Megoldás 2013. 11. 25.

① $y' - \frac{1}{x+1} y = x^3$, $x > 0$

2p Hom: $(y \neq 0)$ $\frac{y'}{y} = \frac{1}{x+1} \Rightarrow \int \frac{dy}{y} = \int \frac{1}{x+1} dx = \ln\underbrace{(x+1)}_{>0} + c_1$

$y > 0 \Rightarrow \int \frac{dy}{y} = \ln y$; $y < 0 \Rightarrow \int \frac{dy}{y} = \ln(-y) \Rightarrow y(x) = \begin{cases} e^{c_1} \cdot (x+1) \\ -e^{c_1} (x+1) \end{cases}$

$\Rightarrow y_{HÁ}(x) = c\,(x+1)$ (c tetsz előjelű)

3p Inhom. part mo: bel:

$y_p(x) = c(x)(x+1) \Rightarrow y_p'(x) = c'(x)(x+1) + c(x)$

$c'(x)(x+1) + c(x) - c(x) = x^3 \Rightarrow c'(x) = \frac{x^3}{x+1}$

$$\begin{array}{l} x^3 : (x+1) = x^2 - x + 1 \\ \underline{x^3 + x^2} \\ \quad -x^2 \\ \quad \underline{-x^2 - x} \\ \qquad x \\ \qquad \underline{x+1} \\ \qquad -1 \end{array}$$

$\Rightarrow c'(x) = x^2 - x + 1 - \frac{1}{x+1}$

$c(x) = \frac{x^3}{3} - \frac{x^2}{2} + x - \ln(x+1)$

1p $y_{Á}(x) = c\,(x+1) + \frac{x^3}{3}(x+1) - \frac{x^2}{2}(x+1) + x(x+1) - (x+1)\ln(x+1)$

---

② $\dot{x} = x - xy + y - 1$; $\dot{y} = x^2 - 2x + 1 - y$

1p EH: $x - xy + y - 1 = (y-1)(1-x) = 0 \Rightarrow x = 1$ vagy $y = 1$

$x^2 - 2x + 1 - y = (x-1)^2 - y = 0 \Rightarrow y = 0$; $x = 2 \vee x = 0$

$\Rightarrow P_0(1,0)$, $P_1(0,1)$, $P_2(2,1)$

1p $Df(x,y) = \begin{pmatrix} 1-y & 1-x \\ 2(x-1) & -1 \end{pmatrix}$

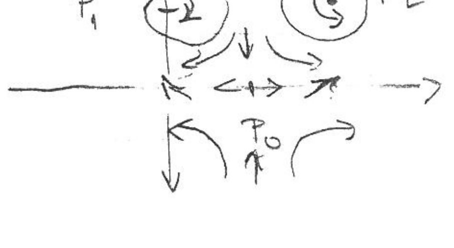
1p $Df|(1,0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow \lambda_{1,2} = \pm 1$ instab. nyereg

2p $Df(0,1) = \begin{pmatrix} 0 & 1 \\ -2 & -1 \end{pmatrix} \Rightarrow \lambda^2 + \lambda + 2 = 0$

$\lambda_{1,2} = \frac{-1 \pm \sqrt{1-8}}{2}$

$\mathrm{Re}\,\lambda_i < 0$ $\mathrm{Im}\,\lambda_i \neq 0$ stab. fókusz asz stab

1p $Df(2,1) = \begin{pmatrix} 0 & -1 \\ 2 & -1 \end{pmatrix} \Rightarrow \lambda^2 + \lambda + 2 = 0$ u.a. kar. egy u.a. típ

1p 

3.) $y_{n+2}-(a+1)y_{n+1}+a y_n = h\left(\frac{5}{12}f_{n+2}+\frac{8}{12}f_{n+1}-\frac{1}{12}f_n\right)$

2p: 1.) $\lambda^2-(a+1)\lambda+a=(\lambda-a)(\lambda-1)=0 \quad \lambda_1=a \quad \lambda_2=1$

$|\lambda_i|\le 1 \quad |a|\le 1$, de $\lambda_2=1$ egys.: $-1\le a<1$ esetén telj. a stab. felt.

