Algebraic curves, error correcting codes and post-quantum cryptography

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Mathematical Modelling Seminar
Oct 29, 2019

Outline

1. Communication on noisy channels
2. Error correction codes
3. Algebraic-geometric codes
4. Post-quantum cryptography

Outline

The scheme of communication

- The message can be: *text, picture, sound, measurement data, etc.*
- The communication channel can be: *one way, two way, data transmission, data storage, etc.*

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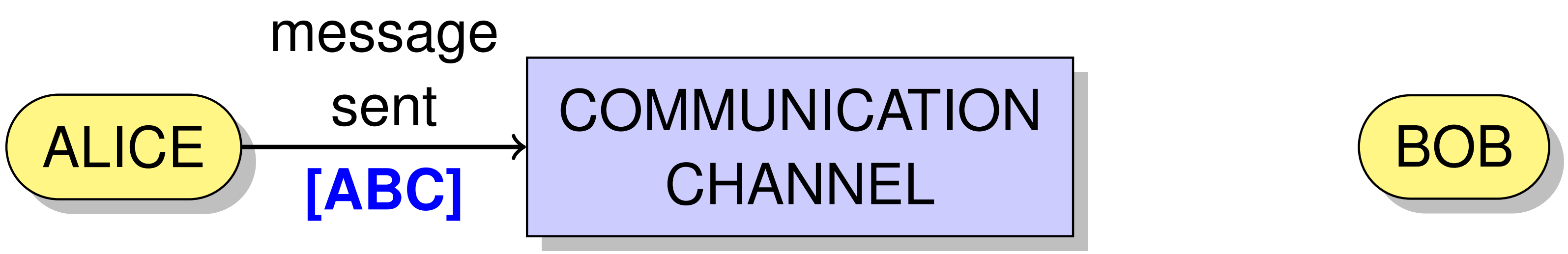

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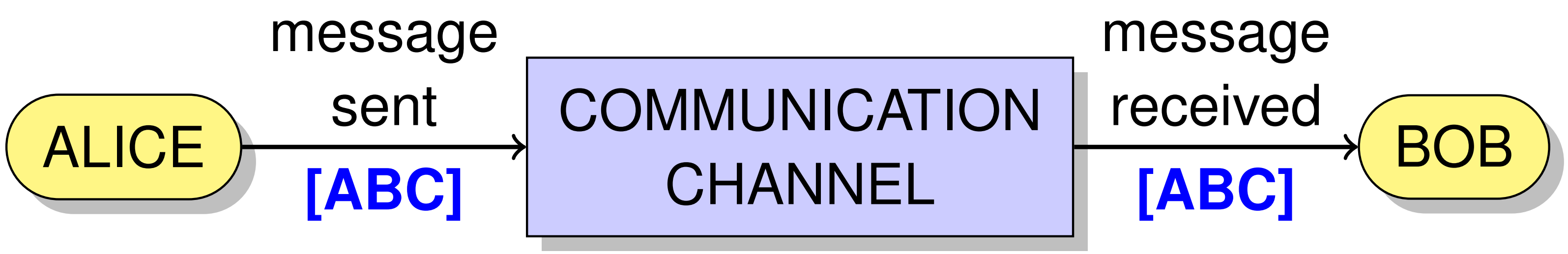

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Digitization, digital reformatting

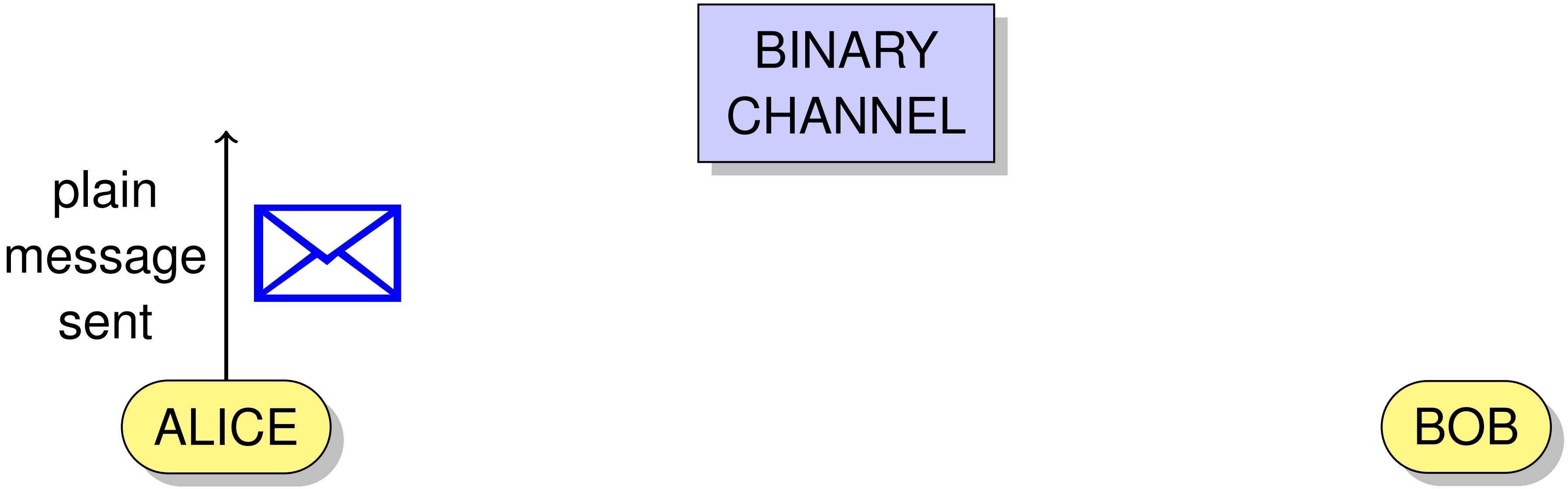
INFORMATION
text, picture, voice, . . .

DIGITALIZATION ⟷

0-1 SEQUENCES
bits, bytes, . . .

John von Neumann
(1903-1957)
Hungarian
mathematician

Digitization, digital reformatting (cont.)



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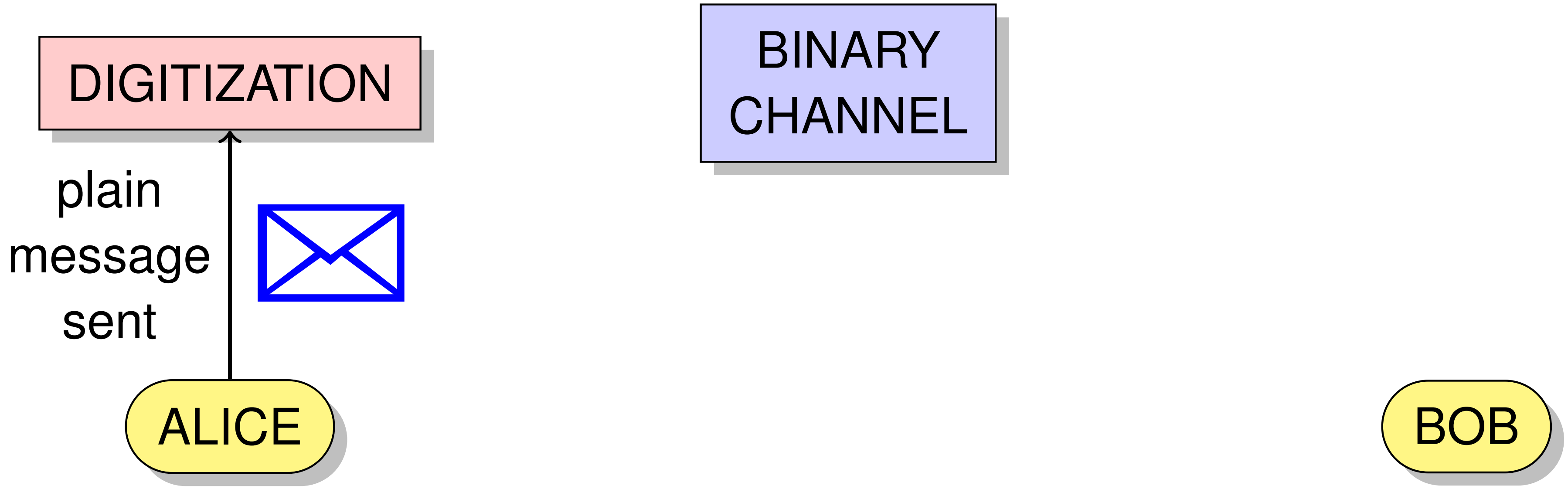

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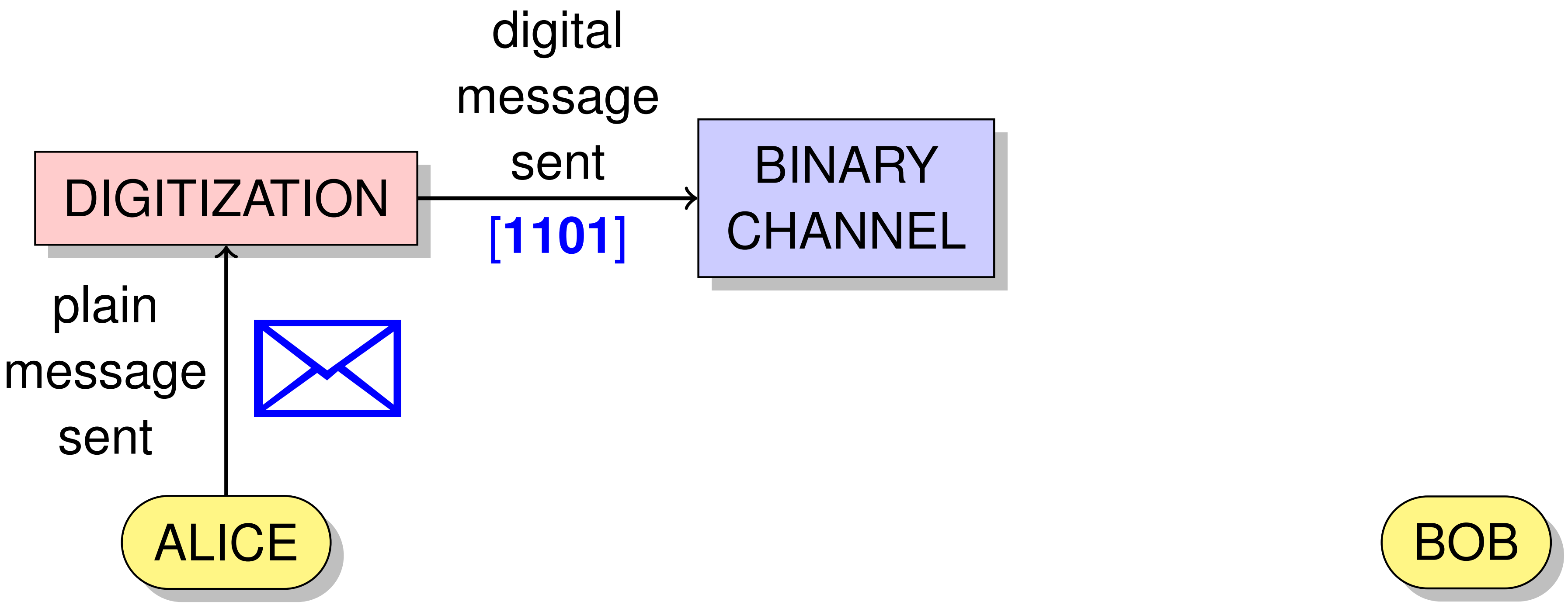

