

8. * Prove that $(\mathbb{Q}, d_p)$ is a metric space, which is in addition *non-archimedean*, that is, for every $x, y, z \in \mathbb{Q}$ one has

$$d_p(x,y) \leq \max\{d_p(x,z), d_p(z,y)\} \ .$$

Conclude that in $(\mathbb{Q}, d_p)$ every triangle is isosceles.