Sketch of the lectures Matematika MC (BMETE92MC11) (Unedited manuscript, full with errors, not to be propagated)

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Notations

$\neg$	no, denial, negation
$\wedge$	and, conjunction
$\vee$	or (permissive), disjunction
$\Longrightarrow$	implies, implication
$\Longleftrightarrow$	if and only if,
iff	if and only if, equivalence
$\Longleftrightarrow$	equivalence
$\forall$	for all
$\exists$	there is, exists
$\exists!$	exists a unique, exists exactly one
!	let
↯	contradiction

Chapter 3

Introduction

The specialities of this course in mathematics given for MSc students in Cognitive Science are as follows.

1. It presents mathematics in modern form, with as few compromises as possible.
2. The mathematical material contained is tiny, much less then the material usually taught for students at universities in two or more semesters (in 6–10 lessons per week): we only have one semester.
3. It contains as much historical and philosophical remarks and additions (and also figures and pictures) as possible, and relatively many application examples.
4. Recurrently, we make remarks on the connections between this subject and the tools provided by *Mathematica* .

3.1 Structure of the course

The requirements to be fulfilled by the student can (or will be) found on the homepage of the Department of Analysis `http://www.math.bme.hu/~analizis/oktatotttargyak/2011osz/oktatotttargyak2011osz.html`.

The structure of the course: See the Contents.

The ingredients of a lecture

- What is this topics good for?
- What has been achieved from the goal(s)?
- Historical remarks
- Philosophical consequences
- Relations to cognitive science, if any
- References, links
- Food for thought

In this file you will find the material of the lectures and also some problems. A few proofs will also be presented.

A generally used textbook at our university is [28], which can also be found in Hungarian, although my lectures will not follow it. You may use it as background material. My major source was [13] which is in Hungarian, and contains much more material than needed here. The *Urtext*, however, is the book by Rudin [24]. You may find further material on my home page `http://www.math.bme.hu/~jtoth`.

Any kind of critical remark is welcome, including those relating my English.

Chapter 4

Tools from logics

The coarse structure of mathematics is as follows. It starts from undefined **basic notion**s, then it formulates unproved statements called **axiom**s using these notions. Next, new concepts are introduced in **definition**s, and using the defined and undefined concepts statements are formulated which are called **theorem**s. (Synonyms with slightly different meaning are **corollary**, **lemma**, **statement**.)

> A corollary is a direct consequence of a statement of any kind. The expression lemma is most often used for a statement which does not belong to the current main line of thought, and also, which may be used elsewhere. It is not excluded that a lemma be of exceptional importance. In the formal use a statement is less important than a theorem.

The theorems are followed by proofs, the proofs are based on the previously proved theorems and on the axioms and they use methods of logics.

One may say that this structure is not so special as it seems to be. The major difference between mathematics and other sciences lies in the exactness and rigorism of formulation. This difference mainly comes from the nature of subject, it is impossible to get really exact evidence e.g. in history.

Example 4.1 The undefined basic concepts of geometry are point, line, fits to. The axioms (also called **postulate**s) with geometric content (also containing defined concepts, for the sake of brevity) are as follows.

1. It is possible to draw a straight line from any point to any point.

2. It is possible to extend a finite straight line continuously in a straight line.

3. It is possible to describe a circle with any center and radius.

4. All right angles are equal to one another.

5. (The parallel postulate) For any given line and point not on the line, there is one parallel line through the point not intersecting the line.

N. Lobachevski (1792–1856)

J. Bolyai (1802–1860)

The last axiom is obviously more complicated and it is not so easy to accept it as truth. There is a long (two thousand year long) story of trials to proof or to confute it, and discard as an axiom. Finally, Nikolai Lobachevski realized that it is possible to construct a geometry based upon the denial of this axiom, and János Bolyai made a systematic study on geometry without his axiom. (There is no authentic picture of him left.) It turned out in the twentieth century that we should reconsider the way we have looked at the concept of a scientific theory earlier, and also at the concept of space. No wonder that this non-Euclidean geometry is a fundamental tool in understanding relativity theory, as well.

There also a few other axioms of the more general character.

1. Things that are equal to the same thing are equal to one another.

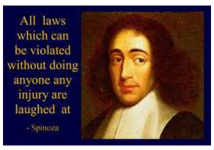
2. If equals are added to equals, then the wholes are equal.

3. If equals are subtracted from equals, then the remainders are equal.

4. Things that coincide with one another, are to equal one another.

5. The whole is greater than the part.

> One of the shock caused by set theory around the turn of the twentieth century came from the fact that this axiom does not hold there. See page 26.



B. Spinoza (1632–1677)

It is an ideal form for other branches of science, including some social sciences and humanities, as well: e.g. Baruch Spinoza used the method (Lat. "more geometrico demonstrata") in his main opus *Ethics, 1677*.

Let us see an example showing that the axioms of geometry can be fulfilled by objects which are far from being "natural".

Definition 4.1 A **finite projective plane** of order n (where n is a positive integer) is formally defined as a set of n^2+n+1 points with the properties that:

1. Any two points determine a line.
2. Any two lines determine a point.
3. Every point has $n+1$ lines on it.
4. Every line contains $n+1$ points.

Homework 4.1 What are the lines of the Fano plane (the finite projective plane for $n = 2$, Fig. 4?

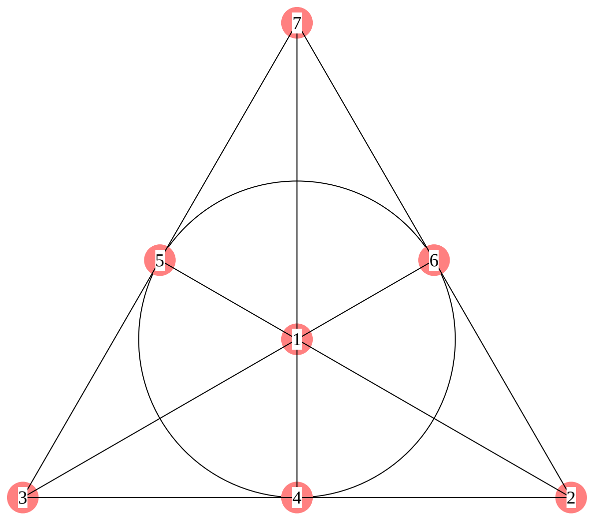


Figure 4.1: Fano plane

4.1 Logical operations

The (undefined) fundamental concepts of mathematical logic are the **mathematical objects**, **statements** or **theorems** or **theses** relating them, and the truth content of the statement: if they are **true** or **false**.

What are the statements? Their simplest property is that if some (one or two) statements are given then applying the logical operations of

negation or denial (NOT),

conjunction or combination (AND),

disjunction or disconnection (OR),

implication or conclusion or inference (IF... THEN),

equivalence or essential equality and interchangeability (... IF AND ONLY IF THEN ...)

one arrives at a statement again.

For example, the statement "It is raining" and the statement "Mary is smiling" can be combined to give the statements

- "It is NOT raining",
- "It is raining AND Mary is smiling",
- "It is raining OR Mary is smiling",
- "IF it is raining THEN Mary is smiling",
- "It is raining IF AND ONLY IF Mary is smiling".

Example 4.2 Find as many as possible alternatives to express the above logical operations.

One may abbreviate statements with a single character and the operations may be denoted by symbols, for example if r denotes the statement "It is raining", and s denotes the statement "Mary is smiling", then the above statements will be denoted as follows: $\neg r$, $r \wedge s$, $r \vee s$, $r \implies s$, $r \iff s$.

There is another way to form statements. We start from the concept of **logical function**s: functions with one or more variables (usually representing mathematical objects) assigning statements to their variables. Let us take the nonmathematical example "A girl is smiling". Here "A girl" is a variable, and if the value "Mary" is given to the variable "A girl" (if the value "Mary" is substituted into the variable "A girl"), then the statement "Mary is smiling" is formed. However, given a logical function, there are other ways to form a statement Truth table, ; verification of identities. The variable can be bound using

- the **universal quantifier**, meaning FOR ALL . . . , or EVERY; or
- the **existential quantifier**, meaning THERE IS . . . , or THERE EXISTS. . . .

Applying these to our examples we get "ALL girls are smiling" and "THERE ARE girls who are smiling".

> Formally, we should take the sentences "FOR ALL girls it is true that they are smiling" and "THERE IS a girl who is smiling", but we do not want to enter into further details.

If $S(x)$ is the shorthand for "x is smiling", then the above statements will be denoted as follows: $\forall x S(x), \exists x S(x)$.

A statement may be true ar false. If the statement is constructed from other statements with the above operations then one can simply „calculate" the truth value of a statement given the truth values of its components.

Example 4.3

$$\forall n (n \text{ even}) \vee (n \text{ odd}) \tag{4.1}$$

$$\neg(\forall n : 2n > n) \iff (\exists n : 2^n \not> n) \tag{4.2}$$

$$\neg(\forall x \exists y x > y) \iff (\exists x \forall y : x \leq y) \tag{4.3}$$

$$\neg(\exists k \forall n > k : n \mid 12) \iff (\forall k \exists n > k : n \nmid 12) \tag{4.4}$$

$$\neg(3 < 5 \vee 10 \geq 20) \iff ((3 \geq 5) \wedge (10 < 20)). \tag{4.5}$$

Necessary condition, sufficient condition, necessary and sufficient condition. All squares are rectangles.

Homework 4.2 Show that all the one- and two-variable logical operations can be expressed by the Sheffer stroke | aka NAND operation which produces a value of false if and only if both of its operands are true. (Or, it produces a value of true if and only if at least one of its operands is false.)

$$(p \implies q) \iff (\neg p \vee q) \tag{4.6}$$

$$\neg(p \implies q) \iff (p \wedge \neg q) \tag{4.7}$$

$$\neg(p \iff q) \iff ((p \wedge \neg q) \vee (q \wedge \neg p)) \tag{4.8}$$

J. R. Kipling (1865–1936)

Homework 4.3 Is this statement true? More than 99% of mankind has more than average number of legs.