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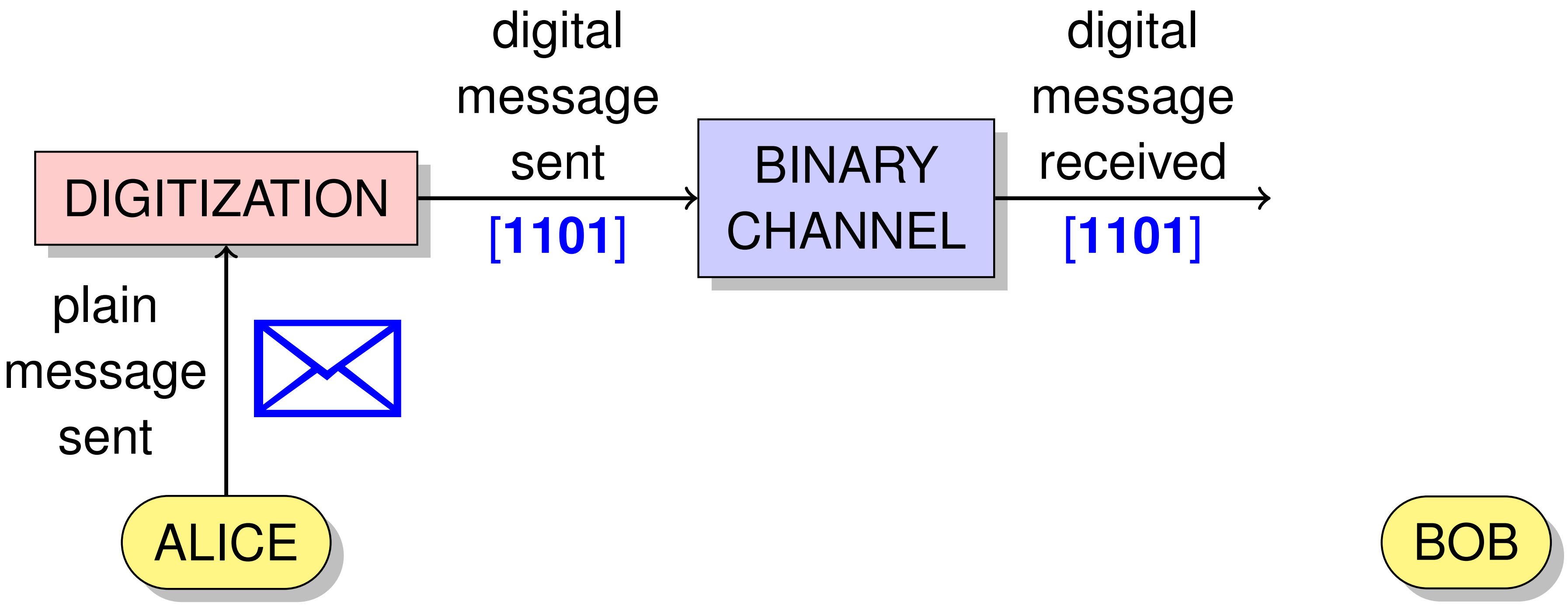

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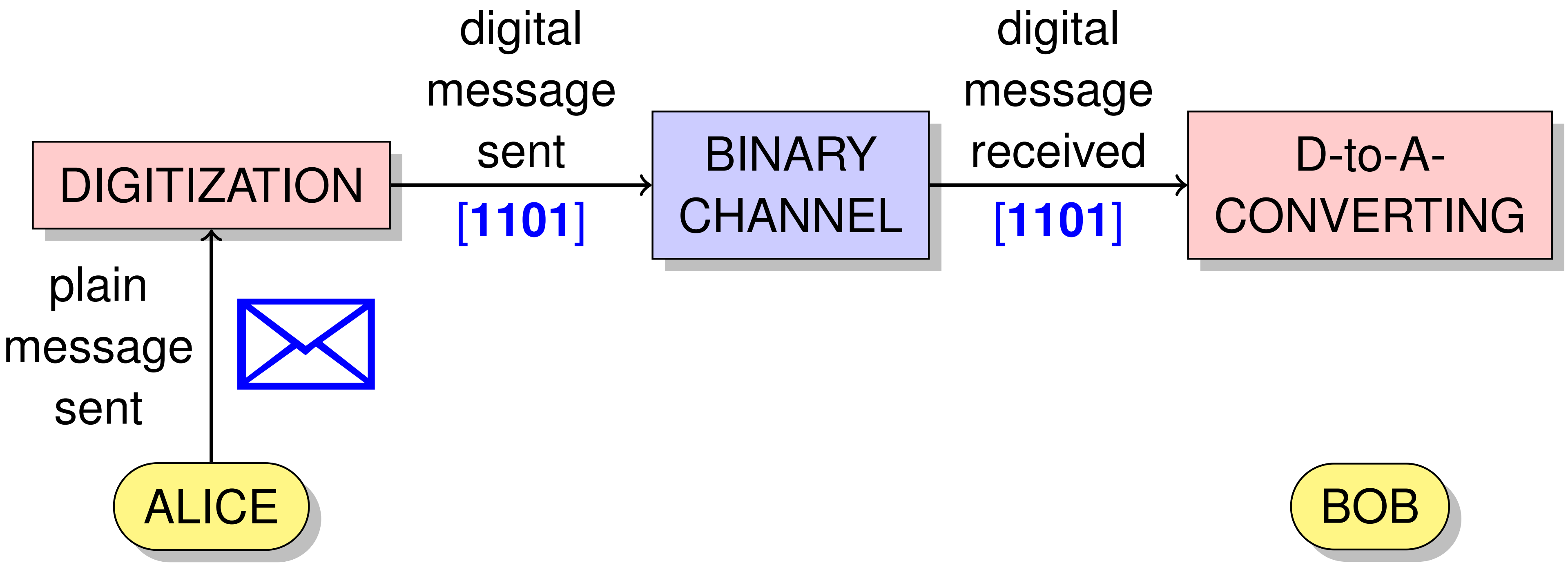

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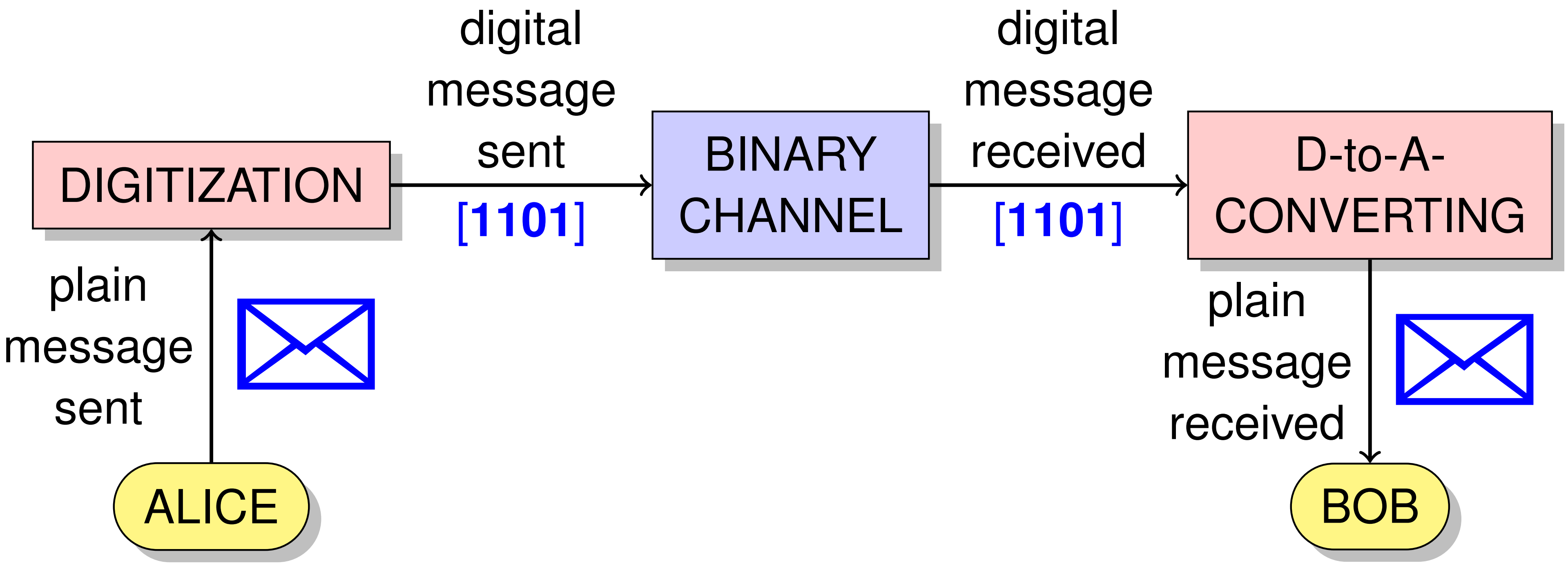

