Topology (TOP) / Alex Küronya / Fall 2005

Homework 6

The problems with an asterisk are the ones that you are supposed to submit. The ones with two asterisks are meant as more challenging, and by solving them you can earn extra credit.

1. Let (X, d) be a metric space, $A \subseteq X$ an arbitrary fixed subset. For $x \in X$ define

$$d(x, A) \stackrel{\text{def}}{=} \inf_{y \in A} d(x, y) .$$

Show that the function $x \mapsto d(x, A)$ is continuous.

2. * (Lebesgue number lemma) Let (X, d) be a compact metric space, $\mathfrak{U}$ an open covering of X. Prove that there is a positive real number δ (depending on $\mathfrak{U}$) such that for every $A \subseteq X$ with diameter less than δ, there exists an element $U \in \mathfrak{U}$ for which $A \subseteq U$. (Note: the diameter of the subset A is is defined as $\sup \{ d(x, y) \,|\, x, y \in A \}$.)

3. Let (X_1, d_X),(Y, d_Y) be metric spaces with X compact, and let $f : X \to Y$ be a continuous function. Then f is uniformly continuous, that is, for every $\epsilon > 0$ there exists $\delta > 0$ such that for every pair of points $x_1, x_2 \in X$

$$d_X(x_1, x_2) < \delta \Rightarrow d_Y(f(x_1), f(x_2)) < \epsilon .$$

4. ** If X is a metric space, then the fact that X is sequentially compact implies that it is compact as well.

5. * Prove that if X, Y are connected topological spaces then so is $X \times Y$.

6. Let X, Y, Z be topological spaces, $f : X \times Y \to Z$ an arbitrary function. Show that f is continuous if and only if it is continuous in both variables separately.

7. Prove that the product of two Hausdorff topological spaces is Hausdorff again.