The Dimension of the $\mathbb{R}$-Disguised Toric Locus of a Reaction Network

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Online seminar, Formal Reaction Kinetics and Related Questions

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History of mass-action kinetics

- In 1972, Horn and Jackson introduced the basic properties and found the complex balancing in mass-action kinetics [1, 2].

- In 2009, Craciun, Dickenstein, Shiu and Sturmfels gave a characterization of the toric locus [3].

[1]: F. Horn. "Necessary and sufficient conditions for complex balancing in chemical kinetics". In: *Arch. Ration. Mech. Anal.* 49.3 (1972), pp. 172–186

[2]: F. Horn and R. Jackson. "General mass action kinetics". In: *Arch. Rational Mech. Anal.* 47 (1972), pp. 81–116

[3]: G. Craciun, A. Dickenstein, A. Shiu, and B. Sturmfels. "Toric dynamical systems". In: *J. Symbolic Comput.* 44.11 (2009), pp. 1551–1565

My work on mass-action kinetics

- In 2022, Craciun, **Jin** and Sorea found a new structural of the toric locus [4, 5].
- In 2023, Craciun, Deshpande and **Jin** discovered more properties on the disguised toric locus [6, 7, 8].

[4]: **J. Jin**, G. Craciun, and M.-S. Sorea. "The structure of the moduli spaces of toric dynamical systems". In: *Submitted* (2023)

[5]: **J. Jin**, G. Craciun, and M.-S. Sorea. "The toric locus of a reaction network is a smooth manifold". In: *Submitted* (2023)

[6]: **J. Jin**, G. Craciun, and Deshpande A. "On the connectivity of the Disguised Toric Locus". In: *Accepted by Journal of Mathematical Chemistry* (2023)

[7]: **J. Jin**, G. Craciun, and A. Deshpande. "A Lower Bound on the Dimension of the Disguised Toric Locus". In: *In revision by SIAM Journal on Applied Algebra and Geometry* (2023)

[8]: **J. Jin**, G. Craciun, and A. Deshpande. "On the Dimension of the R-Disguised Toric Locus of Reaction Networks". In: *Preprint* (2023)

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- The key processes in biological and chemical systems are described by **biochemical reaction networks**.

- A biochemical reaction network comprises a set of **complexes** (**reactants** and **products**), and a set of **reactions**.

$$\text{Complexes:} \quad \{H_2, O_2, H_2O\}$$

$$\text{A reaction:} \quad \underbrace{2H_2 + O_2}_{reactant} \to \underbrace{2H_2O}_{product}$$

Mass-action kinetics and Euclidean embedded graph

- **Standard deterministic mass-action kinetics** says that the rate at which a reaction occurs is proportional to the concentrations of the reactant species.

$$\text{Reaction: } \underbrace{X_1 + X_2}_{reactant} \xrightarrow{k} \underbrace{X_3 + X_4}_{product}$$

$$x_i : \text{the concentration of species } X_i,$$

$$k : \text{the reaction rate constant},$$

$$\text{Reaction rate: } k x_1 x_2.$$

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$$\text{Reaction rate: } kx_1x_2.$$

- A reaction network can be regarded as a **Euclidean embedded graph** $G = (V, E)$, where $V \subset \mathbb{R}^n_{\geq 0}$ is the set of vertices of the graph, and $E \subset V \times V$ is the set of oriented edges of G.

Example: The Lotka-Volterra systems can be considered as a reaction network in XY-plane with 6 complexes and 3 reactions.

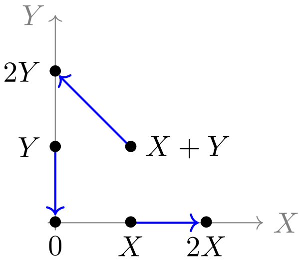


Figure 1: A reaction network of the Lotka-Volterra system.

$$\begin{aligned} \text{Species:} \quad & \mathcal{S} = \{X, Y\}, \\ \text{Complexes:} \quad & \mathcal{C} = \{X, X+Y, Y, 2X, 2Y, 0\}, \\ \text{Reactions:} \quad & \mathcal{R} = \{X \to 2X, X+Y \to 2Y, Y \to 0\}. \end{aligned}$$

Example: We can translate the Lotka-Volterra systems into a Euclidean embedded graph G in $\mathbb{R}^2$ with 6 vertices and 3 reactions.

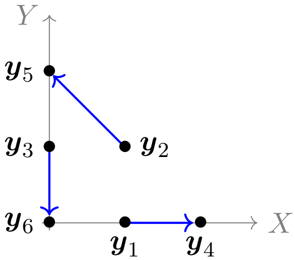
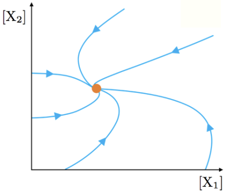


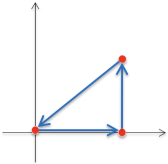
Figure 2: A Euclidean embedded graph $G = (V, E)$ of the Lotka-Volterra system.

The set of vertices $\quad V = \{\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3, \boldsymbol{y}_4, \boldsymbol{y}_5, \boldsymbol{y}_6\}$,

The set of edges $\quad E = \{\boldsymbol{y}_1 \to \boldsymbol{y}_4, \boldsymbol{y}_2 \to \boldsymbol{y}_5, \boldsymbol{y}_3 \to \boldsymbol{y}_6\}$.

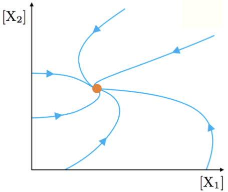


$$\varnothing \xrightarrow{0.7} X_1 \xrightarrow{1} X_1 + X_2 \xrightarrow{1} \varnothing$$



(a)

Figure 3: Reaction networks and Euclidean embedded graphs.



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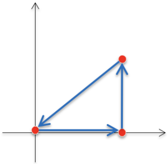
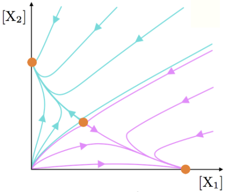
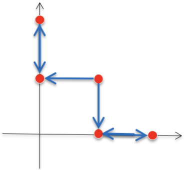


(a)



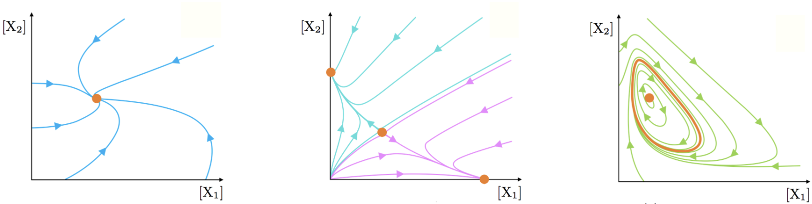
$$X_1 + X_2 \xrightarrow{1} X_1 \underset{1}{\overset{3}{\rightleftharpoons}} 2X_1$$

$$X_1 + X_2 \xrightarrow{2} X_2 \underset{1}{\overset{2}{\rightleftharpoons}} 2X_2$$



(b)

Figure 3: Reaction networks and Euclidean embedded graphs.