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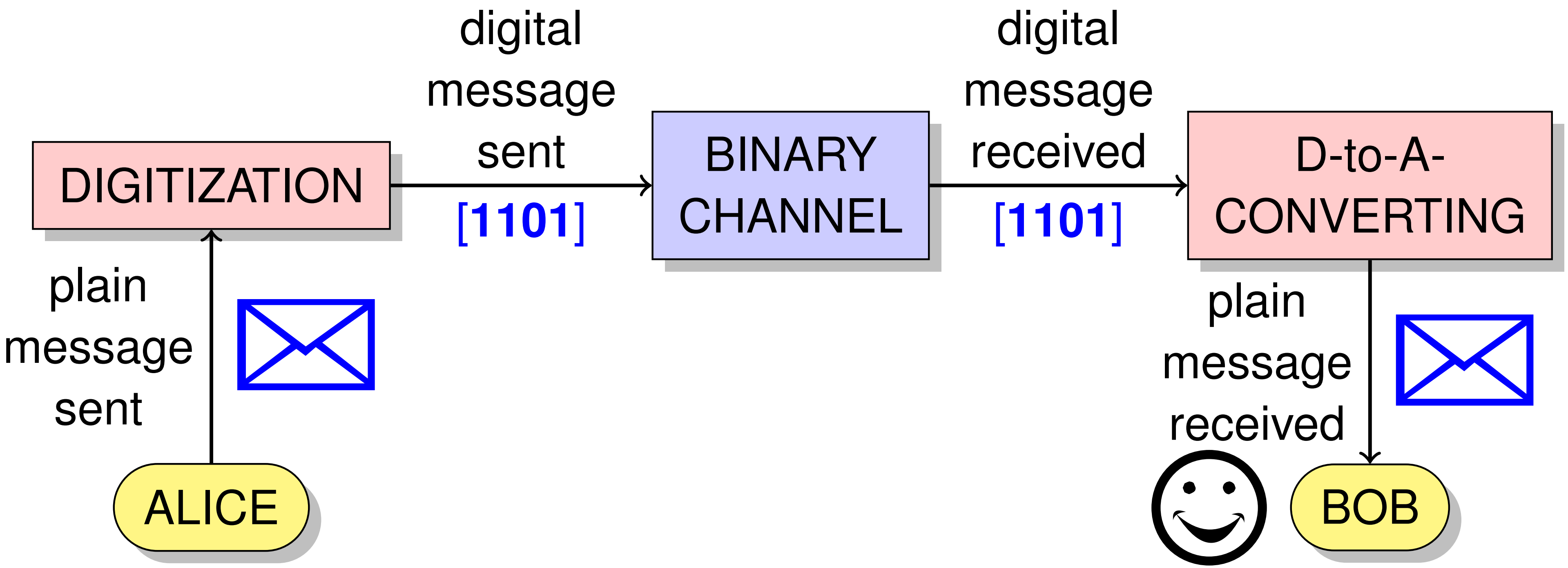

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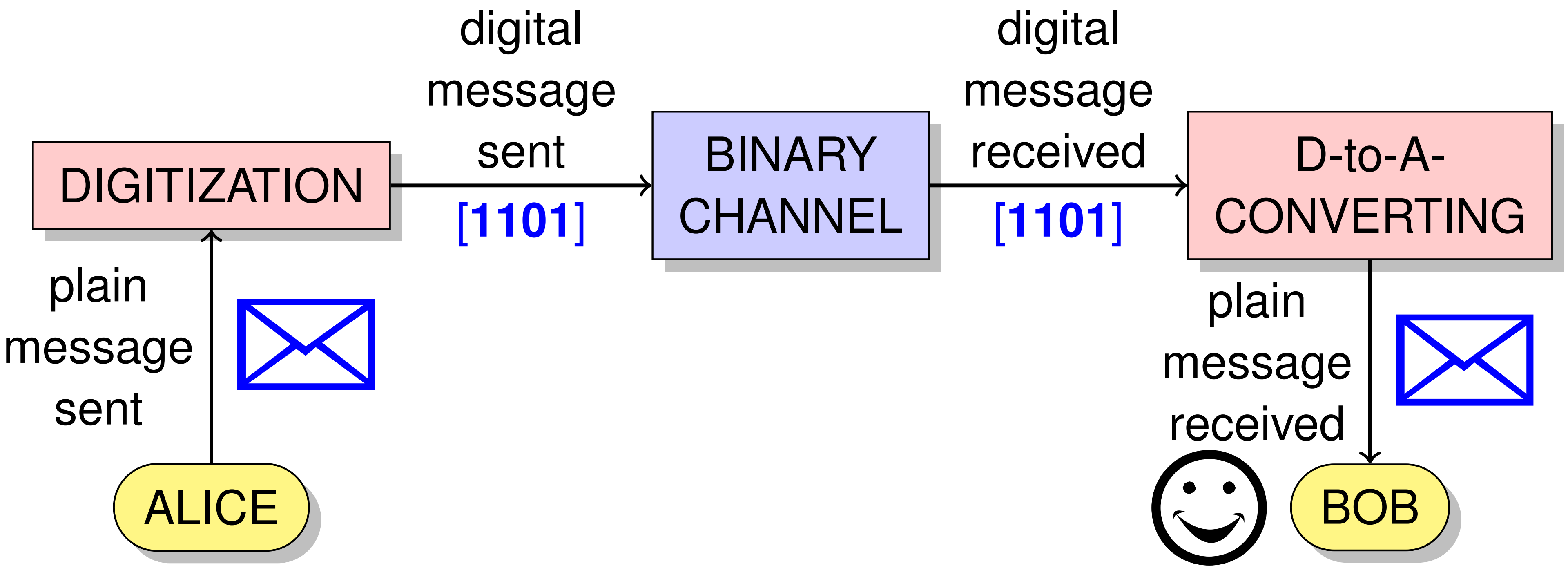

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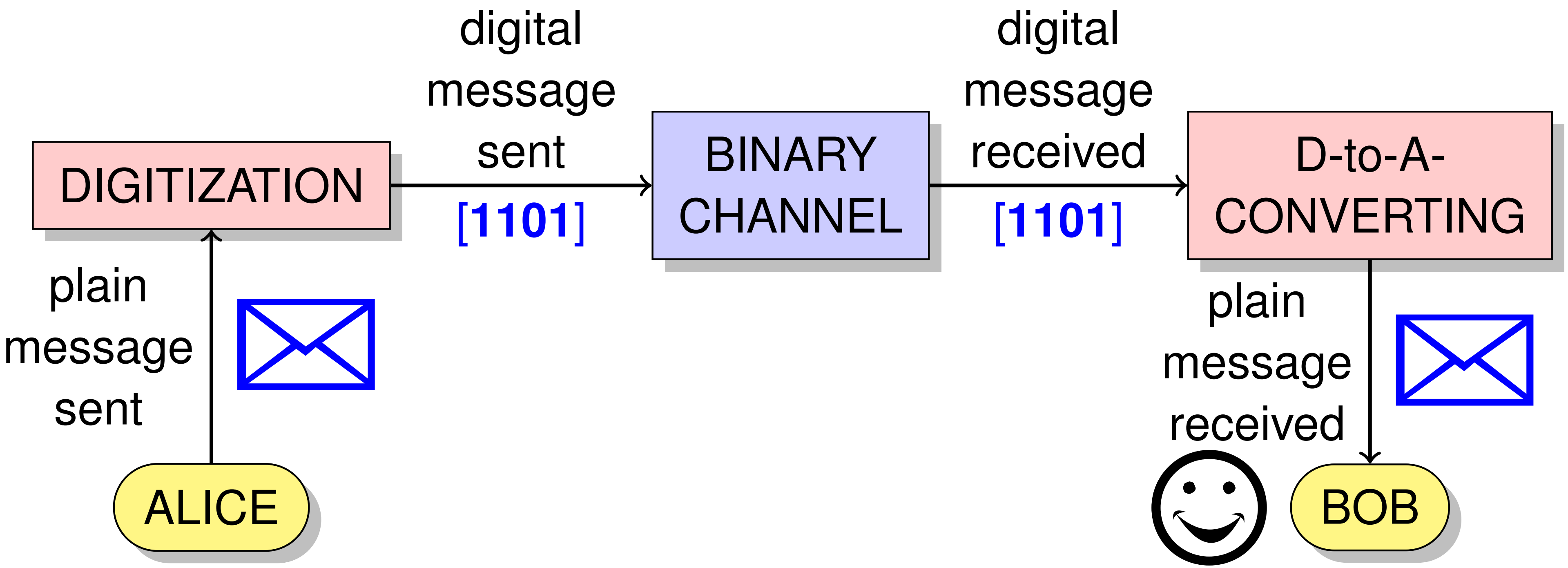

