# Computations with Low Rank Matrices in Logarithmic Space

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# Outline



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# Why low rank matrices?

Matrices that are low rank or can be approximated well by low rank matrices are common in many areas.

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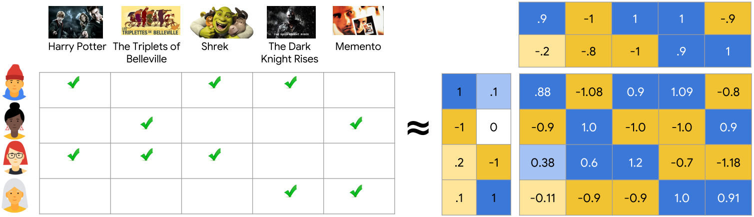
- Example: recommendation systems




Source: Google `developers.google.com/machine-learning/recommendation/collaborative/matrix`

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- $\mathbf{U}^{\mathsf{T}}\mathbf{U} = \mathbf{V}^{\mathsf{T}}\mathbf{V} = \mathbb{1}$
- $\Sigma = \operatorname{diag}(\sigma_1(\mathbf{M}), \sigma_2(\mathbf{M}), \ldots, \sigma_r(\mathbf{M}))$
  where $\sigma_1(\mathbf{M}) \geqslant \sigma_2(\mathbf{M}) \geqslant \ldots \geqslant \sigma_r(\mathbf{M}) > 0$

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- $U^T U = V^T V = \mathbb{1}$
- $\Sigma = \text{diag}(\sigma_1(M), \sigma_2(M), \ldots, \sigma_r(M))$ where $\sigma_1(M) \geqslant \sigma_2(M) \geqslant \ldots \geqslant \sigma_r(M) > 0$
- The $\sigma_i(M)$'s are called the singular values of $M$ and the columns of $U$ and $V$ are called the left and right singular vectors of $M$.

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- The Frobenius norm of $\mathbf{M}$ is

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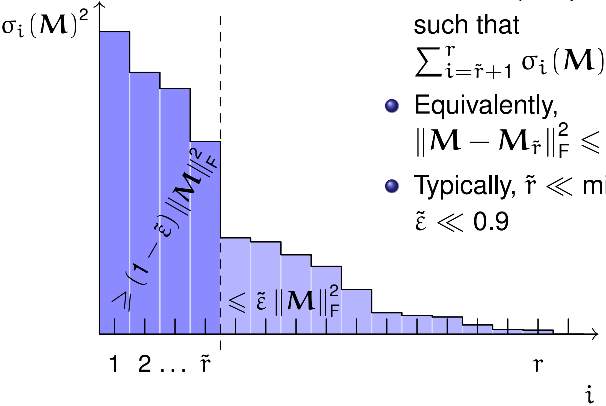
- The Frobenius norm of $M$ is

$$\|M\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} M(i, j)^2} = \sqrt{\sum_{i=1}^{r} \sigma_i(M)^2}$$

# Approximately low rank matrices

If $\mathbf{M}$ can be approximated well by a low rank matrix then its singular values look like this.




- $\exists \tilde{r} \in \mathbb{Z}^+$, $\tilde{r} \leqslant r$ and $0 \leqslant \tilde{\varepsilon} \leqslant 0.9$ such that $\sum_{i=\tilde{r}+1}^{r} \sigma_i(\mathbf{M})^2 \leqslant \tilde{\varepsilon}\|\mathbf{M}\|_F^2$
- Equivalently, $\|\mathbf{M} - \mathbf{M}_{\tilde{r}}\|_F^2 \leqslant \tilde{\varepsilon}\|\mathbf{M}\|_F^2$
- Typically, $\tilde{r} \ll \min\{m, n\}$ and $\tilde{\varepsilon} \ll 0.9$

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$$\Pr[J = j] = \frac{\|M(:, j)\|^2}{\|M\|_F^2}$$

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$$\mathbb{E}\left[\mathbf{c}\mathbf{c}^\mathsf{T}\right] = \sum_{k=1}^{n} \Pr[J = k] \frac{\mathbf{M}(:, k)\,\mathbf{M}^\mathsf{T}(k, :)}{p_k}$$

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$$\mathrm{Var}\left(\mathbf{c}\mathbf{c}^{\mathsf{T}}\right) \stackrel{\text{def}}{=} \mathbb{E}\left[\left\|\mathbf{c}\mathbf{c}^{\mathsf{T}} - \mathbf{M}\mathbf{M}^{\mathsf{T}}\right\|_{\mathsf{F}}^{2}\right]$$

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$$\mathrm{Var}\left(cc^{\mathsf{T}}\right) \stackrel{\text{def}}{=} \mathbb{E}\left[\left\|cc^{\mathsf{T}} - MM^{\mathsf{T}}\right\|_F^2\right] = \sum_{i,j\in[m]} \mathrm{Var}\left(\left(cc^{\mathsf{T}}\right)(i,j)\right)$$