1. Formulate the sentences below using logical operators.
   (a) What therefore God has joined together,
   let no man separate.
   (b) A smile is truly the only thing that can be understood in any language.
   (c) You can fool some of the people all of the time,
   and all of the people some of the time,
   but you can not fool all of the people all of the time.
   (d) If you can dream—and not make dreams your master;
   If you can think—and not make thoughts your aim,
   If you can meet with Triumph and Disaster
   And treat those two impostors just the same...
   Yours is the Earth and everything that's in it,
   And - which is more - you'll be a Man my son!
   (e) You don't marry someone you can live with,
   you marry the person who you cannot live without.
   (f) Not only A, but B, as well.
   (g) Neither A, nor B.
   (h) B, assuming A.
   (i) B is a sufficient condtion for A.
2. Translate into plain English:
   (a) $A \wedge B \wedge \neg C$,
   (b) $\neg A \implies B$.
3. How do you negate/deny these statements:
   (a) To be, or not to be.
4. Let $Q(x)$ denote that x is a rational number. How do you describe by formula that

(a) There exist rational numbers.

(b) Not all the numbers are irrational.

(c) There does not exist a number which is, if it is rational, then it is irrational.

4.2 Methods of proof

Here and below we collect the tools directly used in mathematics beyond logical operations learned above. However, this does not mean that we neglect the rules for the direction of the mind by Descartes, such as

1. You accept only that which is clear to mind.

2. You split large problems into smaller ones.

3. You argue from the simple to the complex.

4. And finally you check everything carefully when you have finished.

It is equally useful to keep in mind the rules of heuristics as formulated by Pólya [18].

4.2.1 Constructive and nonconstructive proof

***********Here and below we use concepts known from high school which we are going to formally introduce later.

The existence of something can be proved by constructing it.

Theorem 4.1 There exist irrational numbers a and b such that a^b is rational.

A constructive proof of the theorem would give an actual example, such as:

$$a = \sqrt{2}\,, \quad b = \log_2(9)\,, \quad a^b = 3\,. \tag{4.9}$$

The square root of 2 is irrational, and 3 is rational. $\log_2(9)$ is also irrational: if it were equal to $\frac{m}{n}$, then, by the properties of logarithms, 9^n would be equal to 2^m, but the former is odd, and the latter is even.

A non-constructive proof may proceed as follows: Recall that $\sqrt{2}$ is irrational, and 2 is rational. Consider the number $q = \sqrt{2}^{\sqrt{2}}$. *Either it is rational or it is irrational.* If q is rational, then the theorem is true, with a and b both being $\sqrt{2}$.

If q is irrational, then the theorem is true, with a being $\sqrt{2}^{\sqrt{2}}$ and b being $\sqrt{2}$, since

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{(\sqrt{2}\cdot\sqrt{2})} = \sqrt{2}^{2} = 2.$$

This proof is non-constructive because it relies on the statement "Either q is rational or it is irrational"—an instance of *the law of excluded middle,* which is not so easy to accept, to say the least.

4.2.2 Indirect proof

Again we use here the law of excluded middle: it is impossible that the negation of a statement is false therefore it should be true.

Assume statement S is false, if this assumption leads to a contradiction than it was a false assumption, thus, the original statement S **is** true.

Theorem 4.2 $\sqrt{2}$ is not a rational number.

Theorem 4.3 There is an infinity (?!) of primes.

Theorem 4.4 Consider a chessboard without two diagonal squares. Show that it is impossible to cover the chessboard with twice by one domino tiles.

Theorem 4.5 $\tan(1^\circ)$ is irrational.

4.2.3 Mathematical induction

`http://en.wikipedia.org/wiki/Mathematical_induction`

$\forall n$ a statement A_n is given (has been formulated). If A_1 is true, and if $\forall n(A_n \implies A_{n+1})$, then the statement A_n is true (holds) for all natural numbers.

A_n is said to be the **induction hypothesis.**

Mathematical induction should not be misconstrued as a form of inductive reasoning, which is considered non-rigorous in mathematics. In fact, mathematical induction is a form of rigorous **deductive** reasoning.

Example 4.4

1. $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ (K. F. Gauss)
2. $1+3+\cdots+(2n+1) = n^2$. (Francesco Maurolico in his *Arithmeticorum libri duo* (1575).)
3. $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$
4. $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2 = n^2(n+1)^2/4$

K. F. Gauss (1777–1855)

Bernoulli inequality

1. $2^N \geq N + 1$ (multiple proofs?)
2. $2 < \left(1 + \frac{1}{N}\right)^N < 4 \quad (N \in \mathbf{N}_2 := \mathbf{N} \setminus \{1\})$
3. Prove using mathematical induction:
   (a) $2!4!\ldots(2N)! > ((N+1)!)^N \quad (N \in \mathbf{N}_2)$
   (b) $\sum_{n=1}^{N}(2n-1)^2 = \frac{N(4N^2-1)}{3}$ (the meaning of $\sum$!)
   (c) $(1 - \frac{1}{4})(1 - \frac{1}{4})\cdots(1 - \frac{1}{(N+1)^2}) = \frac{N+2}{2N+2}$
4. Use the indirect method to prove that $\tan(1^\circ)$ is irrational.
5. If the product of three positive numbers is larger than one and their sum is less than the sum of their reciprocals, then none of them can be larger than one.

Arithmetic and geometric mean

Theorem 4.6 (Arithmetic and geometric mean) For all $N \in \mathbf{N}$; $b_1, b_2, \ldots, b_N \in \mathbf{R}^+$ one has

$$G_N := \left(\prod_{n=1}^{N} b_n\right)^{\frac{1}{N}} \leq \frac{\sum_{n=1}^{N} b_n}{N};$$

and equality holds if and only if the numbers $b_1, b_2, \ldots, b_N \in \mathbf{R}^+$ are equal.

Self-answering problems

1. What fraction of the letters in **one-third** are vowels?
2. Twenty-nine is a **prime** example of what kind of number?

Could you formulate similar problems?

4.2.4 Pigeonhole principle

Eight holes for 9 pigeons

Schubfachprinzip ("drawer principle" or "shelf principle"). For this reason it is also commonly called Dirichlet's box principle, Dirichlet's drawer principle

4.2.5 Invariants

Chapter 5

Sets, relations, functions

G. F. L. Ph. Cantor
(1845–1918)

5.1 Fundamentals of set theory

Fundamentals of set theory. Axiom of extensionality. http://en.wikipedia.org/wiki/Zermelo_set_theory In order to avoid some complications it is safe to consider the subset of given set (called **universal set**) in our investigations.

Operations on sets.

5.1.1 Properties of set operations

De Morgan identities. Descartes product.

1. Show using direct and indirect methods:
   If $A \subset B \subset C$, then $(A \setminus B) \cup (B \setminus C) = \emptyset$.

2. Show (and draw a Venn diagram, as well): $A \cup B = A \cap B \Longrightarrow A = B$.

3. Solve these systems of equations:

   (a) $A \cup X = B \cap X \quad A \cap X = C \cup X$

   (b) $A \setminus X = X \setminus B \quad X \setminus A = C \setminus X$

4. Give a necessary and sufficient condition for: $A \setminus B = B \setminus A$?

5. Calculate (using a Venn diagram) $A \cap (B \cup A)$; calculate using a truth table $A \vee (B \wedge A)$.

First of all: those who do not visit the lecture because of a higher than average background in math should learn the material outside classical calculus for the exam. (Topics like chaos, networks etc. to be defined later.)

Gellért bath (around 1930)

5.1.2 Applications

Example 5.1 The following facts are known:

1. There exists a tvset-owner who is not a painter.
2. Those who visit the Gellért bath but are no painters have no tv set.

Do these facts imply that: Not all tv-owners visit the Gellért bath?

Example 5.2 The following facts are known:

1. Nonsmoking bachelors collect stamps.
2. Either all stamp-collectors living in Cegléd do smoke, or there is no such nonsmoking stamp-collector who does not live in Cegléd.
3. Paul Kiss living in Budapest has as his most important hobby collecting stamps since he has stopped smoking as a result of his wife's stimulation.

Do all bachelors living in Cegléd smoke?

Euclid of Alexandria (cca. 325 BC–cca. 265 BC)

5.1.3 Cardinality

Euclides, Galileo Actual and potential infinity.

The part is smaller than the whole.

Sets of equal cardinality. Finite, countable and uncountable sets. Cardinality of the continuum. Finite and denumerable sets. The cardinality of the power set is always larger than that of the original set. Relations between cardinalities. The cardinality of real numbers is larger than that of the integers. The continuum hypothesis.

Galileo Galilei (1564–1642)

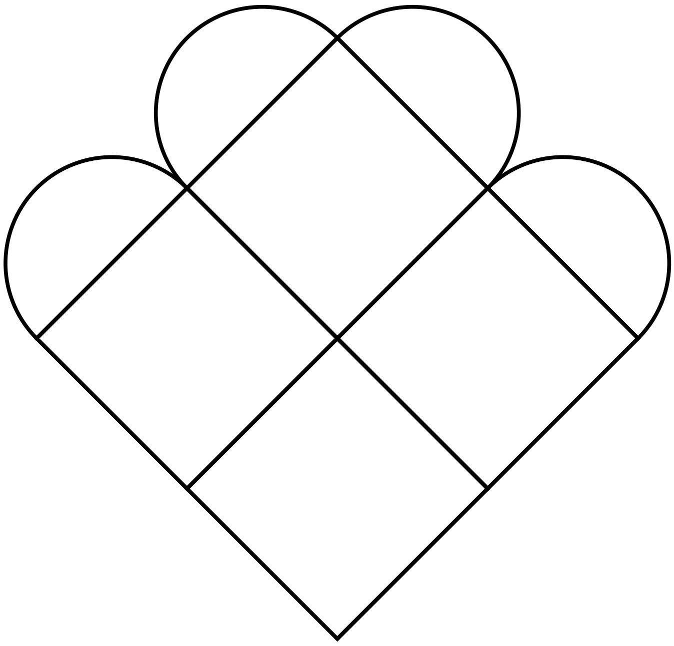


Figure 5.1: Smokers, bachelors, stamp collectors and Cegléd inhabitants can be represented by the four sets in the figure. Denote what the assumptions say about the relationships of the sets.

