List of formulas for the A3 final

Mathematics A3 in English for Civil Engineering students

$\sin^2 x + \cos^2 x = 1$

$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \cdot \tan y}$

$\sin 2x = 2 \sin x \cos x$

$\cos 2x = \cos^2 x - \sin^2 x$

$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$

$\sin^2 x = \frac{1 - \cos 2x}{2}$, $\cos^2 x = \frac{1 + \cos 2x}{2}$

$\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$

$\sin x - \sin y = 2 \cos \frac{x+y}{2} \sin \frac{x-y}{2}$

$\cos x + \cos y = 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}$

$\cos x - \cos y = -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}$

$\sin x \cos y = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$

$\cos x \cos y = \frac{1}{2}[\cos(x+y) + \cos(x-y)]$

$\sin x \sin y = -\frac{1}{2}[\cos(x+y) - \cos(x-y)]$

$\sinh x = \frac{e^x - e^{-x}}{2}$, $\cosh x = \frac{e^x + e^{-x}}{2}$

$\cosh^2 x - \sinh^2 x = 1$

$\sinh 2x = 2 \sinh x \cosh x$

$\cosh 2x = \cosh^2 x + \sinh^2 x$

$\cosh^2 x = \frac{\cosh 2x + 1}{2}$, $\sinh^2 x = \frac{\cosh 2x - 1}{2}$

Derivatives:

$(\sinh x)' = \cosh x$

$(\cosh x)' = \sinh x$

$(\log_a x)' = \frac{1}{x \ln a}$

$(x^\alpha)' = \alpha x^{\alpha - 1}$

$(e^x)' = e^x$

$(a^x)' = a^x \ln(a)$

$(\sin x)' = \cos x$

$(\cos x)' = -\sin x$

$(\tan x)' = \frac{1}{\cos^2 x}$

$(\cot x)' = -\frac{1}{\sin^2 x}$

$(\ln x)' = \frac{1}{x}$

$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$

$(\arctan x)' = \frac{1}{1+x^2}$

$(\operatorname{ar}\sinh x)' = \frac{1}{\sqrt{1+x^2}}$

$(\operatorname{ar}\cosh x)' = \frac{1}{\sqrt{x^2-1}}$

$(\operatorname{ar}\tanh x)' = \frac{1}{1-x^2}$

$(\operatorname{ar}\coth x)' = \frac{1}{1-x^2}$

$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$

$(\operatorname{arc}\cot x)' = -\frac{1}{1+x^2}$

Rules of Differentiation:

$(cu)' = cu'$ (c is constant)

$(u+v)' = u' + v'$

$(uv)' = u'v + uv'$

$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

$\frac{d}{dx} f(g(x)) = \frac{df}{dg} \frac{dg}{dx}$

Rules of Integration:

$\int cf \, dx = c \int f \, dx$ (c is constant)

$\int (f+g) \, dx = \int f \, dx + \int g \, dx$

$\int f(ax+b) \, dx = \frac{1}{a} F(ax+b) + c$,

where F is the antiderivative of f

$\int f(g(x)) g'(x) \, dx = F(g(x)) + c$,

where F is the antiderivative of f

$\int f^\alpha f' \, dx = \frac{f^{\alpha+1}}{\alpha+1} + c$, ha $\alpha \neq -1$

$\int \frac{f'}{f} \, dx = \ln|f| + c$

$\int uv' \, dx = uv - \int u'v \, dx$

Useful Substitutions:

$R(e^x)$	$e^x = t$
$R(\sqrt{ax+b})$	$\sqrt{ax+b} = t$
$R\left(\frac{\sqrt{ax+b}}{\sqrt{cx+d}}\right)$	$\frac{\sqrt{ax+b}}{\sqrt{cx+d}} = t$
$R(\sin x, \cos x)$	$\sin x, \cos x, \tan x, \tan \frac{x}{2} = t$
$R(x, \sqrt{a^2 - x^2})$	$x = a \sin t,\ x = a \cos t$
$R(x, \sqrt{a^2 + x^2})$	$x = a \sinh t$
$R(x, \sqrt{x^2 - a^2})$	$x = a \cosh t$

Integrals:

$\int x^\alpha \, dx = \frac{x^{\alpha+1}}{\alpha+1} + c$ ($\alpha \neq -1$)