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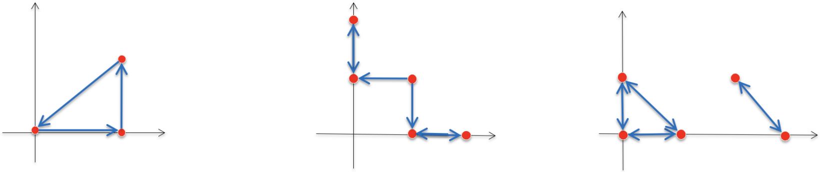
$$\varnothing \rightleftharpoons X_1 \rightleftharpoons X_2 \rightleftharpoons \varnothing$$
$$2X_1 + X_2 \rightleftharpoons 3X_1$$



(a) (b) (c)

Figure 3: Reaction networks and Euclidean embedded graphs.

Weakly reversible

Let $G = (V, E)$ be a Euclidean embedded graph.

(a) The set of vertices is partitioned by its **connected components**, and we identify them by the subset of vertices that belong to that connected component.

Weakly reversible

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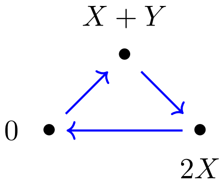
(a) The set of vertices is partitioned by its **connected components**, and we identify them by the subset of vertices that belong to that connected component.

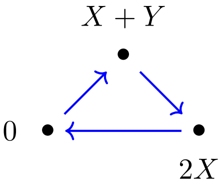
(b) A graph $G = (V, E)$ is **weakly reversible**, if every edge in any connected component is part of an oriented cycle.

Example: Two Euclidean embedded graphs G and G'.



(a) $G = (V, E)$

Example: Two Euclidean embedded graphs G and G'.



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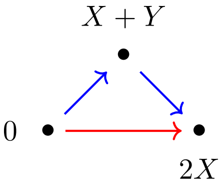


(b) $G' = (V', E')$

Figure 4: G is weakly reversible, but G' isn't weakly reversible

Mass-action system

Let $G = (V, E)$ be a Euclidean embedded graph.

- Let $\boldsymbol{k} = (k_{\boldsymbol{y}\to\boldsymbol{y}'})_{\boldsymbol{y}\to\boldsymbol{y}'\in G} \in \mathbb{R}^{E}_{>0}$ be a vector of *rate constants*. We call $(G, \boldsymbol{k})$ a ***mass-action system***, and its ***associated dynamical system*** is given by

$$\frac{d\boldsymbol{x}}{dt} = \sum_{\boldsymbol{y}\to\boldsymbol{y}'\in E} \underbrace{k_{\boldsymbol{y}\to\boldsymbol{y}'}\boldsymbol{x}^{\boldsymbol{y}}}_{\text{reaction rate}} \times \underbrace{(\boldsymbol{y}' - \boldsymbol{y})}_{\text{change of species}},$$

  where $\boldsymbol{x}^{\boldsymbol{y}} = x_1^{y_1} x_2^{y_2} \cdots x_n^{y_n}$ with $\boldsymbol{x} \in \mathbb{R}^n_{>0}$ is the vector of *concentrations* of the chemical species in the system.

Invariant polyhedron

Given the mass-action system

$$\frac{d\boldsymbol{x}}{dt} = \sum_{\boldsymbol{y} \to \boldsymbol{y}' \in E} k_{\boldsymbol{y} \to \boldsymbol{y}'} \boldsymbol{x}^{\boldsymbol{y}} (\boldsymbol{y}' - \boldsymbol{y}).$$

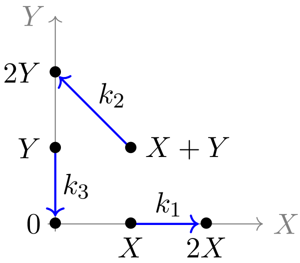
- ***The stoichiometric subspace*** is the vector space spanned by the reaction vectors with $\mathcal{S} = \text{span}\{\boldsymbol{y}' - \boldsymbol{y} : \boldsymbol{y} \to \boldsymbol{y}' \in E\}$.
- For any positive vector $\boldsymbol{x}_0 \in \mathbb{R}^n_{>0}$, the set $\mathcal{S}_{\boldsymbol{x}_0} := (\boldsymbol{x}_0 + S) \cap \mathbb{R}^n_{>0}$ is known as the ***(affine) invariant polyhedron*** of $\boldsymbol{x}_0$.

Example: Recall a reaction network of the Lotka-Volterra system in XY-plane. Given a rate constants vector $\boldsymbol{k} = (k_{\boldsymbol{y}\to\boldsymbol{y}'})_{\boldsymbol{y}\to\boldsymbol{y}'\in G} \in \mathbb{R}^E_{>0}$, the mass-action system $(G, \boldsymbol{k})$ is given by

$$X \xrightarrow{k_1} 2X, \quad X + Y \xrightarrow{k_2} 2Y, \quad Y \xrightarrow{k_3} 0.$$

Then the associated dynamical system is

$$\frac{d\boldsymbol{x}}{dt} = k_1 x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2 x_1 x_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + k_3 x_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} k_1 x_1 \quad - k_2 x_1 x_2 \\ k_2 x_1 x_2 - k_3 x_2 \end{pmatrix}.$$



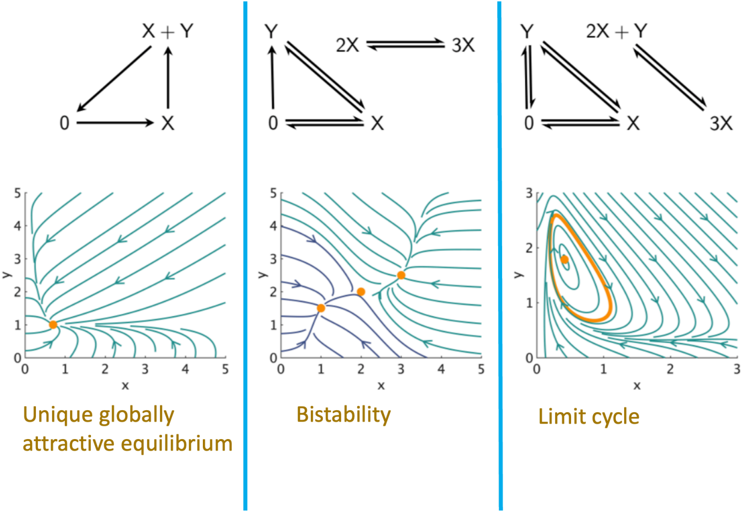


Figure 5: Possible dynamic for mass-action systems in two dimensions.

Complex-balanced system

- Let $(G, \boldsymbol{k})$ be a mass-action system, a state $\boldsymbol{x}_0 \in \mathbb{R}^n_{>0}$ is a ***positive steady state*** if

$$\mathbf{0} \ = \sum_{\boldsymbol{y} \to \boldsymbol{y}' \in G} k_{\boldsymbol{y} \to \boldsymbol{y}'} \boldsymbol{x}_0^{\boldsymbol{y}} (\boldsymbol{y}' - \boldsymbol{y}).$$

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- A positive steady state $\boldsymbol{x}_0 \in \mathbb{R}^n_{>0}$ is ***complex-balanced*** if for every vertex $\boldsymbol{y}_0 \in V_G$, we have

$$\underbrace{\sum_{\boldsymbol{y}_0 \to \boldsymbol{y}' \in G} k_{\boldsymbol{y}_0 \to \boldsymbol{y}'} \boldsymbol{x}_0^{\boldsymbol{y}_0}}_{\text{outgoing flux on } \boldsymbol{y}_0} \ = \ \underbrace{\sum_{\boldsymbol{y} \to \boldsymbol{y}_0 \in G} k_{\boldsymbol{y} \to \boldsymbol{y}_0} \boldsymbol{x}_0^{\boldsymbol{y}}}_{\text{incoming flux on } \boldsymbol{y}_0}.$$

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- If $(G, \boldsymbol{k})$ has a complex-balanced steady state, then it is called **a complex-balanced system** or **toric dynamical system**.

