SHOKUROV'S REDUCTION THEOREM TO PRE LIMITING FLIPS

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This is an expanded version of a talk given in the Fall 2007 Forschunggseminar at the Universität Duisburg-Essen. It is written for non-specialists, and hence might contain more details as experts would think necessary. Naturally, no claims regarding originality are made.

The purpose of this short note is to provide a mostly self-contained exposition of Shokurov's reduction theorem. We will follow closely the presentations found in [8, Chapter 18], [1, Section 4.3] and an early version of [2][1] . In what follows X will denote a normal variety of dimension n.

Definition 0.1. (pre limiting flips and elementary pre limiting flips) Let (X, Δ) be a dlt pair, $f : X \to Z$ a small extremal contraction. Assume that

(a) $-(K_X + \Delta)$ is f-ample[2]

(b) there exists an irreducible component $S \subseteq \lfloor \Delta \rfloor$ such that S is f-negative.

Then f is called a *pre limiting flipping contraction*, and the flip of f (if it exists) is called a *pre limiting flip*.

If in addition one assumes that X is $\mathbb{Q}$-factorial and $\rho(X/Z) = 1$, then f is called an *elementary pre limiting flipping contraction*, and its flip an *elementary pre limiting flip*.

In this note we will always refer to 'pre limiting' as 'pl'.

Definition 0.2. (The birational transform) Let $f : X \dashrightarrow Y$ be a birational map, $\{E_1, \ldots, E_r\}$ the f^{-1}-exceptional divisors, and D an $\mathbb{R}$-Weil divisor on X. Then the *birational transform* of D is defined to be

$$D^Y \stackrel{\text{def}}{=} f_* D + \sum_j E_j \ .$$

Remark 0.3. It is very important to distinguish the birational transform of D from the proper transform (commonly denoted by D_Y in the above situation. Note that this choice deviates from that of the sources [8, 1, 2].

Definition 0.4. (Special termination) Let (X, Δ) be a dlt pair, and

$$(X, \Delta) = (X_0, \Delta_0) \stackrel{\phi_1}{\dashrightarrow} (X_1, \Delta_1) \stackrel{\phi_2}{\dashrightarrow} \cdots$$

[1]We are grateful to the authors of [2] for making an early version of their paper available to us.

[2]A $\mathbb{Q}$-Cartier divisor D on X is f-ample if $D \cdot C > 0$ holds for every effective 1-cycle f contracts.

a sequence of flips. We say that *special termination holds for the sequence* ϕ_m, if there exists a natural number m_0 for which $\mathrm{Ex}(\phi_m)\cap\lfloor\Delta_{m-1}\rfloor = \emptyset$ whenever $m \geq m_0$.

One says that *special termination holds in dimension* n if special termination holds for any sequence of flips on any n-dimensional dlt pair.

Remark 0.5. The question why we care about pl flips arises naturally. The answer we can give is twofold: first of all, the definition of pl flips gives some hope that questions about n-dimensional pl flips can be reduced to questions about flips on $(n-1)$-dimensional varieties. More precisely, consider the following situation: let $S \subseteq X$ an irreducible divisor, $B \geq 0$ $\mathbb{Q}$-divisor, $\lfloor B \rfloor = 0$, $K_X + S + B$ $\mathbb{Q}$-Cartier. Then by Inversion of Adjunction $(X, S+B)$ is plt in a neighbourhood of S if and only if $(S, K_S + \mathrm{Diff}(B))$ is klt[3]

The second argument highlights the connection between special termination and pl flips: namely, if

$$X_0 \overset{\phi_1}{\dashrightarrow} X_1 \overset{\phi_2}{\dashrightarrow} \ldots$$

is a sequence where each rational map is either a divisorial contraction a pl flip, then if special termination holds for the sequence ϕ_m then the sequence must come to a halt after finitely many steps.