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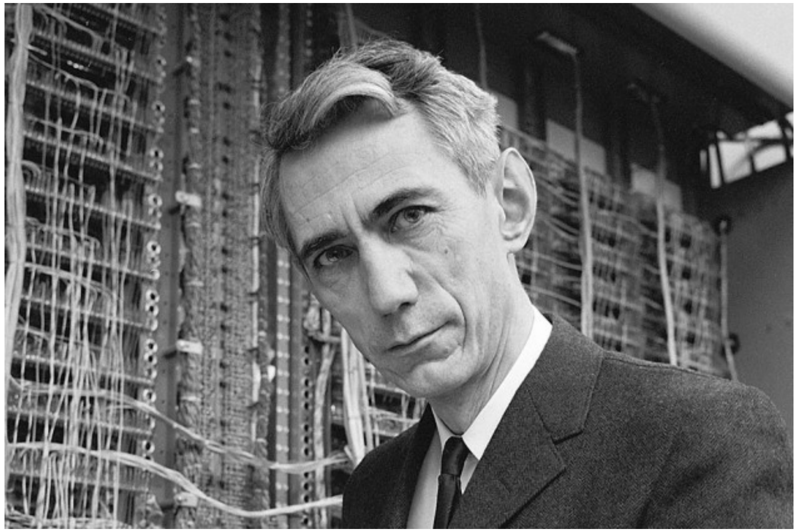
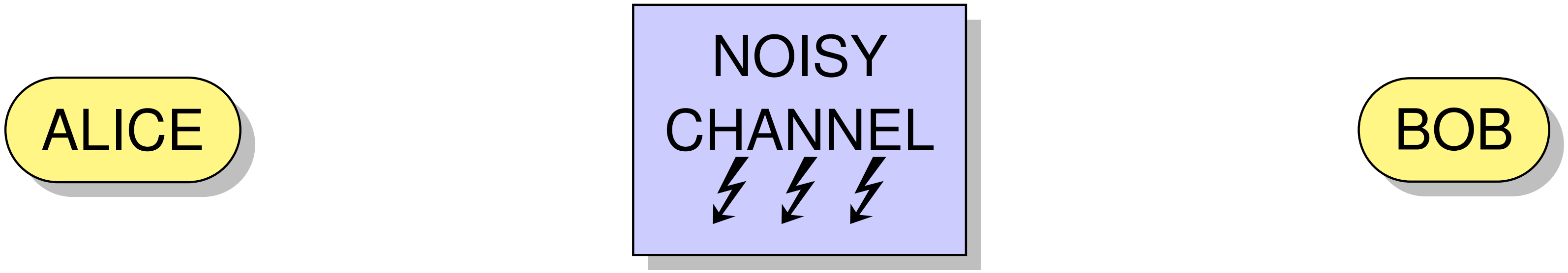
Communication on noisy channel

Claude Shannon
(1916-2001)
US mathematician





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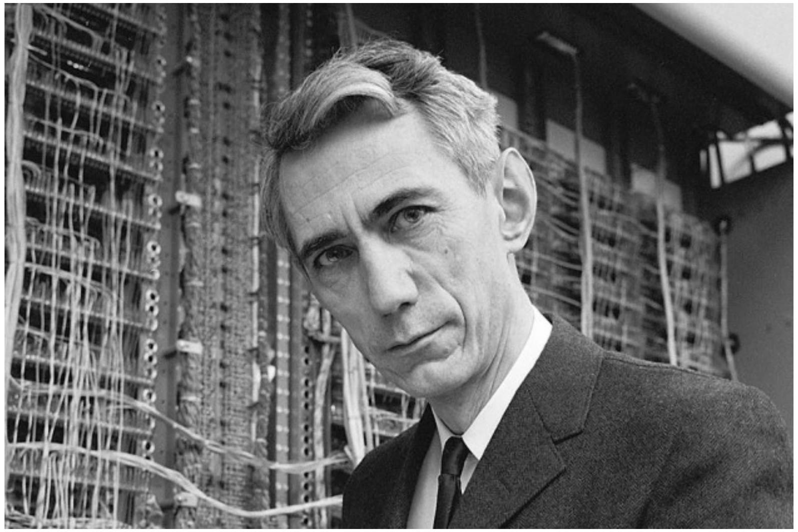

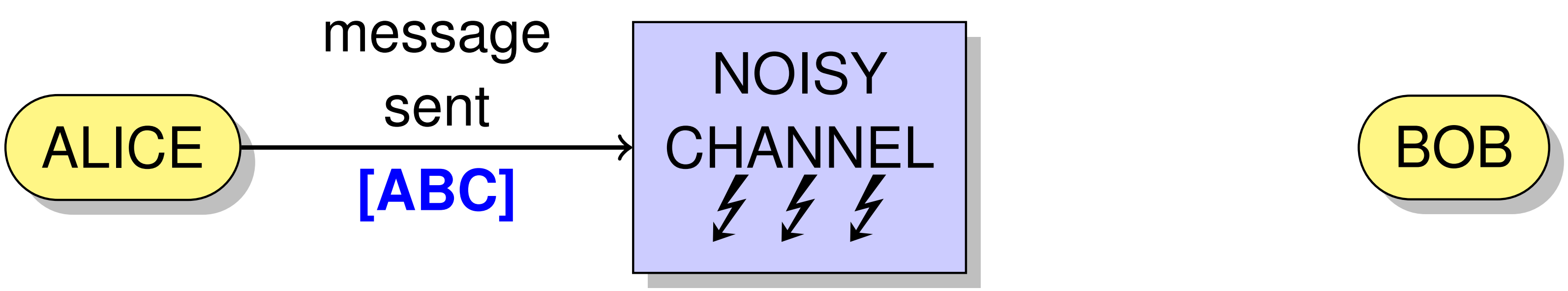

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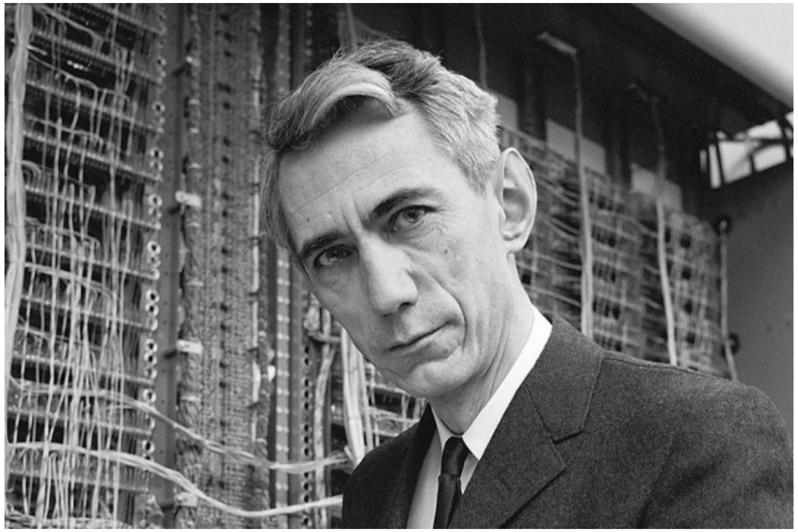

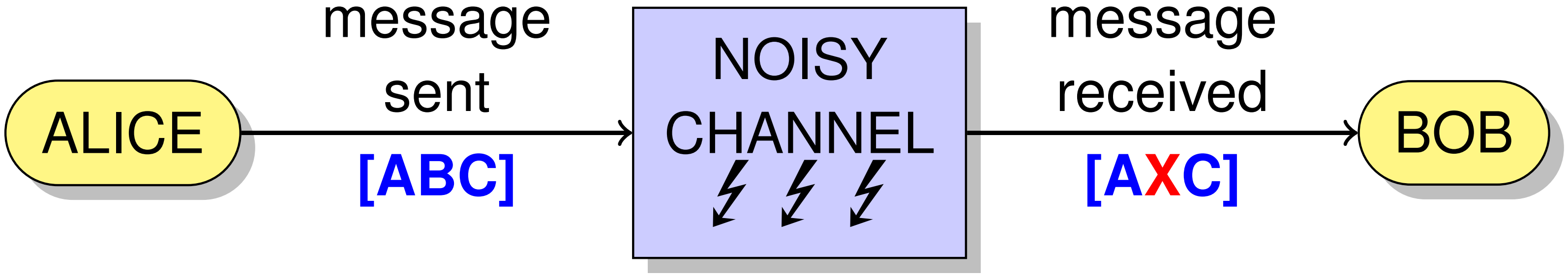

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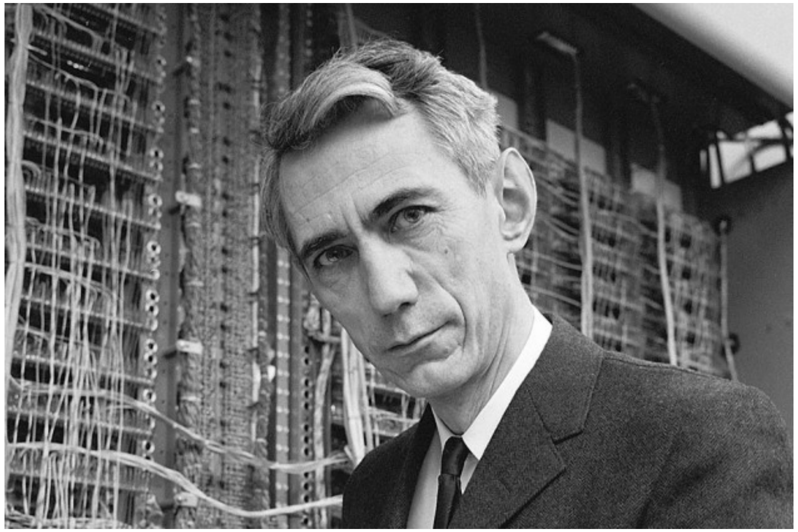