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&= \sum_{i,j \in [m]} \operatorname{Var}\left(\left(\mathbf{c}\mathbf{c}^{\mathsf{T}}\right)(i,j)\right) \\
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&= \sum_{i,j\in[m]} \mathrm{Var}\left(\frac{M(i,J)\,M(j,J)}{p_J}\right) \\
&\leqslant \sum_{i,j\in[m]} \mathbb{E}\left[\frac{M(i,J)^2\,M(j,J)^2}{p_J^2}\right]
\end{aligned}$$

# Variance cont.

$$\mathbb{E}\left[\left\|\mathbf{c}\mathbf{c}^{\mathsf{T}} - \mathbf{M}\mathbf{M}^{\mathsf{T}}\right\|_{\mathsf{F}}^{2}\right] \leqslant \mathbb{E}\left[\sum_{i,j\in[m]} \frac{\mathbf{M}(i,J)^2\,\mathbf{M}(j,J)^2\,\|\mathbf{M}\|_{\mathsf{F}}^4}{\|\mathbf{M}(:,J)\|^4}\right]$$

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$$= \|\mathbf{M}\|_{\mathsf{F}}^{4} \cdot \mathbb{E}\left[\left(\sum_{i=1}^{m} \frac{\mathbf{M}(i,J)^{2}}{\|\mathbf{M}(:,J)\|^{2}}\right)\left(\sum_{j=1}^{m} \frac{\mathbf{M}(j,J)^{2}}{\|\mathbf{M}(:,J)\|^{2}}\right)\right]$$

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$$= \|\mathbf{M}\|_{\mathsf{F}}^4$$

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# Why does $\mathbf{M}$ have to be low rank?

We have the following bound: $\mathbb{E}\left[\left\|\mathbf{C}\mathbf{C}^{\mathsf{T}} - \mathbf{M}\mathbf{M}^{\mathsf{T}}\right\|_{\mathsf{F}}^{2}\right] \leqslant \frac{\|\mathbf{M}\|_{\mathsf{F}}^{4}}{s}$

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$$\frac{1}{s}\left(\sum_{k=1}^{r} \sigma_\mathbf{M}(k)^2\right)^2 < \sum_{k=1}^{r} \sigma_\mathbf{M}(k)^4$$

which holds if $s > r$.

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- Perform random walk on an expander graph to reduce randomness.
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  - We need: good expanders exist for all $N = n^4$.

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# Labeling vertices with columns

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- Let $v_1, v_2, \ldots, v_s$ be the vertices visited by the walk. Then

$$\mathbf{C} = \frac{1}{\sqrt{s}} \left[ \frac{\mathbf{M}(:, f(v_1))}{\sqrt{p_{f(v_1)}}} \quad \frac{\mathbf{M}(:, f(v_2))}{\sqrt{p_{f(v_2)}}} \quad \ldots \quad \frac{\mathbf{M}(:, f(v_s))}{\sqrt{p_{f(v_s)}}} \right].$$

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- Randomness required: $O(\log(n) + s)$.

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# Approximation with expanders

**Theorem**

*For $\mathbf{M} \in \mathbb{R}^{m \times n}$, let $\mathbf{C} \in \mathbb{R}^{m \times s}$ be the matrix we get by the sampling procedure. It holds that*

$$\mathbb{E}\left[\left\|\mathbf{C}\mathbf{C}^{\mathsf{T}} - \mathbf{M}\mathbf{M}^{\mathsf{T}}\right\|_{\mathsf{F}}^{2}\right] = O\left(\frac{\|\mathbf{M}\|_{\mathsf{F}}^{4}}{s}\right).$$

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*This procedure uses* $\mathrm{O}(s + \log(mn))$ *space including the number of random bits.*

- The constant in the big $\mathrm{O}$ notation only depends on the expansion parameter of the graph.
- Only the "work space" counts in the space complexity, reading the input and writing the output does not.

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We can iterate over all the possible random bits.

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- $\left\|\mathbf{A}\mathbf{A}^{\mathsf{T}} - \mathbf{B}\mathbf{B}^{\mathsf{T}}\right\|_{\mathsf{F}}$ is a distance measure between $\mathbf{A}$ and $\mathbf{B}$.

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- $\left\|\mathbf{A}\mathbf{A}^{\mathsf{T}} - \mathbf{B}\mathbf{B}^{\mathsf{T}}\right\|_{\mathsf{F}}$ is a distance measure between $\mathbf{A}$ and $\mathbf{B}$.
- For different random bits we get different $\mathbf{C}\mathbf{C}^{\mathsf{T}}$ matrices. We just have to pick one that is close to many others.