Example: This system is complex-balanced. For example, at the vertex $(0, 1)$, there is one reaction going into it with flux value 3, and there are two reactions leaving this vertex, with sum of fluxes being $2 + 1 = 3$.

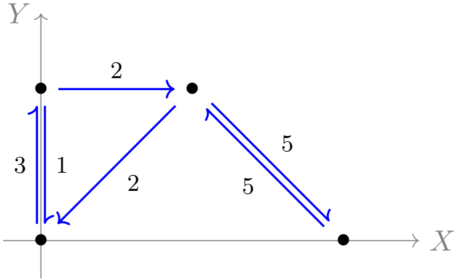

Figure 6: An example of a complex-balanced system. The positive numbers on any edge is the flux of that reaction $\boldsymbol{y} \to \boldsymbol{y}'$.

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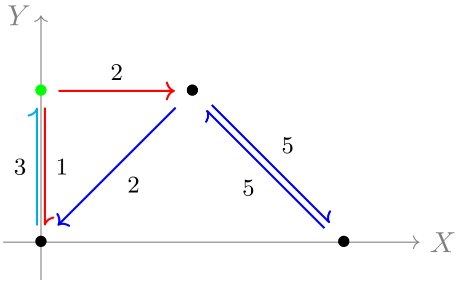

Figure 6: An example of a complex-balanced system. The positive numbers on any edge is the flux of that reaction $\boldsymbol{y} \to \boldsymbol{y}'$.

Dynamical properties of complex-balanced systems

In [2], it shows that given a complex-balanced system $(G, \boldsymbol{k})$ with the stoichiometric subspace $\mathcal{S}$. Let $\boldsymbol{x}^* \in \mathbb{R}^n_{>0}$ be a complex-balanced steady state, then

(a) All positive steady states are complex-balanced. There is exactly one steady state within each invariant polyhedron.

(b) Any complex-balanced steady state $\boldsymbol{x}$ satisfies the following relation: $\ln(\boldsymbol{x}) - \ln(\boldsymbol{x}^*) \in \mathcal{S}^\perp$.

(c) Every complex-balanced steady state is locally asymptotically stable within its invariant polyhedron.

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[2]: F. Horn and R. Jackson. "General mass action kinetics". In: *Arch. Rational Mech. Anal.* 47 (1972), pp. 81–116

Toric locus

- Consider a E-graph $G = (V, E)$, let $\mathcal{V}(G) \subseteq \mathbb{R}^{E}_{>0}$ denote the set of parameters $\boldsymbol{k} \in \mathbb{R}^{E}_{>0}$, for which the dynamical system generated by $(G, \boldsymbol{k})$ is toric (i.e., complex-balanced).

[1]: F. Horn. "Necessary and sufficient conditions for complex balancing in chemical kinetics". In: *Arch. Ration. Mech. Anal.* 49.3 (1972), pp. 172–186

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- $\mathcal{V}(G)$ is called the **toric locus** of toric dynamical systems given by the Euclidean embedded graph G.

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- $\mathcal{V}(G)$ is called the **toric locus** of toric dynamical systems given by the Euclidean embedded graph G.
- In [1], it shows that given an E-graph $G = (V, E)$,
  (a) If $G = (V, E)$ is weakly reversible, then $\mathcal{V}(G) \neq \emptyset$.
  (b) If $G = (V, E)$ is not weakly reversible, then $\mathcal{V}(G) = \emptyset$.

[1]: F. Horn. "Necessary and sufficient conditions for complex balancing in chemical kinetics". In: *Arch. Ration. Mech. Anal.* 49.3 (1972), pp. 172–186

Deficiency and Deficiency Zero Theorem

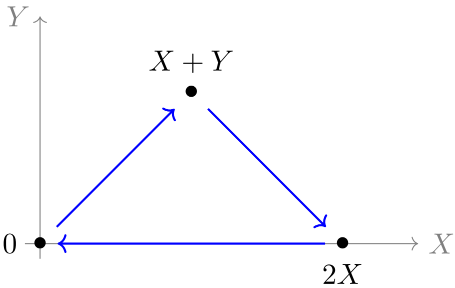
Let $G = (V, E)$ be a reaction network with ℓ connected components and the stoichiometric subspace $\mathcal{S}$. Suppose that $s = \dim S$, then the **deficiency** of the network G is given by

$$\delta = |V| - \ell - s \geq 0.$$

Theorem 1.1 ([1], Deficiency Zero Theorem)

A mass-action system is complex-balanced for every set of positive rate constants if and only if it is weakly reversible and deficiency zero.

Example: This system is weakly reversible and deficiency zero.



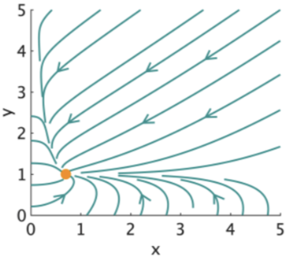


Figure 7: An example of a weakly reversible and deficiency zero system. It is complex-balanced for any positive rate constants

Characterization of the complex-balanced equilibria

The following proposition proved in [3] gives a characterization of the complex-balanced equilibria.

Proposition 1.2

Consider a weakly reversible mass-action system $(G, \boldsymbol{k})$*. For any two vertices* $\boldsymbol{y_i}$ *and* $\boldsymbol{y_j}$*, let* $K_i = \sum_{\mathcal{T} \text{ an } i\text{-tree}} \boldsymbol{k}^{\mathcal{T}}$ *and construct the following equation:*

$$K_i \boldsymbol{x}^{\boldsymbol{y_j}} - K_j \boldsymbol{x}^{\boldsymbol{y_i}} = 0. \tag{1}$$

Then $\boldsymbol{x}$ *is a complex-balanced equilibrium for the reaction rate vector* $\boldsymbol{k}$ *if and only if Equations (1) are satisfied for every pair of vertices in the same connected component in* G*.*

[3]: G. Craciun, A. Dickenstein, A. Shiu, and B. Sturmfels. "Toric dynamical systems". In: *J. Symbolic Comput.* 44.11 (2009), pp. 1551–1565

Example: Consider a strongly connected mass-action system $(G, \boldsymbol{k})$ in Figure 8, with three vertices:

$$\boldsymbol{y}_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad \boldsymbol{y}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \boldsymbol{y}_3 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

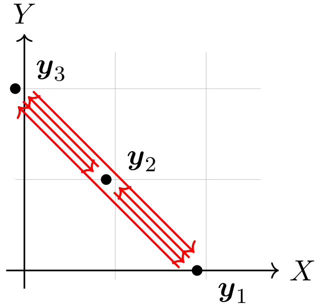
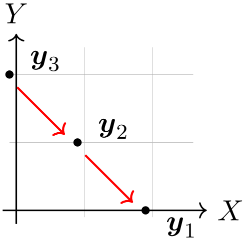
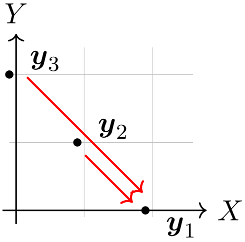


Figure 8: Complete bidirected graph with three vertices.