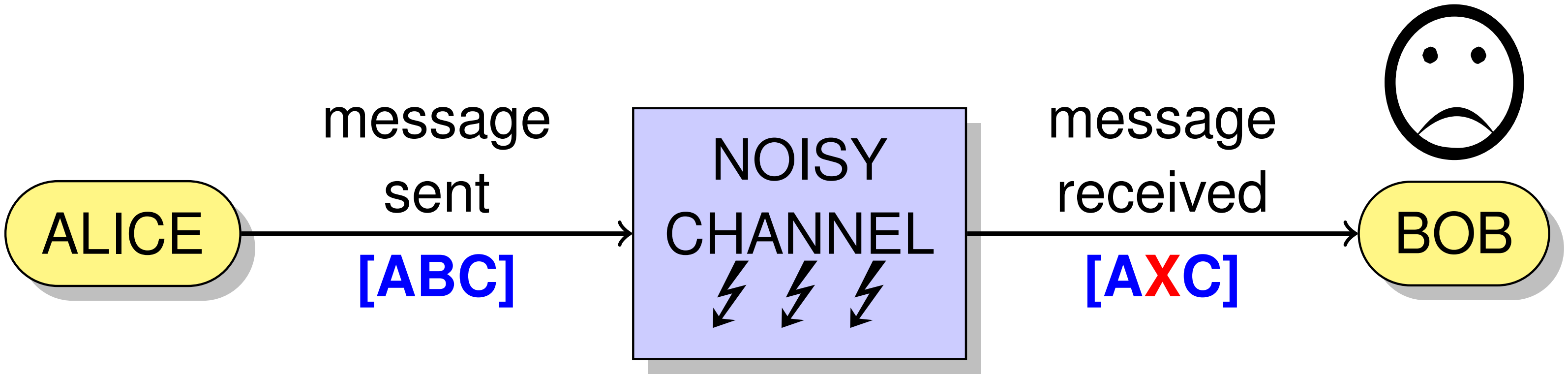

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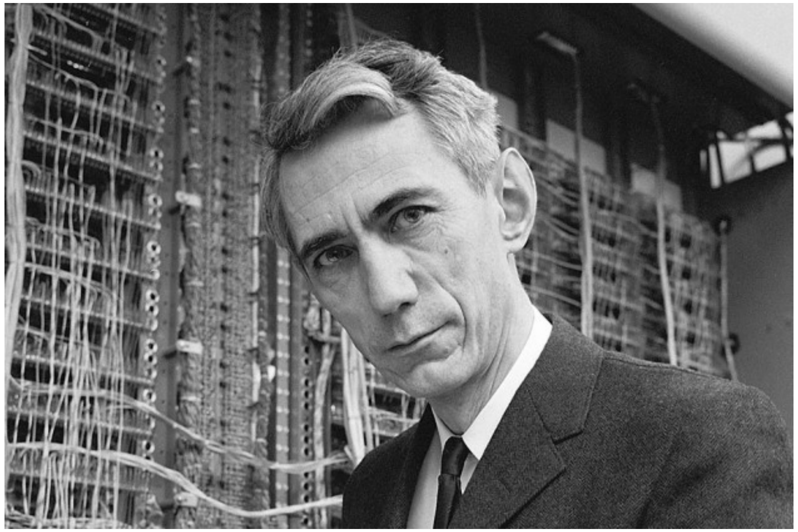

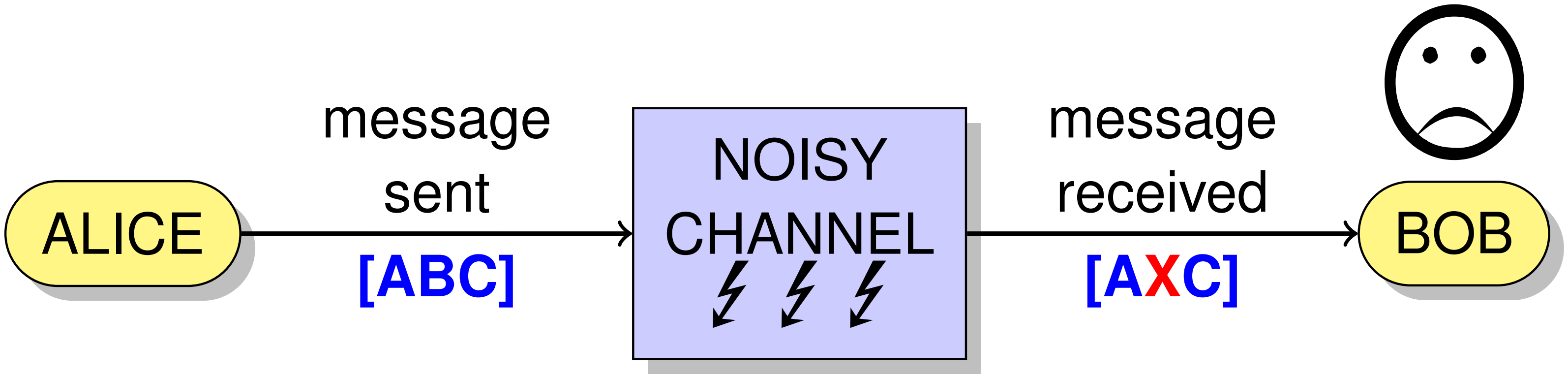

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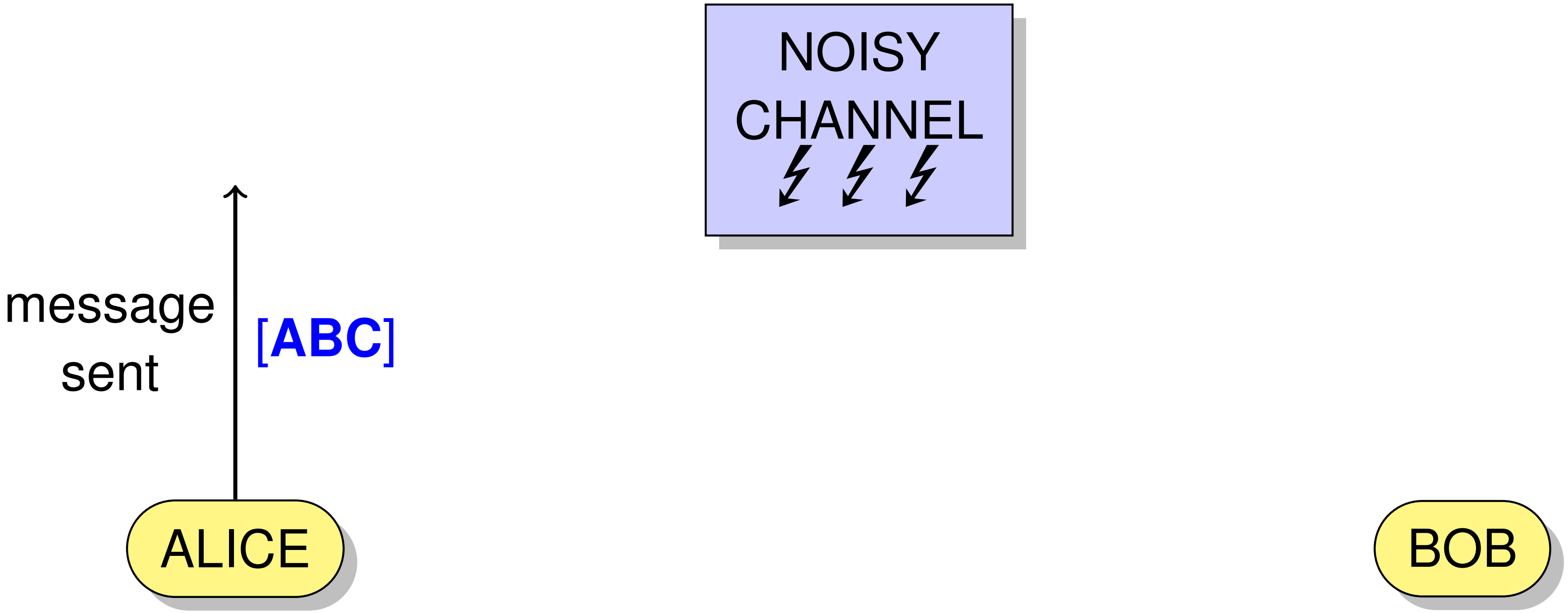
Error correction on noisy communication channel



- Example: **3-fold repetition code:** $0 \mapsto 0|00$, $1 \mapsto 1|11$.
- **Majority/Nearest neighbor/Maximum likelihood** Encoding:
  $0|00, 1|00, 0|10, 0|01 \mapsto 0|00 \mapsto \mathbf{0}$
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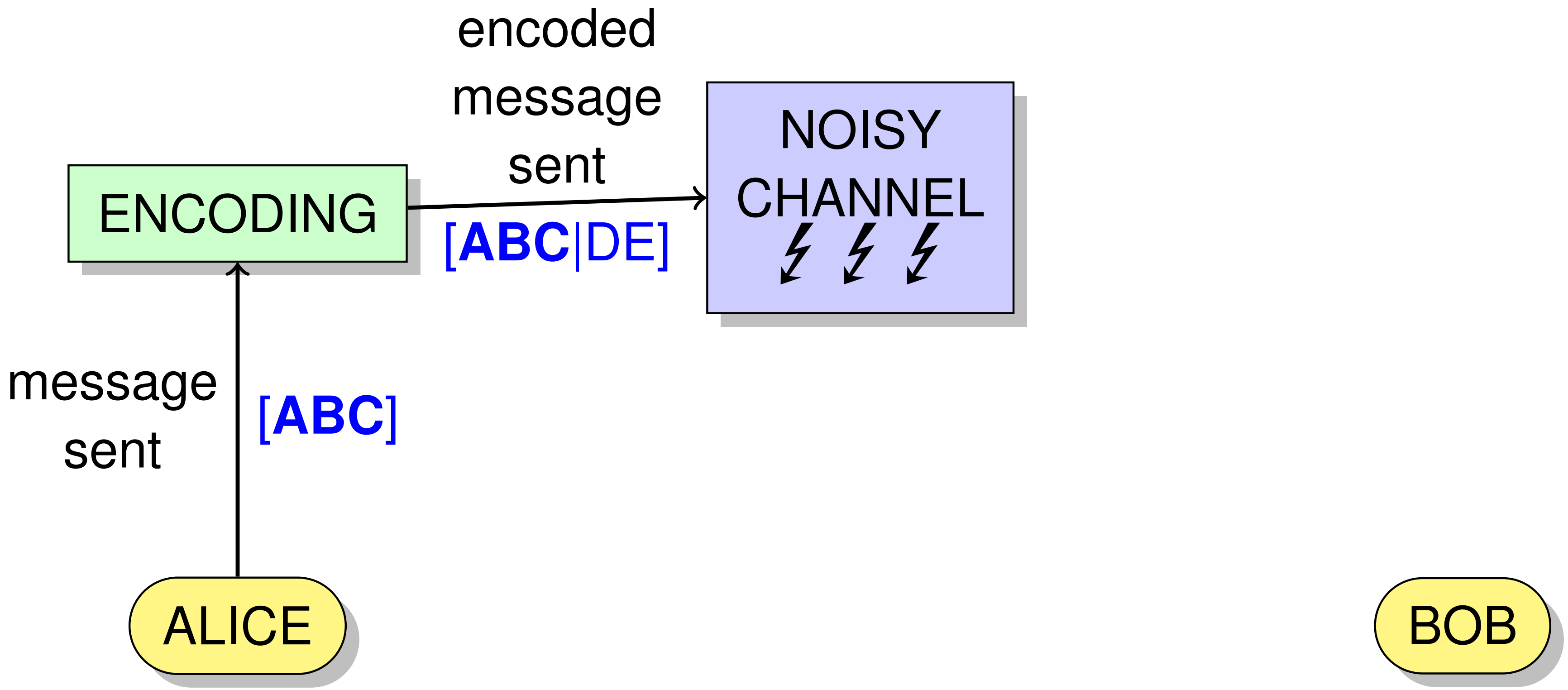