$\cup$	finite	countable	uncontable
finite	finite	countable	uncontable
countable	countable	countable	uncontable
uncontable	uncontable	uncontable	uncontable

$\cap$	finite	countable	uncontable
finite	finite	finite	finite
countable	finite	$\leq$countable	$\leq$uncontable
uncontable	finite	contable	$\leq$uncontable

5.2 Relations

First we give the formal definition of a relation which seems to be quite abstract at the beginning; hopefully the examples below will show that it is a useful formalization.

Definition 5.1 Let A be a nonempty set, then any subset of $A \times A$ is said to be a **relation**.

5.2.1 Properties of relations

Reflexive, transitive, symmetric, antisymmetric relations. Orderings. Equivalence relations. Classes.

5.2.2 Operations on relations

1. Describe the properties of the relations on the set **N**:
    (a) $a = b$
    (b) $a < b$
    (c) $a \neq b$
    (d) $a|b$
    (e) a is a proper divisor of b.
2. How to chose the base set to get a transitive relation from these:
    (a) a is a brother of b,
    (b) a is the mother of a?

5.3 Functions

Definition 5.2 The relation $f \subset A \times B$ is said to be a **function**, if it has the property $(a, b); (a, c) \in f \Longrightarrow b = c$. Notation: $(a, b) \in f$ is usually denoted as $b = f(a)$, and in this case we also say that the **value** of the function at the **argument** a is b. The domain and range of a function is the domain, respectively the range in the sense as defined for relations. Composition of two functions are defined to be their composition as relations.

Remark 5.1 Conventional, prefix, postfix, infix notations. Notations used in *Mathematica* .

Example 5.3

1. If $A = \{1, 2, \dots, n\}$ with some $n \in \mathbf{N}$, the function $f \subset A \times B$ is said to be a (n-dimensional) vector.

2. If $A = \mathbf{N}, B = \mathbf{R}$, then the function $f \subset A \times B$ is said to be a sequence of real numbers.

3. If $A = \mathbf{R}, B = \mathbf{R}$, then the function $f \subset A \times B$ is said to be a a real valued function of a (single) real argument.

Definition 5.3

1. The function $f \subset A \times B$ is said to be a **surjective function,** or **surjection**, if $\mathcal{R}_f = B$.

2. The function $f \subset A \times B$ is said to be an **injective function,** or **injection**, if it has the property that $(a, b); (c, b) \in f \Longrightarrow a = c$.

3. A function which is both surjective and injective is a **bijective function,** or **bijection** (a one-to-one correspondence).

Theorem 5.1 The inverse of a bijective function (as that of a relation) is a function itself.

Theorem 5.2 The identity relation is a function; it is called the **identity function**.

Remark 5.2 Cf. the vector with **`List`**, the identity function with pure functions and **`Slot`**.

Definition 5.4 The function g is said to be the **restriction** of the function f onto the set C, if $f, g \subset A \times B$ are functions, $C \subset A$, and $\mathcal{D}_f = A, \mathcal{D}_g = C$. In this case f is said to be and **extension** of g.

Mapping a set. Pair of functions. Projections. Composition. Inverse. Operations with functions.

1. *Show that $f(\bigcup_{i \in I} A_i = \bigcup_{i \in I} f(A_i))$. (What does the notation mean?)
2. *Suppose that $\varphi : A \longrightarrow B$ is a bijection. Show that
   (a) $\varphi^{-1} : B \longrightarrow A$ is a bijection,
   (b) $\varphi^{-1} \circ \varphi = \mathrm{id}_A$,
   (c) $\varphi \circ \varphi^{-1} = \mathrm{id}_B$.
3. Are the below relations defined on $\mathbf{R}$ functions or not? What about their properties?
   (a) $x \rho y :\Longleftrightarrow x = y^2$
   (b) $x \rho y :\Longleftrightarrow y = x^2$
   (c) $x \rho y :\Longleftrightarrow y = -x^3$
   (d) $x \rho y :\Longleftrightarrow y = 2x - x^2$
4. Find the largest set A, for which it is possible to restrict the function g so as to define $f \circ (g|_A)$, if
   (a) $g(x) := \mathrm{sign}(x) := \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -1, & \text{if } x < 0. \end{cases} \quad (x \in \mathbf{R})$ and
   $f(x) := \frac{1}{x^2-1}$, if $x \in \mathbf{R}$, and $x^2 \neq 1$.
   (b) $*g(z) := \mathrm{Re}(z) \quad (z \in \mathbf{C})$ and $f(x) := \frac{x}{x^2-1}$, if $x \in \mathbf{R}$, and $x^2 \neq 1$.

Chapter 6

Graphs and networks

6.1 The Bridges of Königsberg

Further details can be found here: `jcu.edu/math/vignettes/bridges.htm`

6.2 Formal definitions

Definition 6.1 A finite nonempty set of **vertices** V and a set $E \subset V \times V$ of ordered pairs of vertices, called **arc**s is said to be a **directed graph**, and it is usually denoted as (V, E). Arcs of the form $(v, v) \in E$ are called **loop**s.

Definition 6.2 A finite nonempty set of vertices V and a set E of unordered pairs of vertices, called **edge**s is said to be a(n undirected) **graph**, and it is also denoted as (V, E). (Multiple edges are not always excluded.) Edges of the form $(v, v) \in E$ are called **loop**s in case of undirected graphs, as well.

Definition 6.3 In both cases, vertices connected by an edge (by an arc) are said to be **adjacent**. Edges (arcs) with a common vertex are also called **adjacent**. The **adjacency matrix** of the graph (V, E) is a $|V| \times |V|$ matrix with an entry $a_{ij} = 1$, if vertex $i \in V$ and vertex $j \in V$ are connected with and edge (arc): if they are adjacent, otherwise the entry is zero.

Obviously, in case of undirected graphs the adjacency matrix is symmetric and redundant: all the edges are represented twice.

Definition 6.4 An edge (an arc) and both of its vertices are said to be **incident**. The edge (v, w) of an undirected graph connects the two vertices v and w, while the arc (v, w) of a directed graph begins at the vertex v and ends at the vertex w. The **incidence matrix** I of the directed graph (V, E) is an a $|V| \times |E|$ matrix with an entry

1. $i_{ve} = 1$, if there is an arc beginning at the vertex v incident at the arc e,
2. $i_{ve} = -1$, if there is an arc ending at the vertex v incident at the arc e,
3. $i_{ve} = 0$ otherwise.

Definition 6.5 A **subgraph** of a graph (V, E) is a graph $(\overline{V}, \overline{E})$ such that $\overline{V} \subset V$ and $\overline{E} \subset E$ holds, and all the edges (arcs) $\overline{E}$ are incident with vertices from $\overline{V}$ only. A **spanning subgraph** of (V, E) is a subgraph that contains all the vertices V. A finite sequence of edges $e_1, e_2, \ldots, e_k$ is called a **path** connecting the vertices e_1 and e_k. In case of a directed graph such a sequence of arcs is called a **directed path**. A closed path, i.e. one for which $e_1 = e_k$ is called a **cycle**, respectively a **directed cycle**. A graph without cycles (without directed cycles) is a **forest**; it is a **tree**, if it is also connected. A **spanning forest** is a spanning subgraph which is a forest.

Definition 6.6 A graph is said to be **connected** if any pair of its vertices is connected by a (directed) path. A maximal connected subgraph of an undirected graph is said to be a **connected component**. In case of a directed graph the connected component is also called **strong component**. The weak components of a directed graph (V, E) are obtained in the following way. First, if either $(v, w) \in E$ or $(w, v) \in E$, then let the (undirected) arc (v, w) be represented by an edge of an undirected graph $(V, \overline{E})$. Then, the connected components of $(V, \overline{E})$ are said to be the **weak components** of (V, E). Those strong components of a directed graph in which no edge starts ending in a vertex outside the component are called **ergodic components**.

Definition 6.7 A **directed bipartite graph** consists of two vertex sets, say V_1 and V_2, and arcs can only go from one vertex set into the other one, but there are no arcs proceeding within the vertex sets. This means that for the arc set E of such a graph one has $E \subset V_1 \times V_2 \cup V_2 \times V_1$.

Since graphs without multiple edges (arcs) can be considered as special relations on finite sets, terminology of relations can also be used for them.

Definition 6.8 A directed or undirected graph (V, E) is **reflexive** if for all $v \in V$ $(v, v) \in E$, i.e. the graphs contains all the possible loops. A directed graph (V, E) is **symmetric** if together with the arc $(v, w) \in E$ it also contains the arc $(w, v) \in E$. It is called **transitive**, if for all pairs of arcs $(v, w), (w, z) \in E$ (v, z) is also an arc. The **transitive closure** of the directed graph (V, E) is obtained in such a way that if there is a directed path beginning in the vertex v and ending in the vertex w, then the arc (v, w) is appended to the set of arcs.

Theorem 6.1 If the transitive closure of a directed graph is symmetric then the same number of its strong components is the same as the number of its ergodic components.

Hamiltonian path, circle

6.3 Applications

6.3.1 Enumeration of Carbohydrates

Pólya

6.3.2 Social networks

It is quite common to ask the pupils in a class of elementary school who are their best friends, and evaluate the answers in a such a way that the pupils are represented by vertices of a graph and (directed) edges going from A to B show that A is a friend of B. When the teacher has a look at the graph (s)he immediately conceives the structure of the class, (s)he will see subgroups, called cliques (also in graph theory!) all the members of which are connected to each other, isolated children with no friends at all etc.

Friendships in a class, six steps (Karinthy)

6.3.3 Internet

6.3.4 Web

6.3.5 Automata

6.3.6 Chemical reaction kinetics

Feinberg–Horn–Jackson graph

Volpert graph

6.3.7 PERT method

6.3.8 Transportation problems

Maximal flow (minimal cut)

6.3.9 Matching

Suppose we are given a bipartite graph, the two vertex sets of which are the set of girls and the set of boys, and edge connects two persons of different gender if they know each other. How can we form the maximal number of girl-boy pairs so that pairs are only made from acquaintances? This is an example of the **matching problem**.