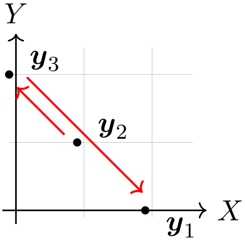
For the vertex $\boldsymbol{y}_1$, we list all spanning $\boldsymbol{y}_1$-trees of G as follows:



(a) $k_{32}k_{21}$



(b) $k_{21}k_{31}$



(c) $k_{23}k_{31}$

Figure 9: Spanning $\boldsymbol{y}_1$-trees of G and $K_1 = k_{32}k_{21} + k_{21}k_{31} + k_{23}k_{31}$.

The toric locus can be written as

$$\mathcal{V}(G) = \Big\{\boldsymbol{k} \in \mathbb{R}^6_{>0} : (k_{21}k_{31} + k_{32}k_{21} + k_{23}k_{31})(k_{13}k_{23} + k_{21}k_{13} + k_{12}k_{23}) - (k_{12}k_{32} + k_{13}k_{32} + k_{31}k_{12})^2 = 0\Big\}.$$

Flux vector

Let $G = (V, E)$ be an E-graph.

- Let $\boldsymbol{J} = (J_{\boldsymbol{y_i} \to \boldsymbol{y_j}})_{\boldsymbol{y_i} \to \boldsymbol{y_j} \in E} \in \mathbb{R}^{E}_{>0}$ denote a **flux vector**, whose component $J_{\boldsymbol{y_i} \to \boldsymbol{y_j}} > 0$ is called the **flux** of the reaction $\boldsymbol{y_i} \to \boldsymbol{y_j}$.

Flux vector

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- Let $\boldsymbol{J} = (J_{\boldsymbol{y_i} \to \boldsymbol{y_j}})_{\boldsymbol{y_i} \to \boldsymbol{y_j} \in E} \in \mathbb{R}^E_{>0}$ denote a **flux vector**, whose component $J_{\boldsymbol{y_i} \to \boldsymbol{y_j}} > 0$ is called the **flux** of the reaction $\boldsymbol{y_i} \to \boldsymbol{y_j}$.
- A flux vector $\boldsymbol{J}$ is called a **complex-balanced flux vector** , if at each vertex $\boldsymbol{y}_0 \in V$,

$$\sum_{\boldsymbol{y} \to \boldsymbol{y}_0 \in E} J_{\boldsymbol{y} \to \boldsymbol{y}_0} = \sum_{\boldsymbol{y}_0 \to \boldsymbol{y}' \in E} J_{\boldsymbol{y}_0 \to \boldsymbol{y}'}.$$

Flux vector

Let $G = (V, E)$ be an E-graph.

- Let $\boldsymbol{J} = (J_{\boldsymbol{y_i} \to \boldsymbol{y_j}})_{\boldsymbol{y_i} \to \boldsymbol{y_j} \in E} \in \mathbb{R}^{E}_{>0}$ denote a **flux vector**, whose component $J_{\boldsymbol{y_i} \to \boldsymbol{y_j}} > 0$ is called the **flux** of the reaction $\boldsymbol{y_i} \to \boldsymbol{y_j}$.
- A flux vector $\boldsymbol{J}$ is called a **complex-balanced flux vector** , if at each vertex $\boldsymbol{y}_0 \in V$,

$$\sum_{\boldsymbol{y} \to \boldsymbol{y}_0 \in E} J_{\boldsymbol{y} \to \boldsymbol{y}_0} = \sum_{\boldsymbol{y}_0 \to \boldsymbol{y}' \in E} J_{\boldsymbol{y}_0 \to \boldsymbol{y}'}.$$

- Recall that $\boldsymbol{x}_0 \in \mathbb{R}^{n}_{>0}$ is a ***complex-balanced steady state*** , if for every vertex $\boldsymbol{y}_0 \in V_G$,

$$\sum_{\boldsymbol{y}_0 \to \boldsymbol{y}' \in G} k_{\boldsymbol{y}_0 \to \boldsymbol{y}'} \boldsymbol{x}_0^{\boldsymbol{y}_0} = \sum_{\boldsymbol{y} \to \boldsymbol{y}_0 \in G} k_{\boldsymbol{y} \to \boldsymbol{y}_0} \boldsymbol{x}_0^{\boldsymbol{y}}.$$

$\mathcal{J}(G)$: set of complex-balanced flux vectors

- We define the **set of complex-balanced flux vectors** on G as

$$\mathcal{J}(G) := \{\boldsymbol{J} \in \mathbb{R}^{E}_{>0} \mid \boldsymbol{J} \text{ is a complex-balanced flux vector on } G\}.$$

$\mathcal{J}(G)$: set of complex-balanced flux vectors

- We define the **set of complex-balanced flux vectors** on G as

$$\mathcal{J}(G) := \{\boldsymbol{J} \in \mathbb{R}^E_{>0} \mid \boldsymbol{J} \text{ is a complex-balanced flux vector on } G\}.$$

- Analogous to complex-balanced systems, given an E-graph $G = (V, E)$, we conclude that

  (a) If $G = (V, E)$ is weakly reversible, then $\mathcal{J}(G) \neq \emptyset$.

  (b) If $G = (V, E)$ isn't weakly reversible, then $\mathcal{J}(G) = \emptyset$.

Example: Revisit the E-graph $G = (V, E)$, the following flux vector is a complex-balanced flux vector in $\mathcal{J}(G) \in \mathbb{R}^6_{>0}$.

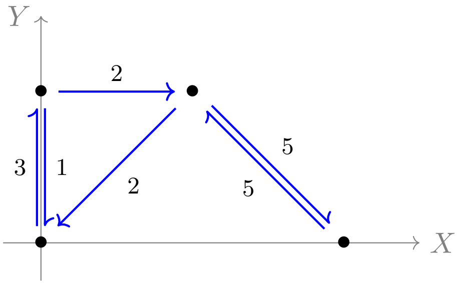

Figure 10: An example of a complex-balanced flux system. The positive numbers on any edge $\boldsymbol{y} \to \boldsymbol{y}'$ is the flux of that reaction.

