Többváltozós diszkrét momentum problémák
(Multivariate Discrete Moment Problems)

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Thesis

The discrete moment problem (DMP) has been formulated as a methodology to find the minimum and/or maximum of a linear functional acting on an unknown probability distribution, the support of which is a known discrete (usually finite) set, where some of the moments are known. The moments may be binomial, power or of more general type. The multivariate discrete moment problem (MDMP) has been initiated by Prékopa who developed a linear programming theory and methodology for the solution of the DMP's and MDMP's under some assumptions, that concern the divided differences of the coefficients of the objective function. The central results in this respect are there that concern the structure of the dual feasible bases.

In the Ph.D. dissertation further results are presented in connection with MDMP's for the case of power and binomial moments. The first two sections are on introduction, then the following two show new results, finally, there are numerical examples in the last one.

In the dissertation we focus on the next type of MDMP. Let $(X_1, \ldots, X_s)$ be a random vector. We assume that the support of X_j is a known finite set $Z_j = \{z_{j0}, \ldots, z_{jn_j}\}$, where $z_{j0} < \cdots < z_{jn_j}$; $j = 1, \ldots, s$ and define

$$p_{i_1 \ldots i_s} = P(X_1 = z_{1i_1}, \ldots, X_s = z_{si_s}),\ 0 \leq i_j \leq n_j,\ j = 1, \ldots, s,$$

$$\mu_{\alpha_1 \ldots \alpha_s} = \sum_{i_1=0}^{n_1} \cdots \sum_{i_s=0}^{n_s} z_{1i_1}^{\alpha_1} \cdots z_{si_s}^{\alpha_s} p_{i_1 \ldots i_s},$$

where $\alpha_1, \ldots, \alpha_s$ are nonnegative integers. The number $\mu_{\alpha_1, \ldots, \alpha_s}$ will be called the $(\alpha_1, \ldots, \alpha_s)$-order moment of the random vector $(X_1, \ldots, X_s)$, and the sum $\alpha_1 + \cdots + \alpha_s$ the total order of the moment. Let $Z = Z_1 \times \cdots \times Z_s$ and $f(\boldsymbol{z})$, $\boldsymbol{z} \in Z$ be a function for which we introduce some assumptions. Let $f_{i_1 \ldots i_s} = f(z_{1i_1}, \ldots, z_{si_s})$. The problem is:

$$\min(\max) \sum_{i_1=0}^{n_1} \cdots \sum_{i_s=0}^{n_s} f_{i_1 \ldots i_s} p_{i_1 \ldots i_s}$$

subject to

$$\sum_{i_1=0}^{n_1} \cdots \sum_{i_s=0}^{n_s} z_{1i_1}^{\alpha_1} \cdots z_{si_s}^{\alpha_s} p_{i_1 \ldots i_s} = \mu_{\alpha_1 \ldots \alpha_s}$$

for $\alpha_j \geq 0,\ j = 1, \ldots, s;\ \alpha_1 + \cdots \alpha_s \leq m$ and

for $\alpha_j = 0,\ j = 1, \ldots, k-1, k+1, \ldots, s,\ m \leq \alpha_k \leq m_k,\ k = 1, \ldots, s;$

$p_{i_1 \ldots i_s} \geq 0$, all $i_1, \ldots, i_s$.

The unknown variables are the $p_{i_1 \ldots i_s}$, all other quantities are known. This means that, in addition to all moments of total order at most m, the at most $m_k th$ order moments $(m_k \geq m)$ of the kth univariate marginal distribution is also known, $k = 1, \ldots, s$.

The above problems serve for bounding

$$E[f(X_1, \ldots, X_s)]$$

under the given moment information.

The purpose of the paper is to find dual feasible bases of the above problem with some assumption on the convexity of function f, and to give bounds by them on the value of the objective function.

In Section 1, beside the introduction of MDMP, it is shown that the power and binomial moment problems are equivalent and there are some words about the connectionship of discrete moment problem and Lagrange interpolation. Section 2 clarifies the convexity of multivariate discrete functions.

The theorem in Section 3 gives a multivariate Lagrange polinomial and its residual term for a special base point structure, with some assumptions on the function. This theorem is common generalization of the former theorems, furthermore the base of the results in Section 4.

Section 4 contains concrete dual feasible structures as well as algorithms to find them. The first subsection there are theorems yielding dual feasible bases directly. Given a dual feasible basis, on one hand it gives bound for the objective function, on the other hand we may look at it as an initial basis and carry out the dual algorithm of linear programming for the MDMP to obtain the best possible bound. The knowledge of an initial dual feasible basis has two main advantages. First it saves roughly half of the running time of the entire dual algorithm. Second, it improves on the numerical accuracy of the computation that we carry out in connection with our LP's.

The following subsection shows a decomposition method for finding the basis corresponding the best bound among the dual feasible structures of the previous subsection. Using this method, we can optimize on each variable independently, hence we can find the closest bound more efficiently.

Finally, the algorithms in the last subsection provide a larger variety of dual feasible basis structures in the bivariate case. This construction is simple and fast, further, given the dual feasible basis and the corresponding Lagrange polynomial $L_I(z_1, z_2)$, the bound is simply $E[L_I(X_1, X_2)]$ which is not difficult to compute, at least in many cases. So, we can test a large number of dual feasible bases in a relatively short time and then choose the best one, to bound and approximate $E[f(X_1, X_2)]$. This method may produce very good results much faster than the execution of the dual algorithm.

In the last section numerical examples are presented for bounding the expectations of functions of random vectors as well as probabilities of Boolean functions of event sequences. The examples can be classified into three groups. Members of the first one show the basis structures of Subsection 4.1 having the best bounds, then write step-by-step the algorithms of Subsection 4.3 yielding the best approximations, finally show the optimal basis structures founded by the dual algorithm. Certainly, numerical results are attached everywhere. The second group handles further problems by the algorithms of Subsection 4.3, showing the usefulness of them.

Finally, the third group in Subsection 5.2 follows another way. They use one of the structures of Subsection 4.1 as an initial basis and then carry out the dual algorithm to obtain the sharpe bounds. In each examples we optimize one problem several times, with different assumptions on the order of the known moments. The results imply that the sharpe bounds can be improved more efficiently by taking into account some new univariate moments, than just setting higher the maximal total order of the considered moments. Hence, this fact confirms the agvantage of the type of MDMP investigated in this paper against the former MDMP's, where just the total order was limited above ($m = m_k,\ j = 1, \ldots, s$).

Papers:

(1) Nagy, G. and A. Prékopa (2000). On Multivariate Discrete Moment Problems and their Applications to Bounding Expectations and Probabilities. *RUTCOR Research Report* 41-2000.

(2) Nagy G. és Prékopa A. (2000). One-sided Approximation of Multivariate Discrete Functions by Polynomials (in Hungarian). *Alkalmazott Matematikai Lapok* **20**, 195-215.

(3) Nagy G. és Prékopa A. (2001). A Multivariate Utility Function (in Hungarian). *Alkalmazott Matematikai Lapok* **21**, to appear.

Submitted paper:

(4) Nagy, G. and A. Prékopa. On Multivariate Discrete Moment Problems and their Applications to Bounding Expectations and Probabilities. *Mathematics of Operations Research* (submitted).