6.3.10 Neural networks

McCulloch and Pitts

6.4 Planar graphs

6.5 Coloring maps

What is a proof, anyway?

Chapter 7

A few words on combinatorics

B. Pascal (1623–1662)

Theory of finite sets.

Erdős, Lovász Permutation, combination, variation, with and without repetition. Binomial coefficients. The Pascal triangle. The binomial theorem by Pascal. (Generalization by Newton).

1. $\sum_{n=0}^{N} \binom{N}{n} = 2^N$
2. $\sum_{n=0}^{N} \binom{N}{n} (-1)^n = 0.$

Theorem 7.1

$$\forall N \in \mathbf{N} : \left(1 + \frac{1}{N}\right)^N < 3.$$

Proof.

$$\begin{aligned}
\left(1 + \frac{1}{N}\right)^N &= \sum_{n=0}^{N} \binom{N}{n} \left(\frac{1}{N}\right)^n = \sum_{n=0}^{N} \frac{N(N-1)\dots(N-n+1)}{1 \cdot 2 \cdot \dots \cdot n} \frac{1}{N^n} \\
&= \sum_{n=0}^{N} \frac{N(N-1)\dots(N-n+1)}{N \cdot N \cdot \dots \cdot N} \frac{1}{1 \cdot 2 \cdot \dots \cdot n} \\
&< 1 + 1 + \sum_{n=2}^{N} 1 \frac{1}{n!} < 2 + \sum_{n=2}^{N} \frac{1}{2^{n-1}} < 3.
\end{aligned}$$

■

Calculate the number of all one-, two-, n-variable logical operations.

Chapter 8

Numbers: Real and complex

Homework 8.1 Could you possibly understand what is going on here: `http://www.mathematika.hu/flash/csoda1.swf`?

8.1 Axioms to describe the set of real numbers

We are given two operations, P and T, and the relation L on the set of real numbers.

Commutativity of addition $\forall x, y \in \mathbf{R} : P(x, y) = P(y, x)$;

Associativity of addition $\forall x, y, z \in \mathbf{R} : P(P(x, y), z) = P(x, P(y, z))$;

Neutral element of addition: zero $\exists 0 \in \mathbf{R} \forall x \in \mathbf{R} : P(x, 0) = x$;

Additive inverse $\forall x \in \mathbf{R} \exists (-x) : (P(x, -x) = 0$;

Commutativity of multiplication $\forall x, y \in \mathbf{R} : T(x, y) = T(y, x)$;

Associativity of multiplication $\forall x, y, z \in \mathbf{R} : T(T(x, y), z) = T(x, T(y, z))$;

Neutral element of multiplication: unity $\exists 1 \in \mathbf{R} \forall x \in \mathbf{R} : T(x, 1) = x$;

Multiplicative inverse $\forall x \in \mathbf{R}, x \neq 0 \Longrightarrow \exists (x^{-1}) : T(x, x^{-1}) = 1$;

Distributivity of multiplication wrt addition

$$\forall x, y, z \in \mathbf{R} : T(x, P(y, z)) = P(T(x, z), T(y, z));$$

Irreflexivity $\forall x \in \mathbf{R} : \neg L(x, x)$;

Asymmetry $\forall x, y \in \mathbf{R} : L(x, y) \Longrightarrow \neg L(y, x)$;

Transitivity $\forall x, y, z \in \mathbf{R} : L(x, y) \wedge L(y, z) \Longrightarrow L(x, z)$;

Trichotomy $\forall x, y \in \mathbf{R}$ exactly one holds: $L(x, y) \vee L(y, x) \vee x = y$;

Monotonicity wrt addition $\forall x, y, z \in \mathbf{R} : L(x, y) \Longrightarrow L(P(x, z), P(y, z))$;

Conditional monotonicity wrt multiplication

$$\forall x, y, z \in \mathbf{R} : L(x, y) \wedge L(z, 0) \Longrightarrow L(T(x, z), T(y, z)).$$

Remark 8.1 From now on we shall mainly use the notations:

$$x + y := P(x, y) \quad xy := T(x, y) \quad x < y := L(x, y) \quad \forall x, y \in \mathbf{R}.$$

The notation $x \leq y$ is used to abbreviate $x < y \vee x = y$.

Closed interval Let $a, b \in \mathbf{R}; a < b$. $[a, b] := \{x \in \mathbf{R}; a \leq x \wedge x \leq b.\}$

Open interval Let $a, b \in \mathbf{R}; a < b$. $]a, b[:= \{x \in \mathbf{R}; a < x \wedge x < b.\}$

Half open intervals Let $a, b \in \mathbf{R}; a < b$.

$$[a, b[:= \{x \in \mathbf{R}; a \leq x < b.\} \quad]a, b] := \{x \in \mathbf{R}; a < x \leq b.\}$$

Remark 8.2 Instead of $a < x \wedge x < b$ one usually writes $a < x < b$, etc.

Let us introduce two **symbols**, not numbers!

Infinity The set of real numbers larger than $a \in \mathbf{R}$ will be denoted as $]a, +\infty[$.

Minus infinity The set of real numbers smaller then $a \in \mathbf{R}$ will be denoted as $] - \infty, a[$.

Remark 8.3 The intervals $[a, +\infty[$ etc. are defined similarly. Note that the side where $-\infty$ or ∞ stands is always open.

The axioms by Archimedes and Cantor. The absolute value function. Which axiom do you use in the individual steps?

1. Solve the inequalities below and present the result as subsets of the real line:

   (a) $5x + 3 \leq 2 - 4x$
   (b) $\frac{5x-1}{4} \leq x + 1 < -2 + 2x$
   (c) $-3(x+1)(x+2) > 0$
   (d) $a^2x^2 - 2x - 5 \leq 0$
   (e) $x^4 - 5x^2 + 4 > 0$
   (f) $\frac{4x-1}{4x+1} < -1$
   (g) $\frac{(x-1)(x+2)}{x+3} < x - 2$
   (h) $\frac{x^2-5x+4}{x^2-6x+7} > 0$

2. Prove that for all $x, y \in \mathbf{R}$ one has

   (a) $|x + y| \leq |x| + |y|$
   (b) $|x - y| \geq ||x| - |y||$

3. Solve the inequalities below and present the result as subsets of the real line:

   (a) $|2x + 3| < 2$
   (b) $|2 - x^2| > 3$
   (c) $||x + 1| - |x - 1|| < 1$
   (d) $|x(1 - x)| < 0.05$
   (e) $|x(1 - x)| < 0.25$

8.1.1 Operations

Smart multiplication

The classical solution to calculate the product of two polynomials is:

$$(a + bx)(c + dx) = ac + (ad + bc)x + bdx^2.$$

Hovever, one can do it smarter, using the **Karatsuba algorithm**.

$$u := (a + b)(c + d)ad + bc = u - ac - bd$$

8.2 Complex numbers

8.2.1 Evolution of the concept of number

The definition of operations are defined as operations on pairs of real numbers.

$i^2 = -1$. Real and imaginary parts. Correspondence between the points of $\mathbf{C}$ and $\mathbf{R}^2$.

. Addition and the parallelogram rule. Absolute value or modulus.

Definition 8.1 The conjugate of the complex number $a + bi$ is defined to be the complex number $a - bi$.

Geometrical meaning. Relations with arithmetical operations and with the calculations of the real and imaginary parts. $\bar{i} = -i$. Now one has square root(s) of -1. Argument. The algebraic form of a complex number.

Moivre formula: $(re^{i\varphi})(se^{i\psi}) = (rs)e^{i(\varphi+\psi)}$. Rotation can be obtained by a multiplication of modulus 1. Unit roots. The vertices of a regular polygon. nth roots of a complex number.

8.3 Operations and relations on real valued functions

8.3.1 Algebraic operations on real valued functions

Definition 8.2 Let $f, g \subset A \times \mathbf{R}$ real valued functions. Then:

1. $(-f)(x) := -f(x) \quad (x \in \mathcal{D}_f)$;
2. $(\lambda f)(x) := \lambda f(x) \quad (x \in \mathcal{D}_f) \wedge \lambda \in \mathbf{R}$;
3. $(f \pm g)(x) := f(x) \pm g(x) \quad (x \in \mathcal{D}_f \cap \mathcal{D}_g)$;
4. $(fg)(x) := f(x)g(x) \quad (x \in \mathcal{D}_f \cap \mathcal{D}_g)$;
5. $(f/g)(x) := f(x)/g(x) \quad (x \in \mathcal{D}_f \cap \mathcal{D}_g) \wedge (g(x) \neq 0)$.

Example 8.1 What do the definitions mean in the case of pairs, vectors, series?

8.3.2 Inequality relations and real valued functions

The definition of $f < g$. This relation is transitive and monotonous with respect to both operations, however, it is not trichotomous. Monotonous functions.

8.4 Polynomials

Definition 8.3 Let $N \in \mathbf{N}_0$ be a natural number, and let $a_0, a_1, \ldots, a_N \in \mathbf{C}$ be real numbers, and suppose $a_N \neq 0$. Then, the function

$$\mathbf{C} \ni x \mapsto p(x) := a_0 + a_1 x + \cdots + a_N x^N \in \mathbf{C} \tag{8.1}$$

is said to be a **polynomial** of **degree** N; the numbers $a_0, a_1, \ldots, a_N \in \mathbf{C}$ are the **coefficient**s of the polynomial. The degree of p is denoted by $\deg(p)$.

As polynomials are special cases of real variable real valued functions, one knows how to carry out operations on them.

Theorem 8.1 1.

2. A polynomial p multiplied by a complex number α is a polynomial as well. If $\alpha = 0$, then $\deg(\alpha p) = 0$, otherwise $\deg(\alpha p) = \deg(p)$.

3. The sum of two polynomials p and q is a polynomial; $\deg(p + q) \leq \max\{\deg(p), \deg(q)\}$.

4. The product of two polynomials p and q is a polynomial; $\deg(pq) = \deg(p) + \deg(q)$.

To find the greatest common divisor of two polynomials it is enough to apply the **Euclidean algorithm**, appropriately modified.