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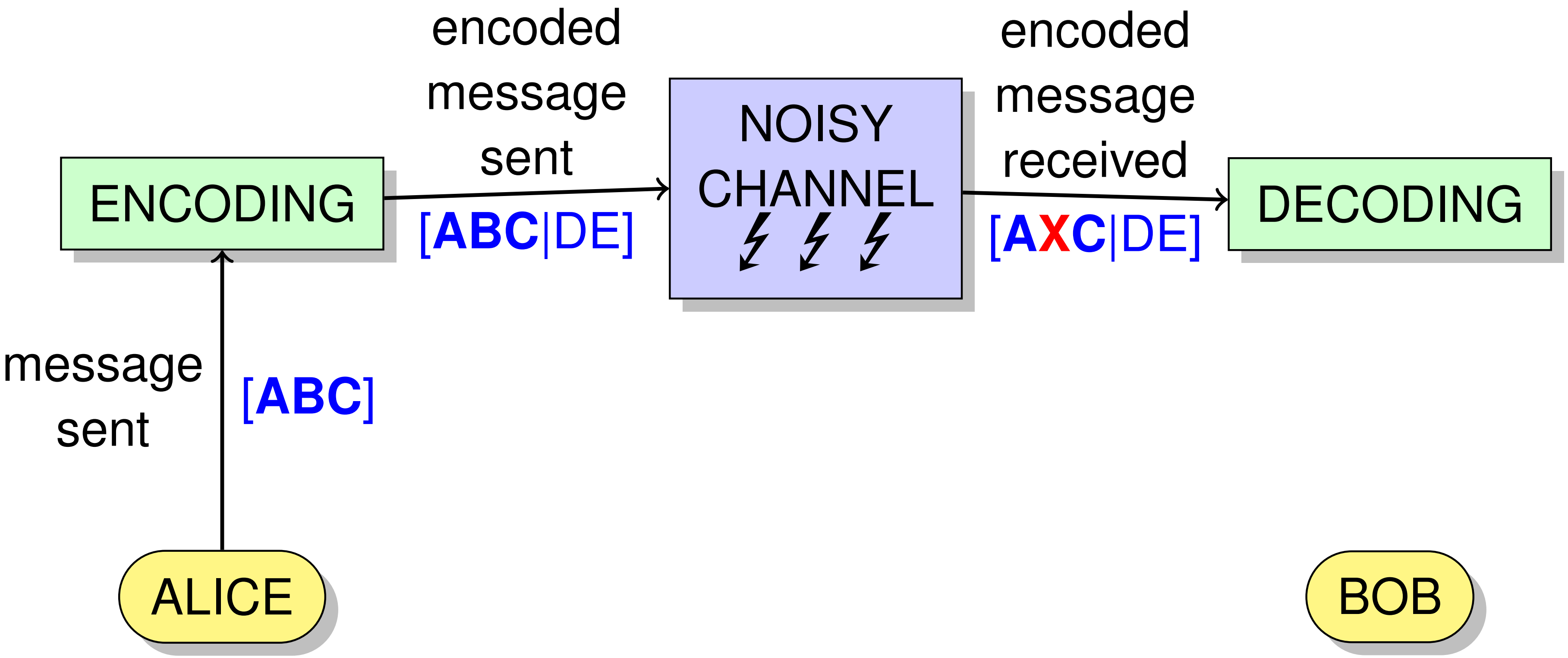

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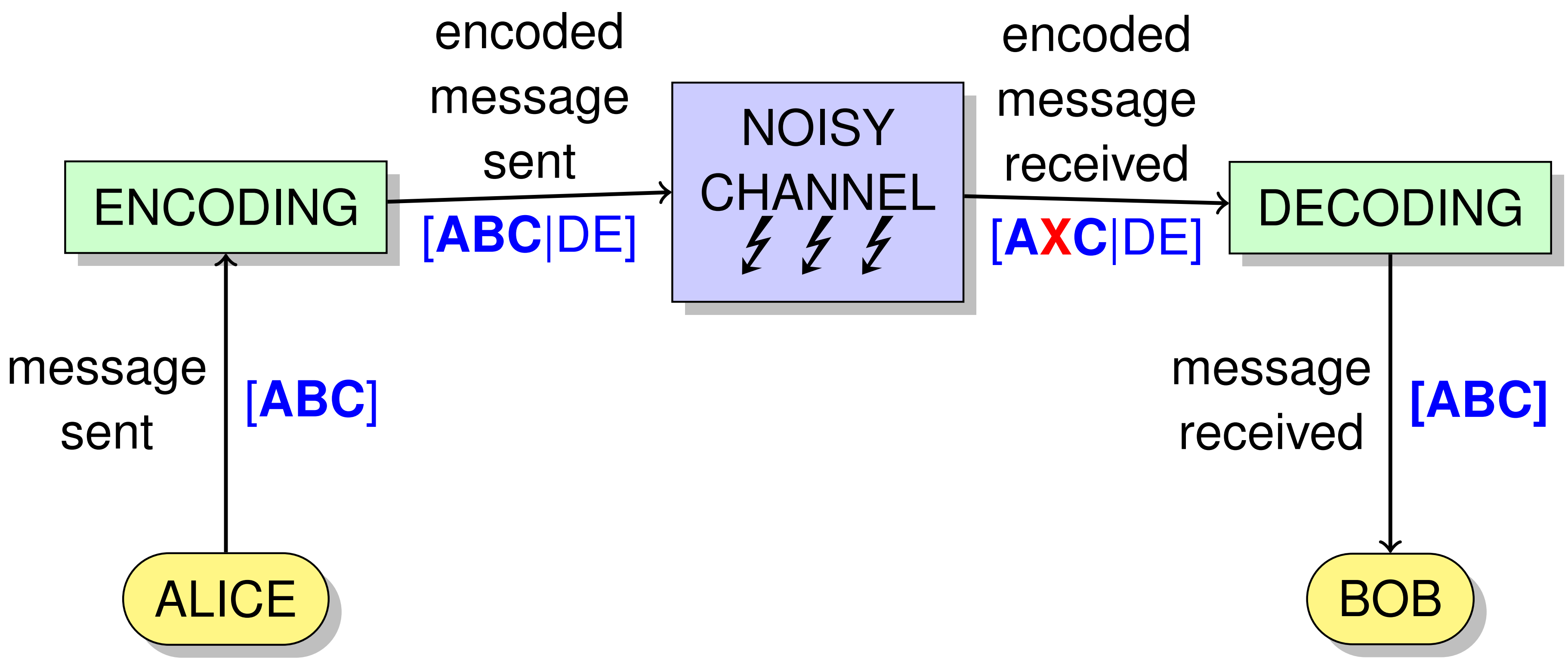

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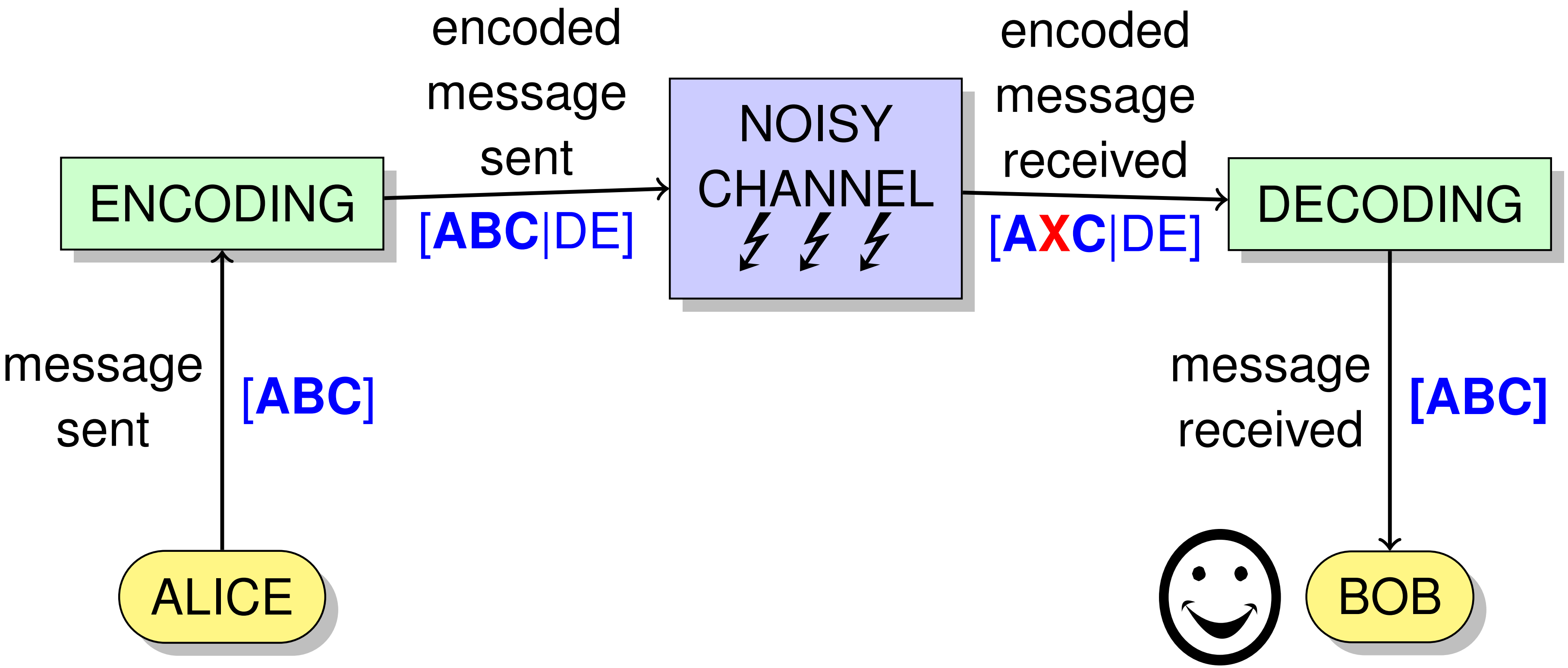