The product structure of the toric locus

Theorem 1 ([4])

Let $G = (V, E)$ be a weakly reversible E-graph with the stoichiometric subspace $\mathcal{S}$. For any state $\boldsymbol{x}_0 \in \mathbb{R}^n_{>0}$, the toric locus $\mathcal{V}(G) \subseteq \mathbb{R}^E_{>0}$ is homeomorphic to the product space $\mathcal{S}_{\boldsymbol{x}_0} \times \mathcal{J}(G)$, that is,

$$\mathcal{V}(G) \simeq \mathcal{S}_{\boldsymbol{x}_0} \times \mathcal{J}(G).$$

[4]: **J. Jin**, G. Craciun, and M.-S. Sorea. "The structure of the moduli spaces of toric dynamical systems". In: *Submitted* (2023)

A map from $\mathcal{S}_{\boldsymbol{x}_0} \times \mathcal{J}(G)$ to $\mathcal{V}(G)$

- Let $G = (V, E)$ be a weakly reversible E-graph. Given a state $\boldsymbol{x}_0 \in \mathbb{R}^n_{>0}$, we define the following map:

$$\varphi : \mathcal{S}_{\boldsymbol{x}_0} \times \mathcal{J}(G) \to \mathcal{V}(G),$$

such that for any $\boldsymbol{x} \in \mathcal{S}_{\boldsymbol{x}_0}$ and $\boldsymbol{J} = (J_{\boldsymbol{y_i} \to \boldsymbol{y_j}})_{\boldsymbol{y_i} \to \boldsymbol{y_j} \in E} \in \mathcal{J}(G)$,

$$\varphi(\boldsymbol{x}, \boldsymbol{J}) := (\varphi_{\boldsymbol{y_i} \to \boldsymbol{y_j}})_{\boldsymbol{y_i} \to \boldsymbol{y_j} \in E}, \text{ with } \varphi_{\boldsymbol{y_i} \to \boldsymbol{y_j}} := \frac{J_{\boldsymbol{y_i} \to \boldsymbol{y_j}}}{\boldsymbol{x}^{\boldsymbol{y_i}}}.$$

A map from $\mathcal{S}_{\boldsymbol{x}_0} \times \mathcal{J}(G)$ to $\mathcal{V}(G)$

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$$\varphi(\boldsymbol{x}, \boldsymbol{J}) := (\varphi_{\boldsymbol{y_i} \to \boldsymbol{y_j}})_{\boldsymbol{y_i} \to \boldsymbol{y_j} \in E}, \text{ with } \varphi_{\boldsymbol{y_i} \to \boldsymbol{y_j}} := \frac{J_{\boldsymbol{y_i} \to \boldsymbol{y_j}}}{\boldsymbol{x}^{\boldsymbol{y_i}}}.$$

- In [4], we show the map φ is a homeomorphism, that is, φ is bijective, continuous and the inverse function φ^{-1} is continuous.

The dimension of the toric locus $\mathcal{V}(G)$

Proposition 2.1 ([4])

Consider an E-graph $G = (V, E)$ with ℓ connected components. Let s be the dimension of the stoichiometric subspace $\mathcal{S}$, then

$$\dim(\mathcal{V}(G)) = |E| - |V| + s + l.$$

The dimension of the toric locus $\mathcal{V}(G)$

Proposition 2.1 ([4])

Consider an E-graph $G = (V, E)$ with ℓ connected components. Let s be the dimension of the stoichiometric subspace $\mathcal{S}$, then

$$\dim(\mathcal{V}(G)) = |E| - |V| + s + l.$$

The following corollary is a direct consequence of Proposition 2.1, which was also proved in [3].

Corollary 2.2 ([4])

Let $G = (V, E)$ be a weakly reversible E-graph. Then the codimension of the toric locus $\mathcal{V}(G) \subseteq \mathbb{R}^E_{>0}$ is δ.

The toric locus $\mathcal{V}(G)$ is a smooth manifold

- In [5], we further show that

$$\varphi : \mathcal{S}_{\boldsymbol{x}_0} \times \mathcal{J}(G) \to \mathcal{V}(G).$$

  is a diffeomorphism

- The toric locus $\mathcal{V}(G)$ is the image of an embedding, thus it is a smoothly embedded manifold.

- The complex-balanced equilibrium depends smoothly on the reaction rate constants in $\mathcal{V}(G)$.

[5]: **J. Jin**, G. Craciun, and M.-S. Sorea. "The toric locus of a reaction network is a smooth manifold". In: *Submitted* (2023)

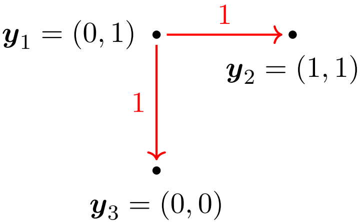
Dynamical Equivalence

Two mass-action systems $(G, \boldsymbol{k})$ and $(G', \boldsymbol{k}')$ are said to be **dynamically equivalent**, if for every vertex $\boldsymbol{y}_0 \in V \cup V'$,

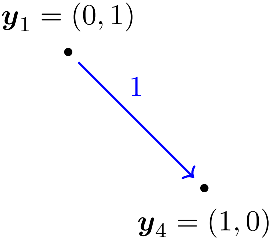
$$\sum_{\boldsymbol{y}_0 \to \boldsymbol{y} \in E} k_{\boldsymbol{y}_0 \to \boldsymbol{y}} (\boldsymbol{y} - \boldsymbol{y}_0) = \sum_{\boldsymbol{y}_0 \to \boldsymbol{y}' \in E'} k'_{\boldsymbol{y}_0 \to \boldsymbol{y}'} (\boldsymbol{y}' - \boldsymbol{y}_0). \tag{2}$$

We let $(G, \boldsymbol{k}) \sim (G', \boldsymbol{k}')$ denote that two systems $(G, \boldsymbol{k})$ and $(G', \boldsymbol{k}')$ are dynamically equivalent.

Example: Figure 11 gives an example of two dynamically equivalent mass-action systems.



(a) $G = (V, E)$



(b) $G' = (V', E')$

Figure 11: The mass-action systems in (a) and (b) are dynamically equivalent.

$\mathcal{D}_{\mathbf{0}}(G)$

- Let $G = (V, E)$ be an E-graph and let $\boldsymbol{d} = (d_{\boldsymbol{y} \to \boldsymbol{y}'})_{\boldsymbol{y} \to \boldsymbol{y}' \in E} \in \mathbb{R}^{|E|}$. We define the set $\mathcal{D}_{\mathbf{0}}(G)$ as

$$\mathcal{D}_{\mathbf{0}}(G) := \{\boldsymbol{d} \in \mathbb{R}^{|E|} \,\Big|\, \sum_{\boldsymbol{y}_0 \to \boldsymbol{y} \in E} d_{\boldsymbol{y}_0 \to \boldsymbol{y}} (\boldsymbol{y} - \boldsymbol{y}_0) = \mathbf{0} \quad \text{for every vertex } \boldsymbol{y}_0 \in V\}.$$

- Let $G = (V, E)$ be an E-graph and let $\boldsymbol{d} = (d_{\boldsymbol{y} \to \boldsymbol{y}'})_{\boldsymbol{y} \to \boldsymbol{y}' \in E} \in \mathbb{R}^{|E|}$. We define the set $\mathcal{D}_{\mathbf{0}}(G)$ as

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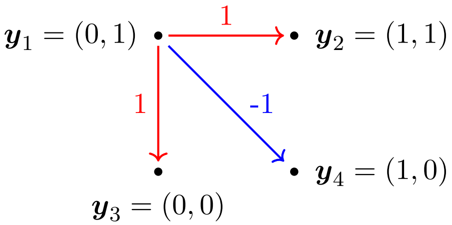


Figure 12: A rate vector $\boldsymbol{d} \in \mathcal{D}_{\mathbf{0}}(G)$.