# Outline

# Low rank approximation with singular vectors

## Theorem (known from earlier)

*For any* $\mathbf{A} \in \mathbb{R}^{m \times s}$, $\mathbf{M} \in \mathbb{R}^{m \times n}$, $k \geqslant 1$, *if the columns of* $\mathbf{U}_k \in \mathbb{R}^{m \times k}$ *are the left singular vectors corresponding to the top* $k$ *singular values of* $\mathbf{A}$ *and* $\mathbf{M}_k$ *is the best rank-*$k$ *approximation to* $\mathbf{M}$ *then*

$$\left\|\mathbf{M} - \mathbf{U}_k \mathbf{U}_k^T \mathbf{M}\right\|_F^2 \leqslant \|\mathbf{M} - \mathbf{M}_k\|_F^2 + 2\sqrt{k}\left\|\mathbf{A}\mathbf{A}^T - \mathbf{M}\mathbf{M}^T\right\|_F .$$

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- $\mathbf{U}_k^T\mathbf{U}_k = \mathbb{1}$ while $\mathbf{U}_k\mathbf{U}_k^T \leqslant \mathbb{1}$ is the projection to the range of the best rank-$k$ approximation to $\mathbf{A}$.

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$$\|M - U_k U_k^T M\|_F^2 \leqslant \|M - M_k\|_F^2 + 2\sqrt{k}\,\|AA^T - MM^T\|_F .$$

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- The first term is the best possible error.
- The second term is what we pay for calculating with $A$.

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  - So, $\mathbf{C}\mathbf{C}^{+} = \mathbf{U}\mathbf{U}^{\mathsf{T}} = \mathbf{U}_{\text{rank}(\mathbf{C})}\mathbf{U}^{\mathsf{T}}_{\text{rank}(\mathbf{C})}$.
  - But calculating the Moore-Penrose pseudoinverse is easier than the SVD. It reduces to inverse calculation which can be done in small space.

# Low rank approximation in small space cont.

**Theorem**

*Let the input matrix be* $\mathbf{M} \in \mathbb{R}^{m \times n}$. *Suppose that, for some* $\tilde{r} \in \mathbb{Z}^{+}$, $\tilde{r} \leqslant \mathrm{rank}(\mathbf{M})$ *and* $0 \leqslant \varepsilon \leqslant 0.9$, *it holds that* $\|\mathbf{M} - \mathbf{M}_{\tilde{r}}\|_F \leqslant \varepsilon \|\mathbf{M}\|_F$ *where* $\mathbf{M}_{\tilde{r}}$ *is the best* $\tilde{r}$*-rank approximation of* $\mathbf{M}$.

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$$\left\|\mathbf{M} - \mathbf{C}\mathbf{C}^+\mathbf{M}\right\|_F \leqslant (\varepsilon + \delta) \|\mathbf{M}\|_F$$

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*and the number of columns of* $\mathbf{C}$ *is* $O(\tilde{r})$. *If* $\tilde{r} = O\left(\frac{\log(mn)}{\log(\log(mn))}\right)$ *then the space bound of the algorithm is* $O(\log(mn))$.

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*and the number of columns of $\mathbf{C}$ is $O(\tilde{r})$. If $\tilde{r} = O\left(\frac{\log(mn)}{\log(\log(mn))}\right)$ then the space bound of the algorithm is $O(\log(mn))$.*

Intuitively, $\mathbf{C}$ must "cover" most of the space that is spanned by the singular vectors corresponding to the top $\tilde{r}$ singular values of $\mathbf{M}$.

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Intuitively, $\mathbf{C}$ must “cover” most of the space that is spanned by the singular vectors corresponding to the top $\tilde{r}$ singular values of $\mathbf{M}$. Then $\mathbf{C}\mathbf{C}^+$ is a projector that projects to this “covered” subspace.

# Note on the space bound

- If $\tilde{r} \ll n$ then $O(\tilde{r} \log \tilde{r} + \log(mn))$ space is less than
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  - storing the small $\mathbf{C}^\mathsf{T}\mathbf{C} \in \mathbb{R}^{s \times s}$ matrix, which would require space $s^2 = \Omega(\tilde{r}^2)$.
- We can't even calculate the rank of $\mathbf{M}$ in this space bound, even if $\varepsilon = 0$.

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- If $\tilde{r} \ll n$ then $O(\tilde{r} \log \tilde{r} + \log(mn))$ space is less than
  - storing $\tilde{r}$ column indices, which would require $\tilde{r} \cdot \log n$ space,
  - storing the small $\mathbf{C}^\mathsf{T}\mathbf{C} \in \mathbb{R}^{s \times s}$ matrix, which would require space $s^2 = \Omega(\tilde{r}^2)$.
- We can't even calculate the rank of $\mathbf{M}$ in this space bound, even if $\varepsilon = 0$.
- Matrix multiplication can be done in this space bound so we can also output $\mathbf{C}\mathbf{C}^+\mathbf{M}$ and $\mathbf{C}^+$.

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    - Calculating the SVD
- Open question:
    - What other interesting matrix properties can we use it for?

# Thank you for your attention!

Got comments, questions, ideas?

Email me at peresz@sztaki.hu