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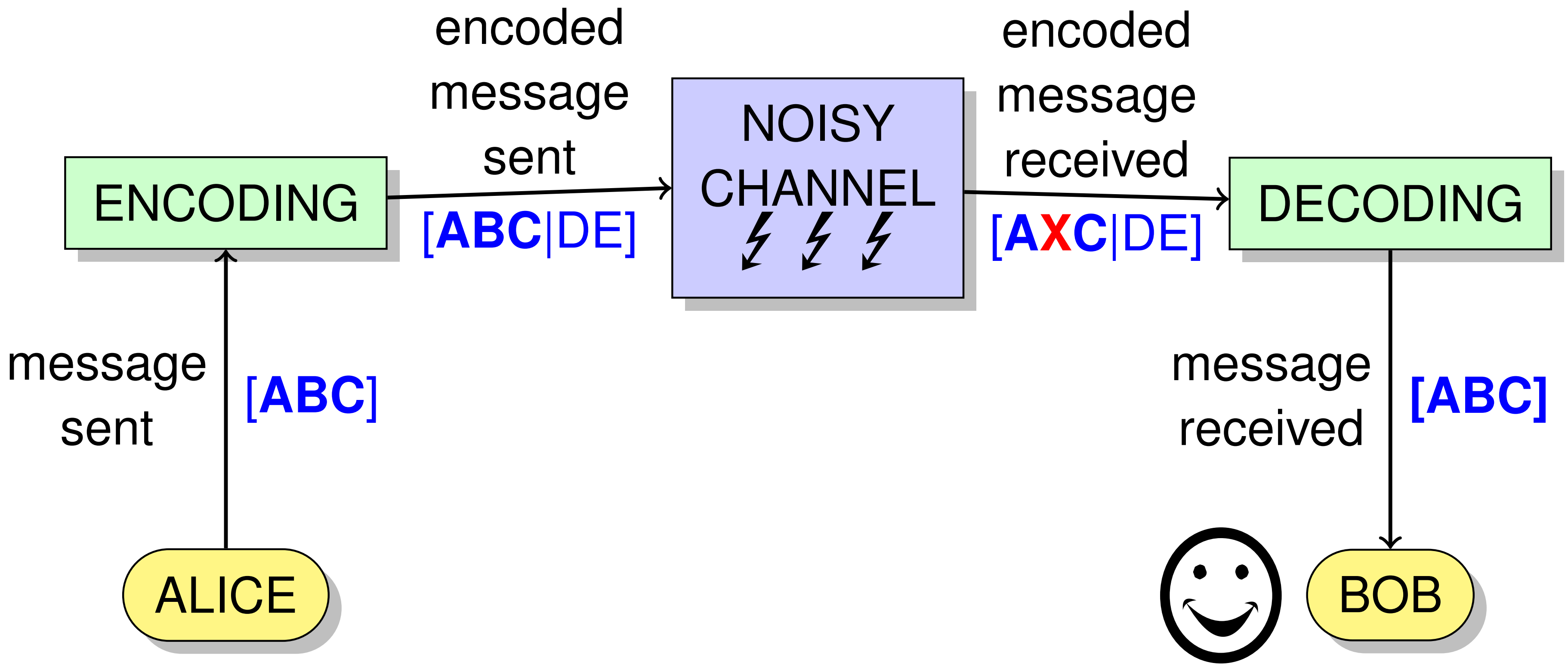

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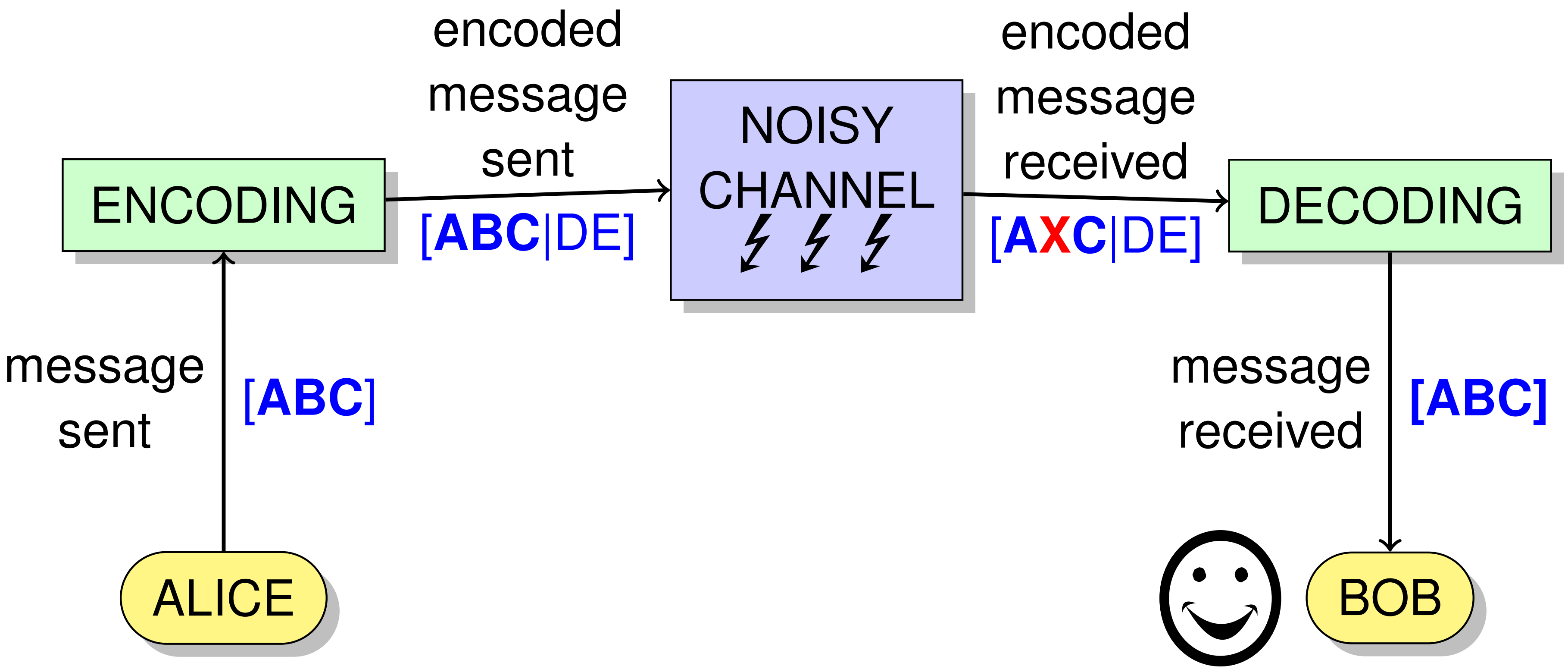

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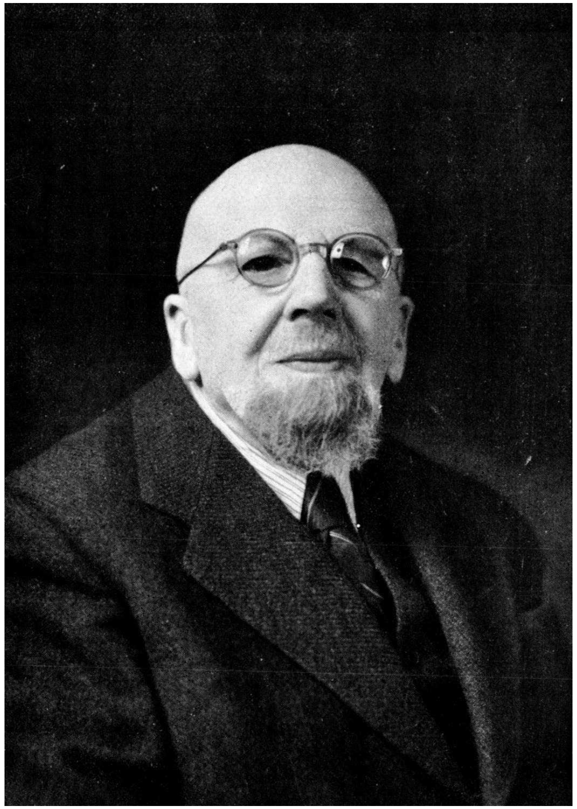
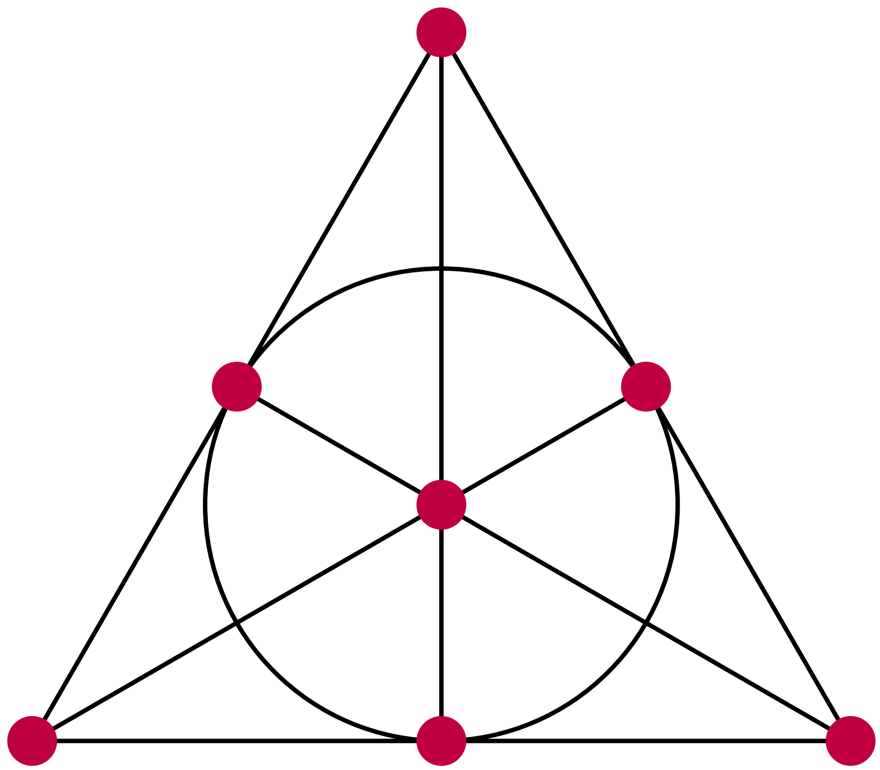