Definition 8.4 Let p be a polynomial, different from the zero polynomial. The set

$$p^{-1}(\{0\}) = \{x \in \mathbf{C} | p(x) = 0\} \tag{8.2}$$

is the set of **root**s or **zero**s of the polynomial p.

Theorem 8.2 (The fundamental theorem of algebra) A nonzero polynomial p has not more than $\deg(p)$ roots.

Theorem 8.3 If two polynomials p and q (as functions) are equal, then all their coefficients are equal too.

Theorem 8.4 (Factorization of polynomials) Let p be a nonzero polynomial, and let $N := \deg(p)$. Then, there are complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_N$, not necessarily different, with which one has:

$$p(x) = (x - \lambda_1)(x - \lambda_2)\ldots(x = \lambda_N). \tag{8.3}$$

Formulas for expressing the roots of polynomials of degree 2 in terms of square roots have been known since ancient times (see quadratic equation), and for polynomials of degree 3 or 4 similar formulas (using cube roots in addition to square roots) were found in the 16th century (see Niccolo Fontana Tartaglia, Lodovico Ferrari, Gerolamo Cardano, and Vieta). But formulas for degree 5 eluded researchers. In 1824, Niels Henrik Abel proved the striking result that there can be no general (finite) formula, involving only arithmetic operations and radicals, that expresses the roots of a polynomial of degree 5 or greater in terms of its coefficients (see Abel-Ruffini theorem). This result marked the start of Galois theory which engages in a detailed study of relationships among roots of polynomials.

Polynomials with real coefficients are of special importance. Let us emphasize that the fact that the coefficients are real does not imply that the roots of the polynomial are real, as the example $x \mapsto x^{+}1$ shows. Still, one can formulate a series of useful statements.

Theorem 8.5 1.

2. A polynomial p with real coefficients multiplied by a real number α is a polynomial with real coefficients as well.

3. The sum of two polynomials p and q with real coefficients is a polynomial with real coefficients.

4. The product of two polynomials p and q with real coefficients is a polynomial with real coefficients.

Let us go a bit farther, let us introduce a larger class of functions.

Definition 8.5 Let p and q be a polynomials, and suppose q is different from the zero polynomial. Let the degree of q be N, and let the roots of q

be $\lambda_1, \lambda_2, \ldots, \lambda_N$. Then, the function

$$\mathbf{R} \setminus \{\lambda_1, \lambda_2, \ldots, \lambda_N\} \ni x \mapsto \frac{p(x)}{q(x)} \in \mathbf{C} \tag{8.4}$$

is said to be a **rational function**.

Example 8.2 The function $x \mapsto \frac{x}{x}$ is not defined at the value 0, at other arguments it is equal to 1.

Rational functions can be represented in a form which is both simple and useful for some purposes.

Theorem 8.6 (Partial fraction decomposition) Let q be a nonzero polynomial which can be represented in the form $q = \prod_{i=1}^{k} r_i^{n_i}$, where the functions r_i are distinct irreducible polynomials (of degree one or two). Then, there are (unique) polynomials b and a_{ij} with $\deg(a_{ij}) < \deg(r_i)$ such that

$$\frac{p}{q} = b + \sum_{i=1}^{k} \sum_{j=1}^{n_i} \frac{a_{ij}}{r_i^j}. \tag{8.5}$$

Furthermore, if $\deg(p) < \deg(q)$,, then $b = 0$.

Example 8.3 $\frac{1}{z^4+1}$.

See how **`Apart`** is working.

Chapter 9

Sequences of real numbers

9.1 Lower and upper limit of a set of real numbers

Definition 9.1 The set $A \subset \mathbf{R}$ of real numbers is said to be **bounded above**, if there is a real number $K \in \mathbf{R}$ such that $\forall x \in A |x| \leq K$. The set $A \subset \mathbf{R}$ of real numbers is said to be **bounded below**, if there is a real number $K \in \mathbf{R}$ such that $\forall x \in A |x| \geq K$. A set is bounded if it is bounded above and bounded below, as well.

Theorem 9.1 Among all the upper bounds of a set bounded above there is a smallest one.

Definition 9.2 The smallest upper bound of the set $A \subset \mathbf{R}$ is called its **supremum**, and this number is denoted as $\sup A$. The largest lower bound of the set $A \subset \mathbf{R}$ is called its **infimum**, and this number is denoted as $\inf A$. The fact that the set A is not bounded above is expressed by the notation $\sup A = +\infty$, while the fact that the set A is not bounded below is expressed by the notation $\inf A = -\infty$,

Real numbers extended. The real line. Accumulation point of a set of real numbers.

9.2 Real sequences

Definition 9.3 The function $a : \mathbf{N} \longrightarrow \mathbf{R}$ is said to be a (real valued) **sequence.** Its values are usually denoted as $a(n)$, or, more often as a_n. a_n is the nth **member** of the sequence, and n is its **index.**

Example 9.1 Let $b, d, q \in \mathbf{R}$ be given.

The sequence $a_n := b + nd \quad (n \in \mathbf{N})$ is an **arithmetic sequence,**

2. the sequence $b_n := bq^n \quad (n \in \mathbf{N})$ is a **geometric sequence,**

3. the sequence $c_n := \frac{1}{n} \quad (n \in \mathbf{N})$ is the **harmonic sequence.**

Example 9.2 Let $b \in \mathbf{R}$ be a given real number. The **constant sequence** $a_n := b \quad (n \in \mathbf{N})$ is both an arithmetic and a geometric sequence. (WHy?)

Definition 9.4 A sequence of real numbers is strictly increasing, increasing, strictly decreasing, decreasing if it has these properties as a real valued function. (See above.)

Definition 9.5 A strictly increasing function $\nu : \mathbf{N} \longrightarrow \mathbf{N}$ is said to be an **index sequence,** if $a : \mathbf{N} \longrightarrow \mathbf{R}$ is a real sequence, then $a \circ \nu : \mathbf{N} \longrightarrow \mathbf{R}$ is its subsequence.

Example 9.3 Let $a_n := (-1)^n n \quad (n \in \mathbf{N})$, and let $\mu n := 2n, \nu(n) := 2n-1 \quad (n \in \mathbf{N})$. Then, μ and ν are index sequences, and the subsequences formed by them have the members: $(2, 4, 6, \ldots)$ and $(-1, -3, -5, \ldots)$, respectively.

Theorem 9.2 All real sequences have a monotonous subsequence.

The statement is not true with the adverb "strictly" added.

9.2.1 Operations on real sequences

Definition 9.6 Let $\alpha \in \mathbf{R}$; and let $a, b : \mathbf{N} \longrightarrow \mathbf{R}$ be real sequences. Then

$$(\alpha a)_n := \alpha a_n; (a \pm b)_n := a_n \pm b_n \quad (n \in \mathbf{N}).$$

Definition 9.7 A sequence is bounded above, below, bounded if its range has the same property. The supremum (infimum) of a sequence is the supremum (infimum) of its range.

The supremum might also be $+\infty$, the infimum might also be $-\infty$.

9.2.2 Relations between real sequences

Definition 9.8 Let $a, b : \mathbf{N} \longrightarrow \mathbf{R}$ be real sequences. Then $a < b$ and $a \leq b$ is defined as follows:

$$a_n < b_n; a_n \leq b_n \quad (n \in \mathbf{N}).$$

The functional sup is monotonously increasing.

9.3 Limit of sequences

Definition 9.9 The real number A is said to be the **limit** of the sequence $a : \mathbf{N} \longrightarrow \mathbf{N}$ if $\forall \varepsilon > 0 \exists N \in \mathbf{N} \forall n > N : |a_n - A| < \varepsilon$. ($N$ is a **threshold index** corresponding to ε.)

Definition 9.10 A sequence can have no more than one limit. If it has one, it is **convergent,** otherwise it is **divergent.**

Theorem 9.3 A convergent sequence is bounded.

Examples.

Three different definitions of convergence. A finite number of components (members) can be changed without effect.

Important convergent sequences.

9.3.1 Arithmetic opearations and limits

Zero sequences: $c_0(\mathbf{R})$, Convergent sequences: $c(\mathbf{R})$. $c_0(\mathbf{R}) \subset c(\mathbf{R}) \subset \mathbf{R}^{\mathbf{N}}$, sőt lineáris alterek. $\lim a = A \longleftrightarrow (a_n - A) \in c_0(\mathbf{R})$.

Operations and convergence, operations and limit, Multiple summands, multiple factors.

9.3.2 Inequality relations and limits

$$\lim a > \lim b \Longrightarrow \text{m. m. } n : a_n > b_n.$$

$$\text{m. m. } n : a_n \geq b_n \Longrightarrow \lim a \geq \lim b.$$

Alkalmazás: nemnegatív tagú sorozatokra és pozitív határértékre. Abszolút értékben nullasorozattal majorálható sorozat nullasorozat. Közrefogási elv.

9.3.3 The limit of monotonous sequences

Theorem 9.4 A monotonously increasing sequence which is also bounded from above is convergent, as well. Its limit is equal to its supremum.

Theorem 9.5 (Bolzano–Weierstrass) All the bounded sequences have a convergent subsequence.

Theorem 9.6 A monotonous sequence is bounded if and only if it is convergent.

Pozitív számok gyöke: értelmezés konstrukcióval (Newton-módszer).

Theorem 9.7 (Cauchy) The sequence $a : \mathbf{N} \longrightarrow \mathbf{R}$ is convergent if and only if

$$\forall \varepsilon > 0 \exists N \in \mathbf{N} : \forall m, n > N |a_n - a_m| < \varepsilon.$$

This criterion of convergence is an inner one, it does not contain the value of the limit.

Some of the divergent sequences are more regular than the others.

Definition 9.11 The sequence $a : \mathbf{N} \longrightarrow \mathbf{R}$ is said to tend to $+\infty$, if $\forall K > 0 \exists N \in \mathbf{N} \forall n > N : a_n > K$. ($N$ is a **threshold index** corresponding to K.)

One may say that such a sequence does have a limit—**in a broader sense**.

Theorem 9.8 A monotonously increasing sequence always has a limit (perhaps in the broader sense). Its limit is equal to its supremum.