1p: $\boxed{\begin{array}{c}2.)\\ a=0\end{array}}$ $L(x,h)=y(x+2h)-y(x+h)-h\left(\frac{5}{12}y'(x+2h)+\frac{8}{12}y'(x+h)-\frac{1}{12}y'(x)\right)$

2p:

$$\begin{aligned}
1\cdot/ \quad & y(x+2h) = y(x) + y'(x)2h + y''(x)(2h)^2/2! + O(h^3)\\
-1\cdot/ \quad & y(x+h) = y(x) + y'(x)h + y''(x)h^2/2! + O(h^3)\\
0\cdot/ \quad & y(x) = y(x)\\
-5h/12\cdot/ \quad & y'(x+2h) = y'(x) + y''(x)(2h) + O(h^2)\\
-8h/12\cdot/ \quad & y'(x+h) = y'(x) + y''(x)h + O(h^2)\\
h/12\cdot/ \quad & y'(x) = y'(x)
\end{aligned}$$

1p:

$$\Rightarrow (1-1)y(x) + \left(2-1-\frac{5}{12}-\frac{8}{12}+\frac{1}{12}\right)h y'(x) +$$

$$+\left(\frac{4}{2}-\frac{1}{2}-\frac{10}{12}-\frac{8}{12}\right)h^2 y''(x) + O(h^3) = O(h^3)$$

$\Rightarrow L(x,h)=O(h^3) \Rightarrow$ a módszer rendje h_a. 2.

1p: 3.) $a=0\in[-1,1)$ és a módszer rendje h_a 2 $\Rightarrow$ konv.

4.) (PDE) $u'_t = 3u''_{xx}$; (PF) $u(0,t)=u(2\pi,t)=0$

1,5p: (KF) $u(x,0)=\sin\frac{x}{2}+\left(\sin\frac{x}{2}\right)^3$

1l. $u(x,t)=X(x)T(t)\ne 0 \Rightarrow \dot T(t)X(x)=3T(t)X''(x) \quad /\cdot\frac{1}{3T(t)X(x)}$

$$\frac{\dot T(t)}{3T(t)}=\frac{X''(x)}{X(x)}=\alpha = \text{konst} \Rightarrow \begin{array}{ll}(1) & X''(x)-\alpha X(x)=0\\ (2) & \dot T(t)-3\alpha T(t)=0\end{array}$$

0,5p: 2.l. (PF) $\Rightarrow X(0)=X(2\pi)=0$

1p: a.) $\alpha>0 \Rightarrow X(x)=c_1 e^{\sqrt{\alpha}x}+c_2 e^{-\sqrt{\alpha}x}$

$X(0)=c_1+c_2=0 \Rightarrow c_1=-c_2$; $X(2\pi)=c_1\left(e^{\sqrt{\alpha}2\pi}-e^{-\sqrt{\alpha}2\pi}\right)=0$

$\Rightarrow c_1=0,\ c_2=0 \Rightarrow$ nem megf.

0,5p b.) $\alpha = 0 \Rightarrow X(x) = c_1 + c_2 x$; $X(0) = c_1 = 0$, $X(2\pi) = 2\pi c_2 = 0$

$\Rightarrow c_2 = 0$ nem megf.

1,5p c.) $\alpha = -\omega^2 < 0$ $(\omega > 0) \Rightarrow X(x) = c_1 \cos\omega x + c_2 \sin\omega x$

$X(0) = c_1 = 0$, $X(2\pi) = c_2 \sin(2\pi\omega) = 0$ $2\pi\omega = k\pi$, $k = 1, 2, \ldots$

$\omega = \frac{k}{2} \Rightarrow X_k(x) = c_k \sin\frac{kx}{2}$

1p $\dot{T}_k(t) + \left(\frac{k}{2}\right)^2 \cdot 3\, T_k(t) = 0 \Rightarrow T(t) = e^{-3\frac{k^2}{4}t}$

$u_k(x,t) = c_k e^{-\frac{3}{4}k^2 t} \sin\frac{kx}{2}$ - e (PDE) & (PF) telj.

1p 3.l. Keressük a mo-t

$$u(x,t) = \sum_{k=1}^{\infty} c_k e^{-\frac{3}{4}k^2 t} \sin\frac{kx}{2}$$

alakban! Terjesszük ki a kezd. fgv-t 4π-n. per. ptlan fgv-ként! A kezd. fgv. már ilyen.

Vegyük a F. sorát:

2p $\sin^3\alpha = \sin\alpha \sin^2\alpha = \frac{1}{2}\sin\alpha - \frac{1}{2}\sin\alpha\cos 2\alpha =$

$= \frac{1}{2}\sin\alpha - \frac{1}{4}(\sin 3\alpha - \sin\alpha) = \frac{3}{4}\sin\alpha - \frac{1}{4}\sin 3\alpha$

$\Rightarrow \sin\frac{x}{2} + \left(\sin\frac{x}{2}\right)^3 = \frac{7}{4}\sin\frac{x}{2} - \frac{1}{4}\sin\frac{3x}{2}$ véges F sor

1p $c_1 = b_1 = \frac{7}{4}$, $c_2 = b_2 = 0$, $c_3 = b_3 = -\frac{1}{4}$, $c_k = b_k = 0$, $k > 3$

$$u(x,t) = \frac{7}{4} e^{-\frac{3}{4}t} \sin\frac{x}{2} - \frac{1}{4} e^{-\frac{27}{4}t} \sin\frac{3x}{2}$$