$\int e^{ax} \, dx = \frac{1}{a} e^{ax} + c$

$\int a^x \, dx = \frac{a^x}{\ln a} + c$

$\int \cos x \, dx = \sin x + c$

$\int \sin x \, dx = -\cos x + c$

$\int \frac{1}{\cos^2 x} \, dx = \tan x + c$

$\int \frac{1}{\sin^2 x} \, dx = -\cot x + c$

$\int \frac{1}{x} \, dx = \ln|x| + c$

$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + c$

$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \arctan \frac{x}{a} + c$

$\int \frac{dx}{\sqrt{x^2 + a^2}} = \operatorname{ar}\sinh \frac{x}{a} + c$

$\int \frac{dx}{\sqrt{x^2 - a^2}} = \operatorname{ar}\cosh \frac{x}{a} + c$

$\int \frac{dx}{a^2 - x^2} = \frac{1}{a} \operatorname{ar}\tanh \frac{x}{a} + c$, ha $\left|\frac{x}{a}\right| < 1$

$\int \frac{dx}{a^2 - x^2} = \frac{1}{a} \operatorname{ar}\coth \frac{x}{a} + c$, ha $\left|\frac{x}{a}\right| > 1$

$\int \tan x \, dx = -\ln|\cos x| + c$

$\int \cot x \, dx = \ln|\sin x| + c$

1. $$e^{it} = \cos t + i \cdot \sin t, \qquad t \in \mathbb{R}.$$

2. The characteristic equation for
$$ay'' + by' + cy = 0 \qquad (a \neq 0)$$
is:
$$ar^2 + br + c = 0.$$

3. In the **Method of Undetermined Coefficients**, if
$$g(t) = e^{ut}\left(A_n(t)\cos(vt) + B_m(t)\sin(vt)\right)$$
in the equation
$$ay'' + by' + cy = g(t), \qquad a \neq 0 \text{ and } t \in I,$$
where $A_n(t)$, $B_m(t)$ are polynomials of degree n and m, respectively, then a particular solution of the inhomogeneous equation is of the form
$$y_{i,p} = t^s e^{ut}\left(P_k(t)\cos(vt) + Q_k(t)\sin(vt)\right)$$
where s is the multiplicity of $u + i \cdot v$ among the roots of the characteristic equation. Here $P_k(t)$, $Q_k(t)$ are general polynomials of degree $k = \max(n, m)$.

4. In the method of **Variation of Parameters**: If the fundamental solution of the homogeneous part $y'' + p(t)y' + q(t)y = 0$ of the equation
$$y'' + p(t)y' + q(t)y = g(t) \qquad t \in I \tag{1}$$
is y_1, y_2, then a particular solution of the inhomogeneous equation (1) is $y_{i,p} = C_1(t) \cdot y_1(t) + C_2(t) \cdot y_2(t)$, where the derivatives of the functions $C_1(t)$, $C_2(t)$ satisfy
$$\begin{aligned} C_1'(t)y_1(t) + C_2'(t)y_2(t) &= 0 \\ C_1'(t)y_1'(t) + C_2'(t)y_2'(t) &= g(t). \end{aligned}$$

5. For an **incomplete second order equation:**
   - If y is missing, the substitution is $p(x) := y'(x)$.
   - If x is missing, then the substitution is $p(y) := y'$.

6. The equation $M(x, y)dx + N(x, y)dy = 0$ is **exact**, if
$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$
To solve it, one has to find the function $F : \mathbb{R}^2 \to \mathbb{R}$ for which
$$\frac{\partial F}{\partial x} = M \qquad \text{and} \qquad \frac{\partial F}{\partial y} = N.$$
Then the solution of the differential equation is $F(x, y) = \text{const}$.

Solution of $\underline{\dot{x}} = \underline{\underline{A}} \cdot \underline{x}$: If $\underline{v}^{(j)}$ is an eigenvector with an eigenvalue $r^{(j)}$ of multiplicity one, then the corresponding part of the homogeneous solution is $C_j \cdot e^{r^{(j)}t} \cdot \underline{v}^{(j)}$.

If $\underline{a} \pm i \cdot \underline{b}$ are eigenvectors with eigenvalues $\lambda \pm i \cdot \mu$ of multiplicity one, then the corresponding part of the homogeneous solution is
$C_j \cdot e^{\lambda t} \cdot [\underline{a}\cos(\mu t) - \underline{b}\sin(\mu t)] + C_{j+1} \cdot e^{\lambda t} \cdot [\underline{a}\sin(\mu t) + \underline{b}\cos(\mu t)]$.

For $\underline{\dot{x}} = \underline{\underline{A}} \cdot \underline{x} + \underline{g}(t)$, make the constants C_j t-dependent.