9.3.4 Explicitly defined sequences, implicitly defined sequences, difference equations

Limes superior and limes inferior of a sequence

Definition 9.12 limes superior (inferior)

Theorem 9.9 (Properties of the limes superior)

1. If $K < \limsup a \Longrightarrow$, then there are an infinite number of members of the sequence greater than K.
2. If $L > \limsup a \Longrightarrow$, then there are only a finite number of the members of the sequence larger than L.
3. If a subsequence b of the sequence a has a limit B, it will necessarily be between $\limsup a$ and $\liminf a : \liminf a \leq B \leq \limsup$.
4. There is always a subsequence b such that $\lim b = \limsup a$.
5. There exists $\lim a$ if and only if $\limsup a = \liminf a (= \lim a)$.
6. For all positive real numbers $\lambda \in \mathbf{R}^+$ $\limsup(\lambda a) = \lambda \limsup a$ holds.

Definition 9.13 An accumulation point of the sequence a is the point A, if for all ε there is a member of a in the interval $]A - \varepsilon, A + \varepsilon[$.

(This may be defined for a set, and transferred to the range of a sequence.)

9.4 Numerical series

The sequence of partial sums.

The map Σ is bijective.

The sum of a series.

Theorem 9.10 (Cauchy criterion)

The change of a finite number of members makes no difference.

Definition 9.14 A series is absolutely convergent if

Theorem 9.11 An absolutely convergent series is convergent, as well.

9.4.1 Important classes of series

Series with positive members

Theorem 9.12 A series with positive members is convergent, if and only if the sequence of its partial sums is bounded.

Theorem 9.13 (Comparison criterion)

Leibniz series

Definition 9.15 (Leibniz series) A series $\sum a$ is a Leibniz series if

Theorem 9.14 A Leibniz series is convergent Error estimate

9.4.2 Operations and series

Operastions. Associativity (putting the parnethesis) Commutativity (re-ordering)

9.4.3 Relations and series

9.4.4 Further criteria of convergence

Theorem 9.15 (The root criterion by Cauchy)

Theorem 9.16 (The ratio criterion by d'Alembert)

Corollaries

Theorem 9.17 (The criterion by Raabe and that of Farkas Bolyai)

Chapter 10

Limit and Continuity of Functions

10.1 Important classes of functions

Here we mainly recollect what is known from high school and also add a few things.

10.1.1 Polynomials

Polynomials have been defined above in Definition **??**. Here we only mention that to evaluate a polynomial it is more economic to use their **Horner form** then the one given in the original definition.

Theorem 10.1 Let p be a polynomial with the degree $N \in \mathbf{N}_0$ and with the (real or complex) coefficients $a_0, a_1, \ldots, a_N$. Then the following equality holds

$$\forall x \in \mathbf{C} : p(x) = (\ldots(a_N x + a_{N-1})x + a_{N-2} + \ldots)x + a_0. \tag{10.1}$$

Remark 10.1 The advantage of the Horner form is that Evaluation of a polynomial in the original form requires N additions and $\frac{n^2+n}{2}$ multiplications (and only $2n-1$ multiplications, if powers are calculated by repeated multiplication). By contrast, Horner's scheme requires only N additions and N multiplications, and its storage requirements is also less that that of the calculations in the original form.

Example 10.1 The polynomial with $N = 1$ and $a_1 = 1$ is the same as the identity function of the real (or: complex) numbers.

Rational functions also have been introduced above: **??**.

There are some other functions known from high school, here we only repeat their definition.

Definition 10.1

1. The **absolute value of a real number** is the number itself if it is non-negative, and it is the negative of the number if it is negative:

$$\mathbf{R} \ni x \mapsto \mathrm{Abs}(x) := |x| := \max\{x, -x\} \in \mathbf{R}^+. \tag{10.2}$$

   The **absolute value of the complex number** $z =: a + bi \quad a, b \in \mathbf{R}$ is defined to be $\sqrt{a^2 + b^2}$.

2. The **sign** of a negative real number is -1, that of a positive number is $+1$, whereas the sign of zero is 0 :

$$\mathbf{R} \ni x \mapsto \mathrm{Sign}(x) := \begin{cases} -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x > 0 \end{cases} \tag{10.3}$$

3. The **integer part of a real number** is the largest integer not larger than the number itself:

$$\mathbf{R} \ni x \mapsto \mathrm{Int}(x) := \max\{n \in \mathbf{Z} | n \leq x\} \in \mathbf{Z}. \tag{10.4}$$

4. The **fractional part of a real number** is the difference of the number and its integer part:

$$\mathbf{R} \ni x \mapsto \mathrm{Frac}(x) := x - \mathrm{Frac}(x) \in [0, 1[. \tag{10.5}$$

5. Re Im

It may be useful to have a look at the *Mathematica* functions **`Abs, Ceiling, Floor, FractionalPart, Im, IntegerPart, Sign, Re, Round`**

intervallum volt már?

10.1.2 Power series

Definition 10.2 power series, coefficients, circle (disk) of convergence

Theorem 10.2 (Cauchy–Hadamard)

Definition 10.3 domain of convergence, convergence radius, sum a power series, analytic function

Power series and operations

Reordering included.

Power series and relations

???

Elementary functions

exp, cos, sin, ch, sh.

Theorem 10.3 (Addition theorems)

Theorem 10.4 (Functional equations)

Parity, oddity

10.2 The limit of functions

10.2.1 Arithmetic operations and limits

10.3 Continuous functions

Continuity at a **point of the domain** of a function Principle of transfer??? Continuity on a set Discontinuity, points of the first and second kind, jump, removable discontinuity Continuity from the left/right

10.3.1 Operations and continuity

Composition

Compact sets

Definition 10.4 The set $F \subset \mathbf{R}$ is a **closed set** if If a closed set is bounded as well, the it is a **compact set**.

Theorem 10.5 If a is a sequence with the property $\mathcal{R}_a \subset K$, where $K \subset \mathbf{R}$ is a compact set, then a has a convergent subsequence b for which $\lim b \in K$.

Theorem 10.6 A closed set contains all of its finite accumulation points.

Theorem 10.7 A compact set contains all of its accumulation points.

Functions continuous on a compact set

Theorem 10.8 If K is a compact set and $f : K \to \mathbf{R}$ is a continuous function, then $f(K)$ is compact as well.

Theorem 10.9 (Weierstrass) If K is a compact set and $f : K \to \mathbf{R}$ is a continuous function, then f has a maximum: $\exists x^* \in K : \sup f = f(x^*)$.

Definition 10.5 (korábbra) The function $f : A \to B$ is said to be **bounded**, if $\mathcal{R}_f$ is bounded.

Theorem 10.10 (Weierstrass) If K is a compact set and $f : K \to \mathbf{R}$ is a continuous function, then f is bounded.

10.3.2 Uniformly continuous fuctions

Heinrich Eduard Heine (1821–1881)

Definition 10.6 The function $f \subset \mathbf{R} \times \mathbf{R}$ is said to be **uniformly continuous**, if for every positive ε there exists a positive δ such that for all $x, y \in \mathcal{D}_f$ for which $|x - y| < \delta$ the relation $|f(x) - f(y)| < \varepsilon$ holds.

Theorem 10.11 If the function $f \subset \mathbf{R} \times \mathbf{R}$ is uniformly continuous, then it is continuous, as well.

Theorem 10.12 (Heine) If the domain of the continuous function $f \subset \mathbf{R}\times\mathbf{R}$ is a compact set, then it is uniformly continuous.

Bernard Placidus Johann Nepomuk Bolzano (1781–1848)

Definition 10.7 The function $f \subset \mathbf{R} \times \mathbf{R}$ is said to have the **Darboux property**, if in case $u, w \in \mathcal{R}_f; u < w$ for all $v \in [u, w]$ there exists an $x \in \mathcal{D}_f$ such that $v = f(x)$.

Theorem 10.13 (Bolzano) If the domain of the function $f \subset \mathbf{R} \times \mathbf{R}$ is a closed interval, then it has the Darboux property.

Theorem 10.14 If the domain of the continuous function $f \subset \mathbf{R} \times \mathbf{R}$ is an interval, then its range is an interval, too.

10.3.3 Continuity of the inverse function


Jean Gaston Darboux (1842–1917)

Theorem 10.15 The inverse of a continuous bijection defined on a compact set is itself continuous.

Theorem 10.16 The inverse of a continuous bijection defined on an interval is itself continuous.

Continuity of the root function. Definition of the functions ln, $\exp_a$, $\log_a$, and that of the power function with real exponent.

Chapter 11

Differential calculus

11.1 Set theoretic preparations

Definition 11.1 Let $A \subset \mathbf{R}$ be an arbitrary set of real numbers. The point $a \in A$ is an **inner point**, if there exists a positive number $\varepsilon \in \mathbf{R}^+$ such that $]a-\varepsilon, a-\varepsilon[\subset A$.

Definition 11.2 The $A \subset \mathbf{R}$ set of real numbers is an **open set**, if all its points are inner points.

Theorem 11.1 The $G \subset \mathbf{R}$ set is open if and only if $\mathbf{R} \setminus A$ is closed.

Proof. A) Suppose that $G \subset \mathbf{R}$ is open, and let $x \in (F := \mathbf{R} \setminus G)^{\mathbf{N}}$ be a convergent sequence with the limit x^*. Then the assumption $x^* \in G$ leads to a contradiction because G being open there should be an $\varepsilon \in \mathbf{R}^+$ such that $]x^*-\varepsilon, x^*+\varepsilon[\subset G$, but this excludes the possibility that a sequence in F has x^* as its limit.

B) Suppose that the set $(F := \mathbf{R} \setminus G)^{\mathbf{N}}$ is closed. If $x \in G$, then the set F cannot contain points arbitrarily close to the point x, because then it would be possible to construct a sequence tending to x, contradicting to the fact that F is closed. ■

11.2 Definition of the derivative and its basic properties

Let the domain of the function f be the set $\mathcal{D}_f \subset \mathbf{R}$, and let $a \in \mathcal{D}_f$ be an inner point of $\mathcal{D}_f$.