Flux Equivalence and $\mathcal{J}_{\mathbf{0}}(G)$

- Two flux systems $(G, \boldsymbol{J}) = (V, E, \boldsymbol{J})$ and $(G', \boldsymbol{J}') = (V', E', \boldsymbol{J}')$ are said to be **flux equivalent**, denoted by $(G, \boldsymbol{J}) \sim (G', \boldsymbol{J}')$, if for every vertex $\boldsymbol{y}_0 \in V \cup V'$

$$\sum_{\boldsymbol{y}_0 \to \boldsymbol{y} \in E} J_{\boldsymbol{y}_0 \to \boldsymbol{y}} (\boldsymbol{y} - \boldsymbol{y}_0) = \sum_{\boldsymbol{y}_0 \to \boldsymbol{y}' \in E'} J'_{\boldsymbol{y}_0 \to \boldsymbol{y}'} (\boldsymbol{y}' - \boldsymbol{y}_0).$$

Flux Equivalence and $\mathcal{J}_{\mathbf{0}}(G)$

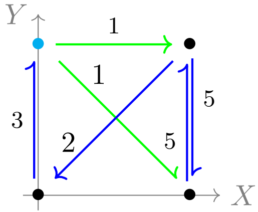
- Two flux systems $(G, \boldsymbol{J}) = (V, E, \boldsymbol{J})$ and $(G', \boldsymbol{J}') = (V', E', \boldsymbol{J}')$ are said to be **flux equivalent**, denoted by $(G, \boldsymbol{J}) \sim (G', \boldsymbol{J}')$, if for every vertex $\boldsymbol{y_0} \in V \cup V'$

$$\sum_{\boldsymbol{y}_0 \to \boldsymbol{y} \in E} J_{\boldsymbol{y}_0 \to \boldsymbol{y}} (\boldsymbol{y} - \boldsymbol{y}_0) = \sum_{\boldsymbol{y}_0 \to \boldsymbol{y}' \in E'} J'_{\boldsymbol{y}_0 \to \boldsymbol{y}'} (\boldsymbol{y}' - \boldsymbol{y}_0).$$

- Let $G = (V, E)$ be an E-graph and let $\boldsymbol{J} = (J_{\boldsymbol{y_i} \to \boldsymbol{y_j}})_{\boldsymbol{y_i} \to \boldsymbol{y_j} \in E} \in \mathbb{R}^E$. We define the set $\mathcal{J}_{\mathbf{0}}(G)$ as

$$\mathcal{J}_{\mathbf{0}}(G) := \{ \boldsymbol{J} \in \mathcal{D}_{\mathbf{0}}(G) \,\Big|\, \sum_{\boldsymbol{y} \to \boldsymbol{y}_0 \in E} J_{\boldsymbol{y} \to \boldsymbol{y}_0} = \sum_{\boldsymbol{y}_0 \to \boldsymbol{y}' \in E} J_{\boldsymbol{y}_0 \to \boldsymbol{y}'} \quad \text{for every vertex } \boldsymbol{y_0} \in V \}.$$

Example: The following two flux systems are flux equivalent.



(a)

Example: The following two flux systems are flux equivalent.

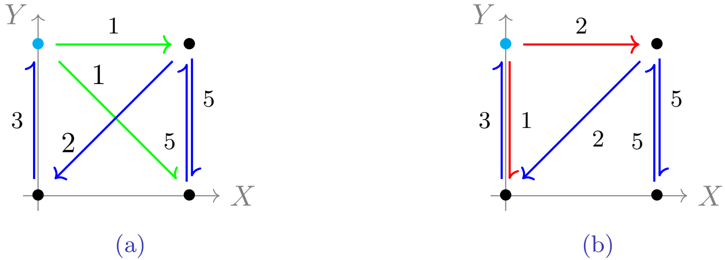


Figure 13: The flux systems in (a) and (b) are flux equivalent. The flux system in (b) is a complex-balanced flux system.

Proposition 3.1 ([9])

Let $(G, \boldsymbol{k})$ *and* $(G', \boldsymbol{k}')$ *be two mass-action systems and let* $\boldsymbol{x} \in \mathbb{R}^n_{>0}$. *Define the flux vector* $\boldsymbol{J}(\boldsymbol{x}) = (J_{\boldsymbol{y} \to \boldsymbol{y}'})_{\boldsymbol{y} \to \boldsymbol{y}' \in E}$ *on* G, *such that for every* $\boldsymbol{y} \to \boldsymbol{y}' \in E$

$$J_{\boldsymbol{y} \to \boldsymbol{y}'} = k_{\boldsymbol{y} \to \boldsymbol{y}'} \boldsymbol{x}^{\boldsymbol{y}}.$$

Further, define the flux vector $\boldsymbol{J}'(\boldsymbol{x}) = (J'_{\boldsymbol{y} \to \boldsymbol{y}'})_{\boldsymbol{y} \to \boldsymbol{y}' \in E'}$ *on* G', *such that for every* $\boldsymbol{y} \to \boldsymbol{y}' \in E$

$$J'_{\boldsymbol{y} \to \boldsymbol{y}'} = k'_{\boldsymbol{y} \to \boldsymbol{y}'} \boldsymbol{x}^{\boldsymbol{y}}.$$

Then the following are equivalent:

(a) $(G, \boldsymbol{k}) \sim (G', \boldsymbol{k}')$.

(b) $(G, \boldsymbol{J}(\boldsymbol{x})) \sim (G', \boldsymbol{J}'(\boldsymbol{x}))$ *for some* $\boldsymbol{x} \in \mathbb{R}^n_{>0}$.

---

[9]: **J. Jin**, G. Craciun, and P. Yu. "An efficient characterization of complex-balanced, detailed-balanced, and weakly reversible systems". In: *SIAM J. Appl. Math.* 80.1 (2020), pp. 183–205

Disguised Toric

- Recall that the **toric locus** on an E-graph G is

$$\mathcal{K}(G) := \{\boldsymbol{k} \in \mathbb{R}^E_{>0} \,\big|\, \text{the mass-action system generated by } (G, \boldsymbol{k}) \text{ is toric}\}.$$

- In [7], a dynamical system of the form

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{f}(\boldsymbol{x}),$$

is called **disguised toric** on G, if it is realizable on G for some $\boldsymbol{k} \in \mathcal{K}(G) \subseteq \mathbb{R}^E_{>0}$, i.e., it has a **complex-balanced realization** on $G = (V, E)$.

---

[7]: **J. Jin**, G. Craciun, and A. Deshpande. “A Lower Bound on the Dimension of the Disguised Toric Locus”. In: *In revision by SIAM Journal on Applied Algebra and Geometry* (2023)

$\mathbb{R}$-Disguised Toric Locus

Let $G = (V, E)$ and $G' = (V', E')$ be two E-graphs.

(a) Define the set $\mathcal{K}_{\mathbb{R}\text{-disg}}(G, G')$ as

$$\mathcal{K}_{\mathbb{R}\text{-disg}}(G, G') := \{\boldsymbol{k} \in \mathbb{R}^E \;\big|\; \text{the dynamical system } (G, \boldsymbol{k}) \text{ is disguised toric on } G'\}.$$

$\mathbb{R}$-Disguised Toric Locus

Let $G = (V, E)$ and $G' = (V', E')$ be two E-graphs.

(a) Define the set $\mathcal{K}_{\mathbb{R}\text{-disg}}(G, G')$ as

$$\mathcal{K}_{\mathbb{R}\text{-disg}}(G, G') := \{ \boldsymbol{k} \in \mathbb{R}^E \mid \text{the dynamical system } (G, \boldsymbol{k}) \text{ is disguised toric on } G' \}.$$

(b) From [9], we define the $\mathbb{R}$**-disguised toric locus** of G as

$$\mathcal{K}_{\mathbb{R}\text{-disg}}(G) = \bigcup_{G' \sqsubseteq G_c} \mathcal{K}_{\mathbb{R}\text{-disg}}(G, G'),$$

where G_c represents the complete graph of G.