Probability

Multiplication rule: $\mathbb{P}\{E_1E_2\dots E_n\} = \mathbb{P}\{E_n \,|\, E_1\dots E_{n-1}\}\cdots\mathbb{P}\{E_3 \,|\, E_1E_2\}\cdot\mathbb{P}\{E_2 \,|\, E_1\}\cdot\mathbb{P}\{E_1\}$,

Law of Total Probability: If $F_1, F_2, \dots$ form a complete system of events that is, $\bigcup_i F_i = S$, and $F_i \cap F_j = \emptyset$ if $i \neq j$, then

$$\mathbb{P}\{E\} = \sum_i \mathbb{P}\{E \,|\, F_i\}\cdot\mathbb{P}\{F_i\}.$$

Bayes Theorem: If $F_1, F_2, \dots$ form a complete system of events that is, $\bigcup_i F_i = S$, and $F_i \cap F_j = \emptyset$ if $i \neq j$, then

$$\mathbb{P}\{F_i \,|\, E\} = \frac{\mathbb{P}\{E \,|\, F_i\}\cdot\mathbb{P}\{F_i\}}{\sum_j \mathbb{P}\{E \,|\, F_j\}\cdot\mathbb{P}\{F_j\}}.$$

The binomial(n, p) distr'n

→ mass function: $p(i) = \binom{n}{i}\cdot p^i\cdot(1-p)^{n-i}, \qquad i = 0, 1, \dots, n$

→ expectation: $\mathbb{E}(X) = np$

→ variance: $\mathbf{Var}(X) = np(1-p)$

→ most probable value: $\lfloor (n+1)p \rfloor$

The Poisson(λ) distr'n

→ mass function: $p(i) = \frac{\lambda^i}{i!}\cdot \mathrm{e}^{-\lambda}, \qquad i = 0, 1, 2, \dots$

→ expectation: $\mathbb{E}(X) = \lambda$

→ variance: $\mathbf{Var}(X) = \lambda$

→ most probable value: $\begin{cases} \lfloor\lambda\rfloor & \text{, if } \lambda \text{ is not integer,} \\ \lambda \text{ and } \lambda - 1 & \text{, if } \lambda \text{ is integer.} \end{cases}$

The geometric(p) distr'n

→ mass function: $p(i) = (1-p)^{i-1}\cdot p, \qquad i = 1, 2, 3, \dots$

→ expectation: $\mathbb{E}(X) = \frac{1}{p}$

→ variance: $\mathbf{Var}(X) = \frac{1-p}{p^2}$

The uniform(a, b) distr'n

→ density function: $f(x) = \begin{cases} \frac{1}{b-a} & \text{, if } a < x < b, \\ 0 & \text{, else.} \end{cases}$

→ distribution function: $F(x) = \begin{cases} 0 & \text{, if } x \leq a, \\ \frac{x-a}{b-a} & \text{, if } a < x < b, \\ 1 & \text{, if } x \geq b. \end{cases}$

→ expectation: $\mathbb{E}(X) = \frac{a+b}{2}$

→ variance: $\mathbf{Var}(X) = \frac{(b-a)^2}{12}$

The normal(μ, σ^2) distr'n

→ density function: $f(x) = \frac{1}{\sqrt{2\pi}\cdot\sigma}\cdot \mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}$

→ distribution function: $F(x) = \Phi\Big(\frac{x-\mu}{\sigma}\Big)$, where Φ is the standard normal distribution function. For all x, $\Phi(-x) = 1 - \Phi(x)$.

→ expectation: $\mathbb{E}(X) = \mu$

→ variance: $\mathbf{Var}(X) = \sigma^2$

The exponential(λ) distr'n

→ density function: $f(x) = \begin{cases} \lambda \cdot e^{-\lambda x} & \text{, if } x > 0, \\ 0 & \text{, if } x \le 0. \end{cases}$

→ distribution function: $F(x) = \begin{cases} 1 - e^{-\lambda x} & \text{, if } x > 0, \\ 0 & \text{, if } x \le 0. \end{cases}$

→ expectation: $\mathbb{E}(X) = 1/\lambda$

→ variance: $\mathbf{Var}(X) = 1/\lambda^2$

DeMoivre-Laplace theorem: If $X \sim$ binomial(n, p), then $\mathbb{P}\Big\{\frac{X - np}{\sqrt{np(1-p)}} \le a\Big\} \underset{n\to\infty}{\longrightarrow} \Phi(a)$.

Central Limit Thoerem: If $X_1, X_2, \ldots, X_n$ are independent identically distributed random variables with expectation μ and variance σ^2, then

$$\mathbb{P}\Big\{\frac{X_1 + X_2 + \cdots + X_n - n\mu}{\sqrt{n} \cdot \sigma} \le a\Big\} \underset{n\to\infty}{\longrightarrow} \Phi(a)$$

The standard normal distribution function

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000