Remark 0.6. Shokurov has proved that the log MMP in dimension $n-1$ implies special termination for $\mathbb{Q}$-factorial dlt pairs in dimension n. Thus, for some purposes the Reduction Theorem could replace the pair

existence of flips, termination of flips in dimension n

by

existence of elementary pl flips, special termination in dimension n ,

which in turn has a fighting chance to become reduced to question on $(n-1)$-dimensional pairs, hence enabling the use of induction on dimension.

Our strategy relies on the following result in a crucial way.

Proposition 0.7. *Let $f : (X, \Delta) \to Z$ be a small extremal contraction of a $\mathbb{Q}$-factorial pair, $g : (Y, \Delta') \to Z$ a log minimal model of $f : (X, \Delta) \to Z$. Then (Y, Δ') is the flip of f.*

Proof. First of all, observe that $\phi^{-1} \overset{\mathrm{def}}{=} g^{-1} \circ f$ has no exceptional divisors, so g is a small contraction.

Next, let A be any g-ample divisor on Y, denote $A_X \overset{\mathrm{def}}{=} \phi^{-1}_* A$. As X is $\mathbb{Q}$-factorial, A_X is $\mathbb{Q}$-Cartier; since the contraction f is extremal the relative Picard number is

[3]the $\mathbb{Q}$-divisor $\mathrm{Diff}(B)$ on S is defined by the formula

$$K_S + \mathrm{Diff}(B) = (K_X + S + B)|_S \ .$$

one, and hence there exists a rational number λ for which

$$\lambda(K_X + \Delta) + A_X \sim_f f^*M$$

for some $\mathbb{Q}$-Cartier divisor M on Z. Taking proper transforms to Y, we obtain that

$$\lambda(K_Y + \Delta') + A \sim_g g^*M \ .$$

As A is g-ample, $\lambda < 0$, hence $K_Y + \Delta'$ is g-ample. This means however that $g : (Y, \Delta') \to Z$ is the unique log canonical model of $f : (X, \Delta) \to Z$, which is in turn the flip of f itself. □

Lemma 0.8. *Let (X, Δ) be a klt pair, $f : (X, \Delta) \to Z$ a small extremal contraction; assume in addition that Z is affine. Then there exist*

- *an effective reduced integral Cartier divisor H' on Z;*
- *a log resolution $h : Y \to X$ of the pair $(X, \Delta + f^*H')$*

such that

(a) the irreducible components of the proper transform of H' and the $h \circ f$-exceptional divisors generate $N^1(Y/Z)$,
(b) the morphism $h \circ f$ is an isomorphism over $Z - \operatorname{Supp} H'$,
(c) $(Y, (\Delta + H)^Y)$ is a log smooth dlt pair.

Corollary 0.9. *It follows that* $\operatorname{Supp}(h \circ f)^*H$ *contains all the h-exceptional divisors.*

Proof. Choose an integral reduced effective Cartier divisor H' on Z such that

(i): the support of $H \overset{\text{def}}{=} f^*H' = f_*^{-1}H$ contains $\operatorname{Ex}(f)$;
(ii): $\operatorname{Supp} H' \supseteq \operatorname{Supp} \operatorname{Sing}(f(D)) \cup \operatorname{Sing}(Z)$;
(iii): fix a resolution of singularities $\pi : Z' \to Z$; let $F_j \subseteq Z'$ be a finite collection of Cartier divisors generating $N^1(Z'/Z)$. We require that H' contains $\pi(F_j)$ for each j.

We note that because of this last property, we can arrange that no $\operatorname{Supp} \pi(F_j)$ contains an irreducible component of $\operatorname{Supp} f(D)$. Therefore we can also assume that H and D have no common components.

First of all if $g : Y \to Z$ is any proper birational morphism with Y $\mathbb{Q}$-factorial then (a) follows for g from (iii): by definition the vector space $N^1(Y/Z)$ is generated by Cartier divisors on Y that do not appear as the pullback of a Cartier divisor on Z. For such a divisor E one of two things will happen: either it is contracted by g, i.e. E is g-exceptional, or its proper transform on Z is a non-Cartier Weil divisor, in which case it is contained in the proper transform of H' by (iii).

