Topology (TOP) / Alex Küronya / Fall 2005

Homework 7

The problems with an asterisk are the ones that you are supposed to submit. The ones with two asterisks are meant as more challenging, and by solving them you can earn extra credit.

1. Let X be a regular topological space, $A \subseteq X$ a closed subset. Show that X/A is Hausdorff. Prove that if X is in addition normal, then X/A is normal as well.

2. * Consider the following three topological spaces:
   (1) $\mathbb{R}^2/\mathbb{Z}^2$, more precisely: $\mathbb{R}^2$ modulo the equivalence relation $(a,b) \sim (c,d)$ if and only if both $a-c$ and $b-d$ are integers,
   (2) $[0,1] \times [0,1]$ with opposite edges identified,
   (3) $\mathbb{S}^1 \times \mathbb{S}^1$.

Show that these are all homeomorphic to each other. (This space is called the *torus*, and is often denoted by $\mathbb{T}^2$.)

Definition 0.1. A *topological group* G is a Hausdorff topological space equipped with a group structure such that
   (1) the multiplication map $G \times G \to G$, $(g,h) \mapsto gh$ is continuous,
   (2) the formation of the inverse $G \to G$, $g \mapsto g^{-1}$ is continuous.

A *subgroup* of a topological group is a subspace $H \subseteq G$, which is a subgroup as well. If G, G' are topological groups, then a function $f : G \to G'$ is called a *homomorphism of topological groups*, if it is a continuous group homomorphism.

3. Let G be a group that is also a Hausdorff topological space. Prove that G is a topological group if and only if the map $G \times G \to G$ sending (x,y) into xy^{-1} is continuous.

4. Verify that the following (with their usual topology) are topological groups: $(\mathbb{Z},+)$,$(\mathbb{Q},+)$, $(\mathbb{R}^+,\cdot)$, (complex numbers of absolute value $1,\cdot$).

5.** Prove that the *general linear group* $GL(n,\mathbb{R})$ with matrix multiplication is a topological group. Here $GL(n,\mathbb{R})$ denotes the set of nonsingular $n \times n$ real matrices, and we give it the topology which it inherits from $\mathbb{R}^{n^2}$.

6. * Let G be a topological group, H a subgroup of G. Show that $\overline{H}$ is a subgroup of G as well.

7. Let G be a topological group, and let G° denote the connected component containing the identity. Prove that G° is a closed normal subgroup of G.