Topology (TOP)
Fall 2005
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Homework 1

The problems with an asterisk are the ones that you are supposed to submit. The ones with two asterisks are meant as more challenging, and by solving them you can earn extra credit.

1. Prove the Cauchy–Schwarz inequality via the following method: start with the inequality

$$\lambda^2|x|^2 + 2\lambda\mu(x \cdot y) + \mu^2|y|^2 \geq 0 \ ,$$

and consider it as an inequality for a quadratic polynomial in $\frac{\lambda}{\mu}$. Prove the statement about equality.

2. Come up with a definition for convergence and Cauchy sequences in metric spaces.

3. (*i*) Show that the functions $s, p : \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by

$$\begin{aligned} s(x,y) &= x+y \\ p(x,y) &= xy \end{aligned}$$

are continuous.
(*ii*) Let $f, g : X \longrightarrow \mathbb{R}$ continuous functions. Then all of $f \pm g, f \cdot g$ are continuous; if $g(x) \neq 0$ for all $x \in X$, then $\frac{f}{g}$ is continuous as well.

4. (*i*) Is the function $f : \mathbb{R}^2 - \{(0,0)\} \longrightarrow \mathbb{R}^2$

$$f(x,y) = \left(\frac{x}{x^2+y^2}, -\frac{y}{x^2+y^2} \right)$$

continuous on $\mathbb{R}^2 - \{(0,0)\}$?
(*ii*) Is there a continuous function $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ for which

$$g|_{\mathbb{R}^2 - \{(0,0)\}} = f \ ?$$

5. * Prove that the function $\varphi : \mathbb{B}^n \longrightarrow \mathbb{R}^n$

$$\varphi(x) = \frac{1}{1-|x|^2} \cdot x$$

is a homeomorphism.

6. Let $f : X \longrightarrow Y$ be a homeomorphism, x_k a sequence in X. Then x_k is convergent in X if and only if $f(x_k)$ is convergent in Y.

7. * Let α, β, γ be arbitrary real numbers. Then the so-called *open half-space*

$$H = \left\{ (x,y,z) \in \mathbb{R}^3 \,|\, \alpha x + \beta y + \gamma z > 0 \right\}$$

is indeed open.

8. Let $X \subseteq \mathbb{R}^n$, $p \in X$, $\delta > 0$. Then $\mathbb{B}_X(p, \delta)$ is open in X.