Next we construct a log resolution of $(X, \Delta + H)$ satisfying the remaining two properties (requirement (a) being now automatic): First resolve the singularities of Z by blowing up inside $\operatorname{Sing}(Z)$, then resolve the singularities of the inverse image of $H' \cup f(\Delta)$, and finally arrange that the proper transforms of Δ,H' and all exceptional divisors be in simple normal crossing. By (ii) this can be arranged so that we don't change anything outside the support of H', and by (i) this can be done starting from X.

By [7, Theorem 2.44] it also follows that the pair $(Y, (\Delta + H)^Y)$ is dlt (remember that Y is smooth), where $(\Delta + H)^Y \stackrel{\text{def}}{=} \Delta_Y + H_Y + \sum_i E_i$, where the E_i's are all the exceptional divisors.

Note moreover that h^*H contains all exceptional divisors. □

Theorem 0.10 (Shokurov's reduction theorem)**.** *Let us assume that elementary pl flips exist and special termination holds in dimension n. Then klt flips exist in dimension n.*

Proof. Let (X, D) be a klt pair, and $f : X \to Z$ a log flipping contraction, denote $T \stackrel{\text{def}}{=} f(\mathrm{Ex}(f)) \subseteq Z$. As the existence of the flip of f is local on the base, we can and will assume that Z is a normal affine variety.

As a preliminary step, construct a log resolution of $(X, \Delta + H)$ as in Lemma 0.8. Observe that if $g \stackrel{\text{def}}{=} h \circ f$, we have

- $\lfloor (\Delta + H)^Y \rfloor \supseteq \mathrm{Supp}\, g^*H'$ (as all components of $g_*^{-1}H'$ as well as the g-exceptional divisors appear with coefficient one),
- $\mathrm{Supp}\, g^*H' \supseteq C$ for any $K_Y + (\Delta + H)^Y$ negative curve C, as g is an isomorphism away from the support of H'.

We run a log MMP for the pair $(Y, (\Delta + H)^Y)$ over Z. More precisely, starting with $g_0 \stackrel{\text{def}}{=} g$ we (hope to) construct a sequence of objects

$$(g_i : Y_i \to Z, (\Delta + H)^{Y_i})$$

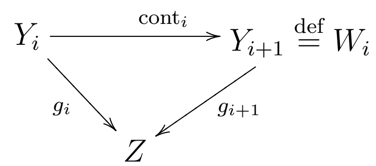
such that

- $\lfloor (\Delta + H)^{Y_i} \rfloor$
- g_i is an isomorphism away from $\mathrm{Supp}\, H'$.

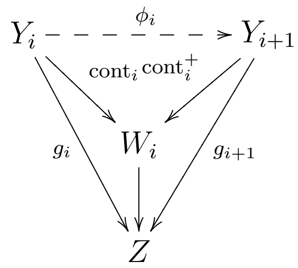
by contracting $K_{Y_i} + (\Delta + H)^{Y_i}$-negative extremal curves C_i with $[C_i] \in \overline{\mathrm{NE}}(Y_i/Z)$ as long as we can. Note that $[C_i] \in \overline{\mathrm{NE}}(Y_i/Z)$ if and only if g_i contracts C_i, hence $C_i \subseteq \mathrm{Supp}\, g_i^*H'$.

At any point, the contraction cont_i of C_i gives rise to either a divisorial contraction or a flipping contraction, If $\mathrm{cont}_i : Y_i \to W_i$ is a divisorial contraction then one quickly checks that the structure morphism $W_i \to Z$ has the properties , so we can call it g_{i+1}, while setting $Y_{i+1} \stackrel{\text{def}}{=} W_i$. It is also immediate that

$$(\mathrm{cont}_i)_*(\Delta + H)^{Y_i} = (\Delta + H)^{Y_{i+1}} .$$



If $\mathrm{cont}_i : Y_i \to W_i$ is a flipping contraction then again the properties hold for g_{i+1} which is the structure morphism of the the flip of cont_i provided it exists.



and

$$\phi_i * (\Delta + H)^{Y_i} = (\Delta + H)^{Y_{i+1}} .$$

holds again.