Definition 11.3 The function

$$\mathcal{D}_f \setminus \{a\} \ni x \mapsto \frac{f(x) - f(a)}{x - a} \tag{11.1}$$

is the **difference ratio function** of the function f at the point a. If the limit $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists and is finite, then the function is said to be **differentiable at the point** a, and the limit $A := \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ is its **derivative at the point** a. The derivative is denoted in the following ways: $f'(a), \dot{f}(a), Df(a), \frac{df}{dx}|_{x=a}$

The difference ratio function shows the value of the slope of the secant to the graph of the function f going through the points $(x, f(x)$ and $(a, f(a)$. Intuitively clear that the in the limit this secant is closer and closer to the tangent line of the graph at the point $(a, f(a)$. However, having no different definition for this concept this is what we accept as a definition of the tangent line.

Definition 11.4 Let the function f be differentiable at the inner point a of its domain $\mathcal{D}_f$. Then the line through $(a, f(a)$ with the slope $f'(a)$ is said to be its **tangent line** at the at the point $(a, f(a)$.

The following fact is something what one would expect and easy to prove, and important, as well.

Theorem 11.2 If the function f be differentiable at the inner point a of its domain $\mathcal{D}_f$ then it is also continuous at the point a.

The example of the function **`Abs`** shows that the converse is not true: this function is continuous, but it is not differentiable at the argument a.

Példák. Egyoldali derivált és érintő.

11.2.1 Arithmetic operations and the derivative

Kapcsolat a műveletekkkel. Deriváltfüggvény. Műveletek. Polinom, racionális függvény, analitikus függvény deriválható.

A deriválhatóság ekvivalens definíciója. (Differenciál, hibaszámítás.)

Közvetett függvény deriváltja. Inverz függvény deriváltja. Példák. Lokális korlátosság, növekedés, fogyás, szélsőérték. Növekedés és derivált. Szélsőérték és derivált.

11.2.2 Mean value theorems

Theorem 11.3 (Rolle)

Theorem 11.4 (Cauchy)

Theorem 11.5 (Lagrange)

Theorem 11.6 (Darboux)

The inverse of some elementary functions

Properties of the functions sin **and** cos

Defining the functions tan **and** cot

The inverse of the trigonometric functions

The inverse of the hyperbolic functions

11.2.3 Applications of differential calculus

Theorem 11.7 (The l'Hospital rule, variations)

Multiple derivatives. Taylor formula

Twice differentiable function at a given point, on a given set. Multiply differentiable functions, infinitely many times differentiable functions. Multiple derivation and arithmetic operations. Leibniz's theorem on the higher derivative of a product. Examples. Derivatives of an analytic function. An analytic function is differentiable infinitely many times. Taylor polynomial, Taylor series, Taylor formula.

Convex and concave functions

Definition 11.5 function convex (concave)

Theorem 11.8 (Convexity and difference ratios)

Theorem 11.9 (Convexity and derivatives)

Definition 11.6 change of sign

Definition 11.7 inflexion point

Analysis of fuctions

Theorem 11.10 (First order necessary condition of the existence of extrema.)

Theorem 11.11 (Second order sufficient condition of the existence of extrema.)

Theorem 11.12 (Second order sufficient condition of the existence of an inflexion poi

Theorem 11.13 (Third order necessary condition of the existence of an inflexion point

Definition 11.8 (Asymptotes)

What to check when analyzing a function?

1. Domain of the function

2. Zeros.
3. Parity.
4. Continuity, differentiability.
5. Monotonicity.
6. Extrema.
7. Inflexion points.
8. Concave and convex parts.
9. Limits at $\pm\infty$.
10. Asymptotes at a finite point and at infinity.
11. Range of the function

Tangent to a curve

The derivative of function with values in $\mathbf{R}^n$. Smooth elementary curve in $\mathbf{R}^n$. Parametrization. Closed smooth elementary curve. The parametrization of a line and of a section. Tangent.

Examples.

`http://www.georgehart.com/bagel/bagel.html` `http://www.dimensions-math.org/Dim_E.htm` `http://2009b.impulsive.hu`
Öveges

Chapter 12

Forms

Operations!! Moebius

Analytic geometry (Descartes) In this way, I should be borrowing all that is best in geometry and algebra, and should be correcting all the defects of the one by the help of the other.

Chapter 13

Integrals

13.1 Antiderivative

13.1.1 Basic notions

Relations with differential equations to be emphasized. Only function defined on intervals are considered here.

AntiderivativeAntiderivative with a given root. Indefinite integral Basic integrals Examples Operations and integrals Integration by parts Examples Integration by substitution Examples

13.1.2 Functions with elementary antiderivatives

Elementary functions. Their historical role. Rational functions Integrals reducing to calculation of the integrals of rational functions Examples

13.2 Definite integral

13.2.1 Basic notions

Divisions. Lower and upper sum. Lower and upper integral of the Darboux-type. Riemann integrability. Area under a curve. Nonintegrable functions: examples. Oscillatory sum. Riemann sum. The limit of Riemann sums.

13.2.2 Operations on integrable functions

The integral is a homogeneous linear functional. Integrability of the product and the ratio. How does the integral depend on the interval? Estimates. Mean value theorems and their consequences.

13.2.3 Classes of integrable functions

Continuous, monotonous, piecewise continuous, piecewise monotonous functions are integrable. Finite exceptional points make no difference.

13.2.4 Newton–Leibniz theorem

The fundamental theorem of calculus Conditions!

Ennek egy része az előadáson fog elhangzani.

Érdemes megemlíteni néhány integráltáblázatot (Bronstejn–Szemengyajev, Korn–Korn, Abramowitz–Stegun, Gradstejn–Rüzsik; egyszer majd megadom a rendes hivatkozást is), valamint azt a tényt, hogy a matematikai programcsomagok elég jól tudnak primitív függvényt számolni. A www.wolfram.com címen működik egy integrátor: a begépelt függvénynek kiszámolja egy primitív függvényét.

Chapter 14

Differential equations

Chapter 15

Discrete dynamical systems

Simple models with chaotic behaviour May

Chapter 16

Algorithms

Chapter 17

Guide to the collateral texts

17.1 How to read a paper (book), how to listen to a lecture?

1. Where was the paper/text/book/lecture published/presented? (Importance, quality, impact factor, SJR etc.)

2. Where do the authors work? (Prestigious institute, multiple places, continents)

3. Number of pages, figures, tables, references (review paper, new results, popularization), length and appearance of video etc.

4. Electronic supplement(s), if any. (Additional data, documents, experiments, proofs etc.)

5. Acknowledgements, (financial) support. (Beware of cancer research supported by a tobacco factory.)

6. Goal of the work in one sentence. Does it reach the purported/declared goal?

7. Methods (experimental, theoretical, mathematical, etc.).

8. Previous knowledge needed (courses).

17.2 Which one to choose?

Do not be frightened, what you see is an abundant list from which you have to select five items (usually papers or book chapters), and I will select the one which you report on.

[2] is an easy to read booklet. Any chapter of [10] (including the one I inserted here), any dialogue by Rényi, perhaps the one here [19] can play the rule of collateral texts.

Interested in the neurobiological background? Read Chapter 4 or 11 from [3].

You will be astonished how smart a Bush can be: [5].

Of the twin books [6] and [12] the first one needs quite a good background in mathematics. The second one is much less ordered, however, it offers a much wider than usual approach to the philosophy of mathematics.

The minimum needed from set theory if you would like to learn mathematics seriously, is here: [11]. Relatively easy to read.

[14] is the best (and probably the shortest) paper to start with if you are interested in chaos theory and its applications at the same time. [15, 16] and [29] are classics of Computer Science. Visiting the sites [8, 26, 25] will give provide you with a lot of interesting things to listen and to read. [23] is a wonderful introduction to mathematics originally written for Marcell Benedek, a literary man. [31] and [30] should be read together, especially if you have a background in economics.

Chapter 18

Sources of information

You find here links which may help you not only in learning mathematics.

Connected to Mathematics The site `http://thesaurus.maths.org/mmkb/view.html?resource=index&msglang=en` contains brief explanations of mathematical terms and ideas in Danish, English, Finnish, Hungarian, Lithuanian, Polish, Slovak and Spanish at a level between elementary and high school.

Oxford Dictionary, Mathematics And not only mathematics. The site `http://www.tankonyvtar.hu/konyvek/oxford-typotex/oxford-typotex-081030-` provides a smooth transition between high school and university in Hungarian.

Eric Weisstein: Wolfram Mathworld `http://mathworld.wolfram.com/` is a university level glossary with *Mathematica* notebooks included.

Michiel Hazewinkel: Encyclopaedia of Mathematics `http://eom.springer.de` is a graduate or research level reference work of mathematics. Originally, it was a five volume set in Russian, then it has been translated and enlarged to consist of 10 volumes, and finally it arrived at the Internet.

Dictionaries All the major English dictionaries are collected here `http://www.onelook.com/`. Pronunciation is also included.

English-Hungarian, Hungarian-English An unfinished still rich English-Hungarian dictionary is here: `http://mek.oszk.hu/00000/00076/html/index.htm`.

Wikipedia Almost every time it is worth starting here: http://en.wikipedia.org/wiki/Main_Page.

Magyarító könyvecske Foreign words translated into Hungarian: http://www.net.klte.hu/~keresofi/mke/a1.htm

If you happen to find something what is useful, important or both, tell me, and I'll include it here.

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Chapter 19

Appendix 1: Assignments

These are the assignments of the previous year, the actual ones you receive from Ágota Busai http://www.math.bme.hu/~bgotti/matMC.html.

19.1 Assignment 1

1. Could you possibly understand what is going on here: http://www.mathematika.hu/flash/csoda1.swf?
2. Prove that $1 \times 0 = 0$ using the axioms for real number in all steps.
3. Find the solution to the inequality $3x - 5 < -2x + 12$. again showing which axiom is used in the individual steps.
4. Calculate (using a Venn diagram) $A \cap (B \cup A)$; calculate using a truth table $A \vee (B \wedge A)$.

19.2 Assignment 2

1. Show that both the additive and the multiplicative inverse is unique.
2. What is the range of the function $x \mapsto x^2 + 3x + 2$?