$\mathbb{R}$-Disguised Toric Locus is connected

Theorem 3.2 ([6])

Let $G = (V, E)$ and $G' = (V', E')$ be two E-graphs. Then

(a) The set $\mathcal{K}_{\mathbb{R}\text{-}disg}(G, G')$ is connected.

(b) The $\mathbb{R}$-disguised toric locus of G is also connected.

[6]: **J. Jin**, G. Craciun, and Deshpande A. "On the connectivity of the Disguised Toric Locus". In: *Accepted by Journal of Mathematical Chemistry* (2023)

$\mathbb{R}$-realizable on flux systems

- Let $(G', \boldsymbol{J}')$ be a flux system. We say it is **$\mathbb{R}$-realizable** on G if there exists some $\boldsymbol{J} \in \mathbb{R}^E$, such that for every vertex $\boldsymbol{y}_0 \in V \cup V'$,

$$\sum_{\boldsymbol{y}_0 \to \boldsymbol{y_j} \in E} J_{\boldsymbol{y}_0 \to \boldsymbol{y_j}} (\boldsymbol{y_j} - \boldsymbol{y}_0) = \sum_{\boldsymbol{y}_0 \to \boldsymbol{y'_j} \in E'} J'_{\boldsymbol{y}_0 \to \boldsymbol{y'_j}} (\boldsymbol{y'_j} - \boldsymbol{y}_0).$$

$\mathbb{R}$-realizable on flux systems

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- Further, we define the set $\mathcal{J}_{\mathbb{R}}(G', G)$ as

$$\mathcal{J}_{\mathbb{R}}(G', G) := \{ \boldsymbol{J}' \in \mathcal{J}(G') \mid \text{the flux system } (G', \boldsymbol{J}') \text{ is } \mathbb{R}\text{-realizable on } G \}.$$

Let $G_1 = (V_1, E_1)$ be a weakly reversible E-graph and let $G = (V, E)$ be an E-graph.

- There exists a set of vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\} \subset \mathbb{R}^{|E_1|}$, such that

$$\mathcal{J}_{\mathbb{R}}(G_1, G) = \{a_1 \boldsymbol{v}_1 + \cdots a_k \boldsymbol{v}_k \mid a_i \in \mathbb{R}_{>0}, \boldsymbol{v}_i \in \mathbb{R}^{|E_1|}\},$$

  and

$$\dim(\mathcal{J}_{\mathbb{R}}(G_1, G)) = \dim(\mathrm{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}).$$

Let $G_1 = (V_1, E_1)$ be a weakly reversible E-graph and let $G = (V, E)$ be an E-graph.

- There exists a set of vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\} \subset \mathbb{R}^{|E_1|}$, such that

$$\mathcal{J}_{\mathbb{R}}(G_1, G) = \{a_1 \boldsymbol{v}_1 + \cdots a_k \boldsymbol{v}_k \mid a_i \in \mathbb{R}_{>0}, \boldsymbol{v}_i \in \mathbb{R}^{|E_1|}\},$$

  and

$$\dim(\mathcal{J}_{\mathbb{R}}(G_1, G)) = \dim(\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}).$$

- Moreover, if $\mathcal{J}_{\mathbb{R}}(G_1, G) \neq \emptyset$, then

$$\mathcal{J}_{\mathbf{0}}(G_1) \subseteq \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}.$$

Orthonormal basis of $\mathcal{D}_{\mathbf{0}}(G)$ and $\mathcal{J}_{\mathbf{0}}(G_1)$

Let $G_1 = (V_1, E_1)$ be a weakly reversible E-graph and let $G = (V, E)$ be an E-graph.

(a) Let b denote the dimension of the linear subspace $\mathcal{D}_{\mathbf{0}}(G)$, and denote an orthonormal basis of $\mathcal{D}_{\mathbf{0}}(G)$ by

$$\{\boldsymbol{B}_1, \boldsymbol{B}_2, \ldots, \boldsymbol{B}_b\}.$$

(b) Let a denote the dimension of the subspace $\mathcal{J}_{\mathbf{0}}(G_1)$, and denote an orthonormal basis of $\mathcal{J}_{\mathbf{0}}(G_1)$ by

$$\{\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_a\}.$$

A map from $\mathcal{J}_{\mathbb{R}}(G_1, G)$ to $\mathbb{R}^a$

- Define the map
$$\psi : \mathcal{J}_{\mathbb{R}}(G_1, G) \to \mathbb{R}^a,$$
such that for $\boldsymbol{J} \in \mathcal{J}_{\mathbb{R}}(G_1, G)$,
$$\psi(\boldsymbol{J}) = (\boldsymbol{J}\boldsymbol{A}_1, \boldsymbol{J}\boldsymbol{A}_2, \ldots, \boldsymbol{J}\boldsymbol{A}_a).$$
Moreover, we define the set Q as
$$Q := \psi(\mathcal{J}_{\mathbb{R}}(G_1, G)) = \{\psi(\boldsymbol{J}) \mid \boldsymbol{J} \in \mathcal{J}_{\mathbb{R}}(G_1, G)\}.$$

A map from $\mathcal{J}_{\mathbb{R}}(G_1, G)$ to $\mathbb{R}^a$

- Define the map
$$\psi : \mathcal{J}_{\mathbb{R}}(G_1, G) \to \mathbb{R}^a,$$
such that for $\boldsymbol{J} \in \mathcal{J}_{\mathbb{R}}(G_1, G)$,
$$\psi(\boldsymbol{J}) = (\boldsymbol{J}\boldsymbol{A}_1, \boldsymbol{J}\boldsymbol{A}_2, \ldots, \boldsymbol{J}\boldsymbol{A}_a).$$
Moreover, we define the set Q as
$$Q := \psi(\mathcal{J}_{\mathbb{R}}(G_1, G)) = \{\psi(\boldsymbol{J}) \mid \boldsymbol{J} \in \mathcal{J}_{\mathbb{R}}(G_1, G)\}.$$

- Suppose $\mathcal{J}_{\mathbb{R}}(G_1, G) \neq \emptyset$, then Q is an open set in $\mathbb{R}^a$.

A map from $\mathcal{J}_{\mathbb{R}}(G_1, G)$ to $\mathcal{K}_{\mathbb{R}\text{-disg}}(G, G_1)$

- Given an E-graph $G = (V, E)$ and $\boldsymbol{x}_0 \in \mathbb{R}^n_{>0}$.

A map from $\mathcal{J}_{\mathbb{R}}(G_1, G)$ to $\mathcal{K}_{\mathbb{R}\text{-disg}}(G, G_1)$

- Given an E-graph $G = (V, E)$ and $\boldsymbol{x}_0 \in \mathbb{R}^n_{>0}$.
- Given a weakly reversible E-graph $G_1 = (V_1, E_1)$ with its stoichiometric subspace $\mathcal{S}_{G_1}$.