We will show that the flipping contractions occurring in this log MMP are all elementary pl flips, whose existence we have assumed. Let C_i be such such a flipping curve, then $C_i \subseteq \operatorname{Supp} g_i^* H'$ and so $g_i^* H' \cdot C_i = 0$. As $N^1(Y_i/Z)$ is generated by the irreducible components of $(H')_Y$ and the g_i-exceptional divisors, which in turn all show up in $g_i^* H'$, there exists an irredcible component F_i of $g_i^* H'$ for which $F_i \cdot C_i \neq 0$. But then $g_i^* H'$ also has an irreducible component with $F_i' \cdot C_i < 0$. Therefore (?) the contraction of C_i is an elementary pl flipping contraction.

Let g_i be the corresponding flip. Note that g_{i+1} also has the property

$$\lfloor (\Delta + H)^{Y_{i+1}} \rfloor \supseteq \operatorname{Supp} {g_{i+1}}^* H' \supseteq C_{i+1}$$

for g_{i+1}-flipping curves.

Since we are assuming Special Termination in the appropriate dimension, this means that after finitely many steps the program stops with a $\mathbb{Q}$-factorial dlt pair $\overline{g} : (\overline{Y}, (\Delta + H)^{\overline{Y}}) \to Z$ where the $\mathbb{Q}$-Cartier divisor $K_{\overline{Y}} + (\Delta + H)^{\overline{Y}}$ is $\overline{g}$-nef, i.e. a log minimal model (but not a log minimal model of $(Y, (\Delta + H)^Y)$ over Z!!!).

Apply the Subtraction Theorem 0.11 with the following setup:

$$f \stackrel{\text{def}}{=} \overline{g} \;\; X \stackrel{\text{def}}{=} Y \;\; Y \stackrel{\text{def}}{=} Z \;, H \stackrel{\text{def}}{=} H_{\overline{Y}} \;, \;\; B \stackrel{\text{def}}{=} \Delta_{\overline{Y}} \;, \;\; S \stackrel{\text{def}}{=} \sum_i E_i \;,$$

where the E_i's are all the exceptional divisors of the rational map $\overline{g} : \overline{Y} \dashrightarrow Z$. The conditions of the Subtraction Theorem are satisfied. The only non-obvious condition to check is that

$$H \sim_{\overline{g}} - \sum_j b_j S_j \;,$$

with nonnegative b_j's, but this is equivalent to

$$H_{\overline{Y}} + \sum_j b_j S_j \sim_{\overline{g}} 0 \sim_{\overline{g}} \overline{g}^* H'$$

, and hence is satisfied.

This provides us with a log minimal model $\tilde{g} : (\tilde{Y}, \Delta_{\tilde{Y}}) \to Z$ of (Y, Δ) where $(\tilde{Y}, \Delta_{\tilde{Y}})$ is $\mathbb{Q}$-factorial dlt, and $K_{\tilde{Y}} + \Delta + \tilde{Y}$ is $\tilde{g}$-nef.

Now it can be proved using the Negativity Lemma 0.12 that $\tilde{g}$ is small, $(\tilde{Y}, \Delta_{\tilde{Y}})$ is klt, and the inverse of the rational map $\tilde{g} \circ f$ has no exceptional divisors. Hence $\tilde{g} : (\tilde{Y}, \Delta_{\tilde{Y}})$ is a log minimal model of f, and hence it is the flip of f. □

Theorem 0.11 (Subtraction Theorem)**.** ([1, Theorem 4.3.8]) *Assume special termination and the existence of elementary pl flips in dimension n.*

Let (X,S+B+H) be an n-dimensional $\mathbb{Q}$-factorial dlt pair with S, B, H being effective $\mathbb{Q}$-divisors such that $\lfloor S \rfloor = S$ and $\lfloor B \rfloor = 0$. Let $f : X \to Y$ be a projective birational morphism satisfying the following properties:

(a) $H \equiv_f -\sum_j b_j S_j$, where the b_j's are nonnegative rational numbers, and the S_j's are the irreducible components of S;
(b) the divisor $K_X + S + B + H$ is f-nef.