3. Show that the composition of two functions is a function. What about the domain of the composition?

4. Let the relation $\rho \subset \mathbf{N} \times \mathbf{N}$ be defined by: $n\rho m :\Longleftrightarrow m = n + 1$.

   (a) Is it a function? Is it surjective, injective, bijective?

   (b) What is the inverse of this relation? Is that a function?

   (c) What is the domain and range of this relation? And those of it inverse?

19.3 Assignment 3

1. Is the relation $\{(x, y) in R^2; y = 2x - x^2\}$ a function? If yes, what is its range, is it injective, surjective, bijective?

2. Let f and g two real valued functions defined on the same set, and let us introduce the relation $f < g$ by the definition for all $x \in \mathcal{D}(f)$: $f(x) < g(x)$. Show that this relation is transitive, monotonous wrt addition and conditionally monotonous wrt multiplication, but is not trichotomous.

3. Could you characterize those first degree polynomials which are surjections? (Additional problems: Second degree etc.? Injections, bijections?)

4. Find anything (methods of proof, logic, set theory, axiomatics, real numbers etc.) connected to the lectures on mathematics on the web page `demonstrations.wolfram.com`

19.4 Assignment 6

Characterize the sequences below (monotonicity, boundedness), find their supremum and infimum. Which of them are convergent? In the case of convergent sequences find their limit, and calculate the threshold index N given $\varepsilon = 0.015$.

1. $a_n := \frac{1-4n}{n+1} \quad (n \in \mathbf{N})$
2. $a_n := \frac{1}{\sqrt{n}} \quad (n \in \mathbf{N})$
3. $a_n := (-n)^3 \quad (n \in \mathbf{N})$
4. $a_n := \frac{(-1)^n}{n} \quad (n \in \mathbf{N})$
5. $a_n := (-1)^n n \quad (n \in \mathbf{N})$

19.5 Assignment 7

1. Solve problem 4 and 5 from the previous set.
2. Characterize the sequences below (monotonicity, boundedness), find their supremum and infimum. Which of them are convergent? In the case of convergent sequences find their limit, and calculate the (if possible, the smallest) threshold index N given $\varepsilon = 0.015$. You may also find that a sequence is divergent but tends to $\pm\infty$. To have an idea about the behaviour of the sequence calculate the first few members of the sequence, make drawings etc.
   (a) $a_n := n - \frac{1}{n} \quad (n \in \mathbf{N})$,
   (b) $a_n := \frac{5n+1}{n-11.5} \quad (n \in \mathbf{N})$,
   (c) $a_n := ((-1)^n + 1) \quad (n \in \mathbf{N})$,
   (d) $a_n := \frac{1+2+\cdots+n}{n(n+1)} \quad (n \in \mathbf{N})$,
   (e) $a_{n+1} := \frac{1}{2}(a_n + \frac{3}{a_n}),\ a_1 := 1 \quad (n \in \mathbf{N})$.

19.6 Assignment 8

1. Give a formal proof of the statement that a monotonously decreasing sequence if it is also bounded from below is convergent.

2. Give a sequence which is monotonously decreasing and not convergent.

3. Show that the sequence $a_n := \sqrt{1 + a_{n-1}}, a_1 = 3 \quad (n \in \mathbf{N})$ is convergent and calculate its limit.

4. Find the tenth member of the Fibonacci sequence defined by $f_n = f_{n-1} + f_{n-2}, f_1 = f_2 = 1$.

5. Characterize the sequences below (monotonicity, boundedness), find their supremum and infimum. Which of them are convergent? In the case of convergent sequences find their limit, and calculate the (if possible, the smallest) threshold index N given $\varepsilon = 0.015$. You may also find that a sequence is divergent but tends to $\pm\infty$. In this case find a threshold index from which on $|a_n| > 1000$ holds. To have an idea about the behaviour of the sequence calculate the first few members of the sequence, make drawings etc.

   (a) $a_n := \frac{n^2-3n+1}{n} \quad (n \in \mathbf{N})$,

   (b) $a_n := \frac{n^2-3n+1}{-n^2+2} \quad (n \in \mathbf{N})$,

   (c) $a_n := \frac{n^2-3n+1}{n^3+n^2+n+1} \quad (n \in \mathbf{N})$,

   (d) $a_n := \frac{n^2-3n+1}{(-1)^n n^2+n+1} \quad (n \in \mathbf{N})$,

19.7 Assignment 9

1. Calculate the values $f(\sqrt{2}), f(\sqrt{8}), f(\sqrt{\log_2 1024})$, if

$$f(x) := \begin{cases} 2x^3 + 1, & \text{if} \quad -1 \le x < 0; \\ \frac{1}{x-2} & \text{if} \quad 0 \le x < \pi; \\ \frac{x}{x^2-2} & \text{if} \quad \pi \le x \le 6. \end{cases}$$

2. Suppose we know $f(x-2) = \frac{1}{x+1} \quad (x \neq -1)$. What do we know about f?

3. Determine the largest intervals which may be taken as the domain of the functions defined by the formulae:
a) $\frac{x^2}{1+x}$ b) $\sqrt{x}+\sqrt{-x}$ c) $\sqrt[3]{\frac{2x}{x^2-2x+2}}$ d) $\lg(\sin(\lg(x)))$.

4. The domain of f is the interval $[0,1]$. What is the domain of $f \circ \tan$?

5. Calculate the limits below
a) $\lim_{x\to 0} \frac{1}{1+x}$ b) $\lim_{x\to 1} \frac{x^4+2x^2-3}{x^2-3x+2}$ c) $\lim_{x\to +\infty} \frac{x^2-1}{2x^2+1}$.

6. Find the points where the function defined by the formula is continuous:
$$f(x) := \begin{cases} 1-x^2, & \text{if} \quad x \le 0; \\ (1-x)^2 & \text{if} \quad 0 < x \le 2; \\ 3-x & \text{if} \quad 2 < x. \end{cases}$$

7. How to choose the parameter a to get a continuous function by the definition
$$f(x) := \begin{cases} ax^2+1, & \text{if} \quad 0 < x; \\ -x & \text{if} \quad x \le 0. \end{cases}$$

8. Calculate the inverse of the function: $[0,1] \ni x \mapsto 3x+5 \in \mathbf{R}$.

19.8 Assignment 10

1. What can you say about the continuity of the functions $f \circ g$ and $g \circ f$, if $f = \mathrm{Sign}$ and $g = 1 + \mathrm{id}^2$?

2. Calculate the limit $\lim_{x\to 0} \frac{\sqrt{1+x+x^2}-1}{x}$

3. Find the points of discontinuity of the function defined by
$$f(x) := \begin{cases} \frac{x^2-5x+6}{x^2-7x+10}, & \text{if} \quad x \ne 2, x \ne 5; \\ 0 & \text{if} \quad x = 2, x = 5. \end{cases}$$

4. Suppose the functions f and g are both discontinuous at the point of their domain a. Is it possible that $f+g, f-g, f/g, f^2$ is continuous at a?

5. Suppose the functions f is continuous at the point of its domain a, and the function g is discontinuous at the point of its domain a. Is it possible that $f+g, f-g, f/g, f^2$ is continuous at a?

6. Calculate the limits below
   a) $\lim_{x \to 0} \frac{\frac{1}{1+x}-1}{x}$ b) $\lim_{x \to 1} \frac{(y+x)^2-y^2}{x}$ c) $\lim_{h \to 0} \frac{(x+h)^2-x^2}{h}$.

Chapter 20

Appendix 2: Mid-terms

20.1 Mid-term 1

20.1.1 Preliminary version

Exercise 1 Formulate the sentences below using logical operators.
"And if you've got to sleep
A moment on the road
I will steer for you
And if you want to work the street alone
I'll disappear for you
If you want a father for your child
Or only wanna walk with me a while
Across the sand
I'm your man" (Leonard Cohen)

Exercise 2 Translate into plain English.
A is a necessary and sufficient condition for B.

Exercise 3 Expand the expressions below using logical identities. Give their truth table.

1. $x \iff y$

2. $(\neg x \vee y) \implies (v \wedge w)$

Exercise 4 Convert the expression using $\cup, \cap$ and complement.

$$A - (C - (B - (B - C)))$$

Exercise 5 Prove the statements below for every A, B and C set. Illustrate the expressions on either side of the equal sign by using Venn diagrams.

1. $(A \cup B) \cap (A \cup C) \cap (B \cup C) = (A \cap B) \cup (A \cap C) \cup (B \cap C)$
2. $A - (B - C) = (A - B) \cup (A \cap C)$

Exercise 6 How can you arrive at Barack Obama through a chain of acquaintances of length not more than 6?

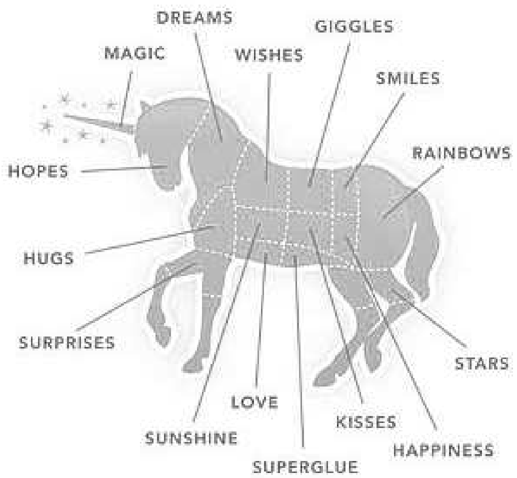

Figure 20.1: Exercise 7.

Exercise 7 Color the "map" of the unicorn in Fig. 20.1. Draw its graph and color the vertices too.

Exercise 8 Describe the properties of the relations below. Are they functions? What is the inverse of the relations, is that a function? What about their properties?

1. $\rho \subset \mathbb{N} \times \mathbb{N},\ n\,\rho\,m \iff m \geq n - 1$
2. $\rho \subset \mathbb{N} \times \mathbb{N},\ n\,\rho\,m \iff m = 7n - 4$
3. $\rho \subset \mathbb{Z} \times \mathbb{Z},\ n\,\rho\,m \iff n = m^2$

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