A map from $\mathcal{J}_{\mathbb{R}}(G_1, G)$ to $\mathcal{K}_{\mathbb{R}\text{-disg}}(G, G_1)$

- Given an E-graph $G = (V, E)$ and $\boldsymbol{x}_0 \in \mathbb{R}^n_{>0}$.
- Given a weakly reversible E-graph $G_1 = (V_1, E_1)$ with its stoichiometric subspace $\mathcal{S}_{G_1}$.
- We consider the following map:

$$\Psi : \mathcal{J}_{\mathbb{R}}(G_1, G) \times [(\boldsymbol{x}_0 + \mathcal{S}_{G_1}) \cap \mathbb{R}^n_{>0}] \times \mathbb{R}^b \to \mathcal{K}_{\mathbb{R}\text{-disg}}(G, G_1) \times Q$$

The map Ψ

The map

$$\Psi : \mathcal{J}_{\mathbb{R}}(G_1, G) \times [(\boldsymbol{x}_0 + \mathcal{S}_{G_1}) \cap \mathbb{R}^n_{>0}] \times \mathbb{R}^b \to \mathcal{K}_{\mathbb{R}\text{-disg}}(G, G_1) \times Q$$

satisfies that for $(\boldsymbol{J}, \boldsymbol{x}, \boldsymbol{p}) \in \mathcal{J}_{\mathbb{R}}(G_1, G) \times [(\boldsymbol{x}_0 + \mathcal{S}_{G_1}) \cap \mathbb{R}^n_{>0}] \times \mathbb{R}^b$,

$$\Psi(\boldsymbol{J}, \boldsymbol{x}, \boldsymbol{p}) := (\boldsymbol{k}, \boldsymbol{q}),$$

The map Ψ

The map

$$\Psi : \mathcal{J}_{\mathbb{R}}(G_1, G) \times [(\boldsymbol{x}_0 + \mathcal{S}_{G_1}) \cap \mathbb{R}^n_{>0}] \times \mathbb{R}^b \to \mathcal{K}_{\mathbb{R}\text{-disg}}(G, G_1) \times Q$$

satisfies that for $(\boldsymbol{J}, \boldsymbol{x}, \boldsymbol{p}) \in \mathcal{J}_{\mathbb{R}}(G_1, G) \times [(\boldsymbol{x}_0 + \mathcal{S}_{G_1}) \cap \mathbb{R}^n_{>0}] \times \mathbb{R}^b$,

$$\Psi(\boldsymbol{J}, \boldsymbol{x}, \boldsymbol{p}) := (\boldsymbol{k}, \boldsymbol{q}),$$

where

$$(G, \boldsymbol{k}) \sim (G_1, \boldsymbol{k}_1) \quad \text{with} \quad k_{1,\boldsymbol{y}\to\boldsymbol{y}'} = \frac{J_{\boldsymbol{y}\to\boldsymbol{y}'}}{\boldsymbol{x}^{\boldsymbol{y}}}.$$

and

$$\boldsymbol{p} = (\boldsymbol{k}\boldsymbol{B}_1, \boldsymbol{k}\boldsymbol{B}_2, \ldots, \boldsymbol{k}\boldsymbol{B}_b), \quad \boldsymbol{q} = (\boldsymbol{J}\boldsymbol{A}_1, \boldsymbol{J}\boldsymbol{A}_2, \ldots, \boldsymbol{J}\boldsymbol{A}_a).$$

Properties of the map Ψ

- The map Ψ is well-defined and injective.
- The map Ψ is continuous.
- There exists a set of vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\} \subset \mathbb{R}^{|E_1|}$, such that

  $$\mathcal{J}_{\mathbb{R}}(G_1, G) = \{a_1 \boldsymbol{v}_1 + \cdots a_k \boldsymbol{v}_k \mid a_i \in \mathbb{R}_{>0}, \boldsymbol{v}_i \in \mathbb{R}^{|E_1|}\},$$

  and $\dim(\mathcal{J}_{\mathbb{R}}(G_1, G)) = \dim(\mathrm{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\})$.
- $\mathcal{K}_{\mathbb{R}\text{-disg}}(G, G_1)$ is a ***semialgebraic*** set, that is, it can be represented as a finite union of sets defined by polynomial equalities and polynomial inequalities.

The dimension of $\mathcal{K}_{\mathbb{R}\text{-disg}}(G)$

Theorem 3.3 ([8])

Let $G_1 = (V_1, E_1)$ be a weakly reversible E-graph with its stoichiometric subspace $\mathcal{S}_{G_1}$. Consider an E-graph $G = (V, E)$ and $\boldsymbol{x}_0 \in \mathbb{R}^n_{>0}$, then

$$\dim(\mathcal{K}_{\mathbb{R}\text{-}disg}(G)) = \max_{G_1 \sqsubseteq G_c} \{\dim(\mathcal{J}_{\mathbb{R}}(G_1, G)) + \dim(\mathcal{S}_{G_1}) + \dim(\mathcal{D}_{\boldsymbol{0}}(G)) - \dim(\mathcal{J}_{\boldsymbol{0}}(G_1))\}.$$

[8]: **J. Jin**, G. Craciun, and A. Deshpande. "On the Dimension of the R-Disguised Toric Locus of Reaction Networks". In: *Preprint* (2023)

Example. We consider the $\mathbb{R}$-disguised toric locus on the pair of E-graphs in Figure 14. Specifically, we show how Theorem 3.3 leads to the lower bound of the $\mathbb{R}$-disguised toric locus.

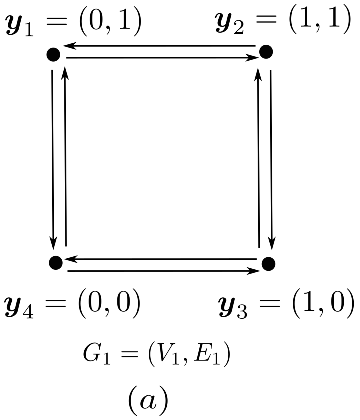

Figure 14: Two E-graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$.

Step 1. We start with computing $\dim(\mathcal{J}_{\mathbb{R}}(G_2, G_1)) \subset \mathbb{R}^{12}$.

- First, being a complex-balanced vector in G_2 gives $4 - 1 = 3$ constraints on $\boldsymbol{J}$.

- Second, being $\mathbb{R}$-realizable in G_1 gives no constraints on $\boldsymbol{J}$ since every flux vector can be transformed into G_1. Thus we obtain

$$\dim(\mathcal{J}_{\mathbb{R}}(G_2, G_1)) \geq 12 - 3 - 0 = 9.$$

Step 2. Recall from previous example, we get

$$\mathcal{D}_{\mathbf{0}}(G_1) = \{\mathbf{0}\} \quad \text{and} \quad \dim(\mathcal{J}_{\mathbf{0}}(G_2)) = 3.$$

Step 3. By Theorem 3.3, we derive that

$$\begin{aligned} & \dim(\mathcal{K}_{\mathbb{R}\text{-disg}}(\mathrm{G}_1, \mathrm{G}_2)) \\ = & \dim(\mathcal{J}_{\mathbb{R}}(G_2, G_1)) + \dim(\mathcal{S}_{G_2}) + \dim(\mathcal{D}_{\mathbf{0}}(G_1)) - \dim(\mathcal{J}_{\mathbf{0}}(G_2)) \\ = & 8. \end{aligned}$$

Therefore we conclude that $\dim(\mathcal{K}_{\mathbb{R}\text{-disg}}(\mathrm{G}_1)) = 8$.

Future projects

- The structure of the disguised toric locus

- Elucidate when does the disguised toric locus have full dimension

- Generalization for reaction networks with toric steady states

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