Then the pair $(X, S + B)$ has a log minimal model over Y.

Proof. Observe that by [7, Complement 3.6][4] there exists a rational number $0 \leq \lambda frm[o]--$ such that $K_X + S + B + \lambda H$ is f-nef, moreover $\lambda > 0$, then in addition there exists a $(K_X + S + B)$-negative extremal ray ρ over Y such that

$$(K_X + S + B) \cdot \rho = 0 \ .$$

As $\lambda = 0$ implies that we are done, we will assume that $\lambda > 0$. Let $\text{cont}_\rho : X \to Z$ denote the contraction of ρ.

If cont_ρ is a divisorial contraction, then we replace $f : X \to Y$ by the structure morphism $Z \to Y$, and S, B, H by their proper transforms. Note that all conditions of the Theorem are again satisfied.

If cont_ρ is a flipping contractions, then we will show that it is an elementary flipping contraction. Granting this, and the existence of elementary pl flips, replace f by $\text{cont}^+_\rho : X^+ \to Y$, and S, B, H by their proper transforms.Again, it is immediate to check that the conditions of the Theorem continue to hold.

As to the fact that cont_ρ is a pl flipping contraction: from the relations

$$(K_X + S + B) \cdot \rho = 0 \text{ and } (K_X + S + B) \cdot \rho < 0$$

it follows that $H \cdot \rho > 0$. Consequently, S has an irreducible component S_j with $S_j \cdot \rho < 0$ according to condition (i) of the theorem, which is what we wanted.

Repeating this process over and over again, we claim that after finitely many steps it will come to a halt, that is, we arrive at $\lambda = 0$. First of all, observe that we can only have finitely many divisorial contractions in this sequence, as they decrease the Picard number by one. Therefore our only concern again is the termination of the

[4]Kollár and Mori require (X, Δ) to be klt. However they only need this for the Rationality/Cone theorems to hold, hence it is safe to relax their hypothesis to (X, Δ) dlt, since these results are true for dlt pairs.

occurring flips. However as all the occurring flips are elementary pl flips, special termination does the trick according to Remark 0.5.

It is now immediate to check that the above process returns a log minimal model of f. □

Lemma 0.12 (Negativity of contraction)**.** ([8, Lemma 2.19.1]) *Let $f : X \to Y$ be a proper birational morphism, assume that X is normal. Let $\{F_i\}$ be the f-exceptional divisors. Let L be a Cartier divisor on Y , M an f-nef Cartier divisor on X, G an effective divisor on X such that none of the f-exceptional divisors is a component of G. If*

$$f^*L \sim_{num} M + G + \sum_i a_i F_i \ ,$$

then

(a) $a_i \geq 0$ for every i;
(b) $a_i > 0$ if M is not numerically f-trivial on some F_j with $f(F_i) = f(F_j)$.

Remark 0.13. By checking the proof in [8], we can convince ourselves that it is okay to take $G = 0$ (although 0 is not an effective divisor.)

References

[1] Alessio Corti: Flips for 3-folds and 4-folds.
[2] Alessio Corti, Anne-Sophie Kaloghiros, Vladimir Ĺazic: Introduction to the minimal model program and the existence of flips
[3] Olivier Debarre: Introduction to higher-dimensional geometry. Springer
[4] Robin Hartshorne: Algebraic geometry. Springer 1977.
[5] Iitaka: Algebraic geometry (an introdcution to birational geometry of algebraic varieties)
[6] Steven L. Kleiman: Toward a numerical theory of ampleness. Annals of Mathematics 84 (1966) No. 3., 293–344.
[7] János Kollár, Shigefumi Mori: Birational geometry of algebraic varieties.
[8] János Kollár et al.: Flips and abundance
[9] Robert Lazarsfeld: Positivity in Algebraic Geometry I.-II. Ergebnisse der Mathematik und ihrer Grenzgebiete, Vols. 48-49., Springer Verlag, Berlin, 2004.

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