Gino Fano
(1871-1952)
Italian mathematician

Richard Hamming
(1915-1998)
US mathematician

The Fano plane: 7 points, 7 „lines”



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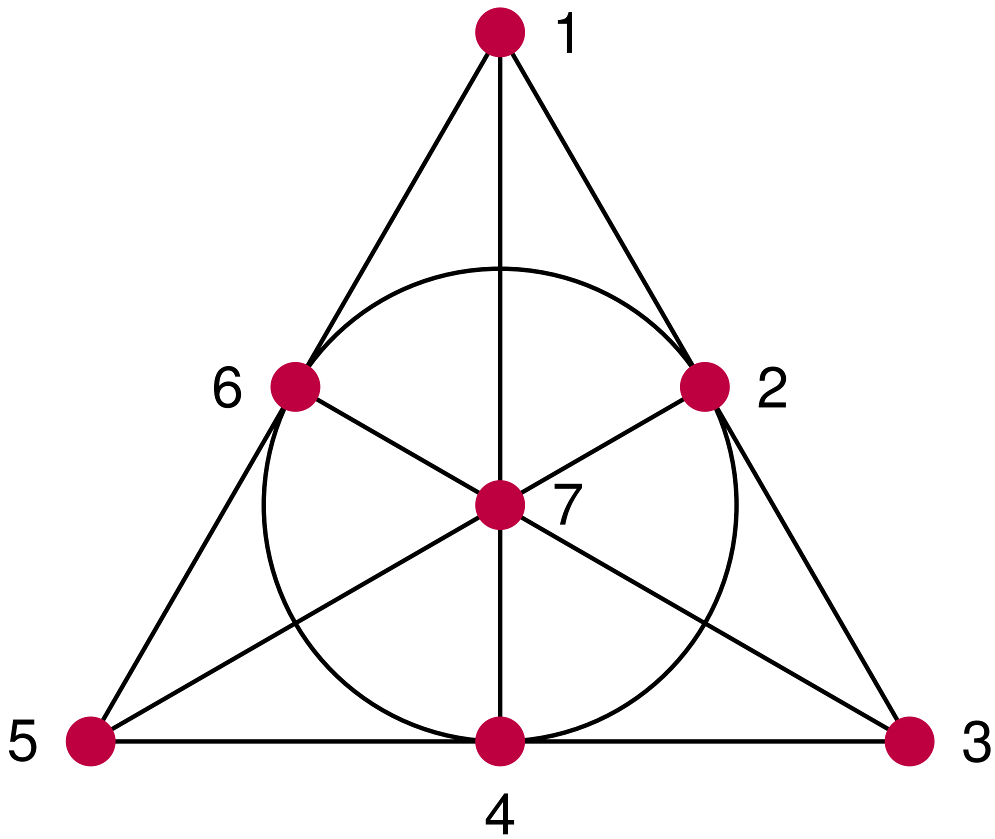

$\{1, 2, 3\}$

$\{3, 4, 5\}$

$\{1, 5, 6\}$

$\{1, 4, 7\}$

$\{2, 5, 7\}$

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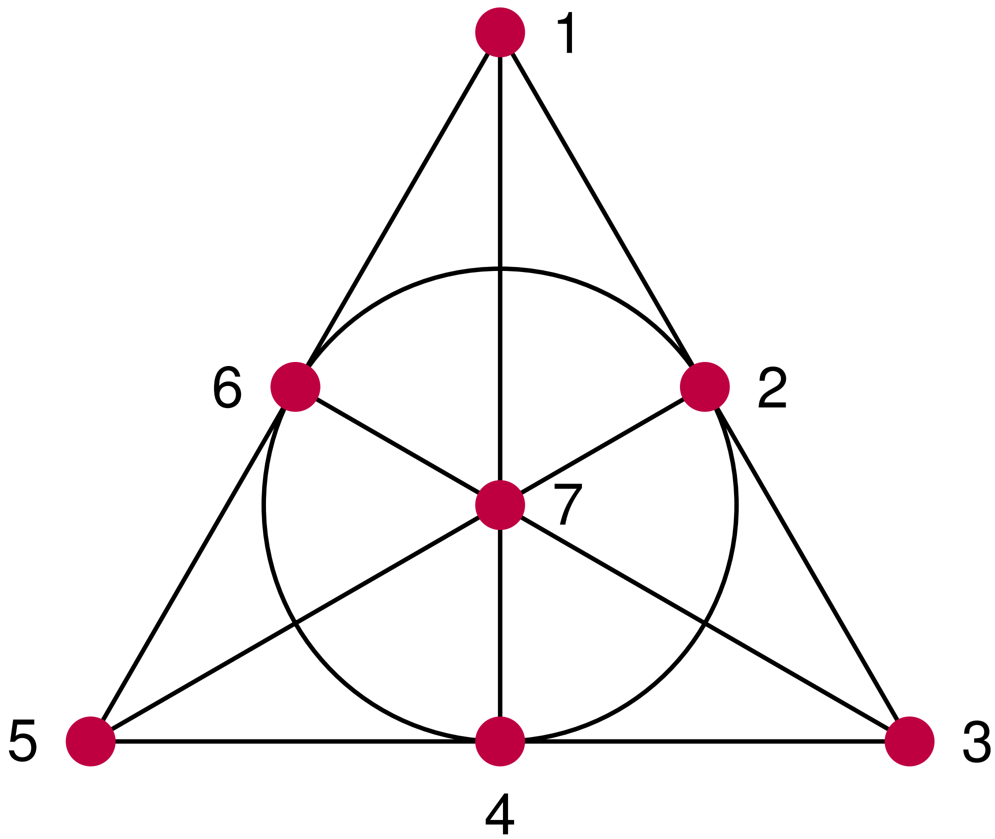

$\{1, 2, 3\}$	**[1 1 1 0 0 0 0]**
$\{3, 4, 5\}$	**[0 0 1 1 1 0 0]**
$\{1, 5, 6\}$	**[1 0 0 0 1 1 0]**
$\{1, 4, 7\}$	**[1 0 0 1 0 0 1]**
$\{2, 5, 7\}$	**[0 1 0 0 1 0 1]**
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$\{2, 4, 6\}$	**[0 1 0 1 0 1 0]**

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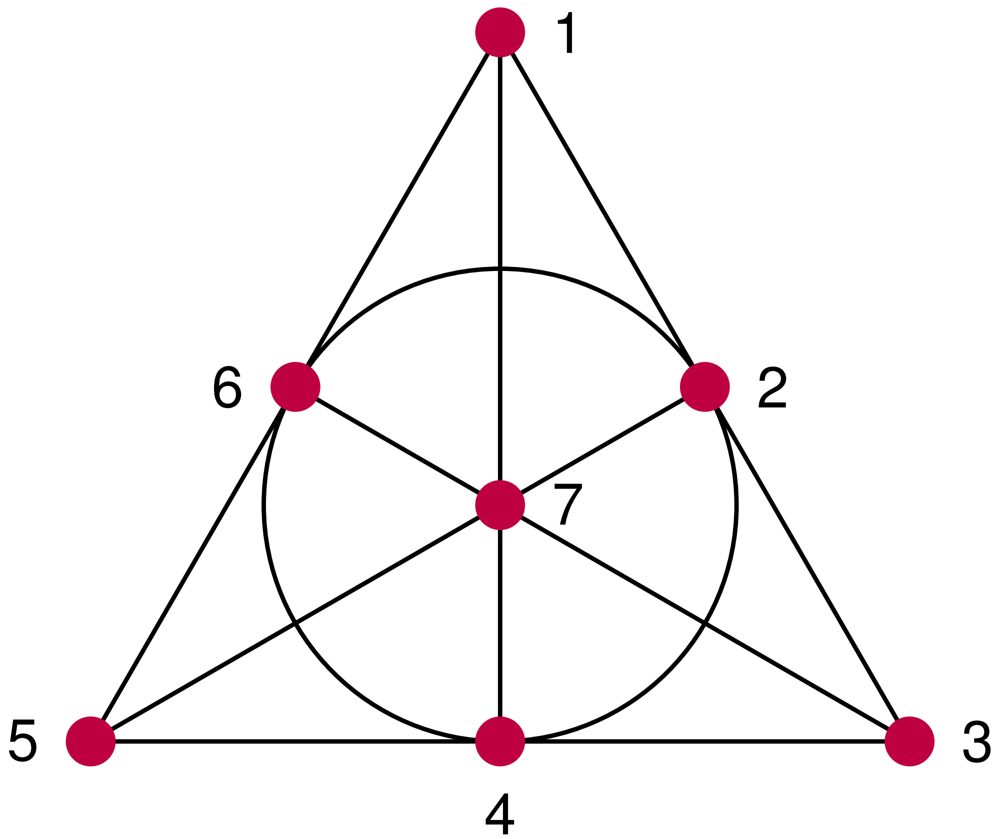


matrix M

$\{1,2,3\}$	[1 1 1 0 0 0 0]
$\{3,4,5\}$	[0 0 1 1 1 0 0]
$\{1,5,6\}$	[1 0 0 0 1 1 0]
$\{1,4,7\}$	[1 0 0 1 0 0 1]
$\{2,5,7\}$	[0 1 0 0 1 0 1]
$\{3,6,7\}$	[0 0 1 0 0 1 1]
$\{2,4,6\}$	[0 1 0 1 0 1 0]

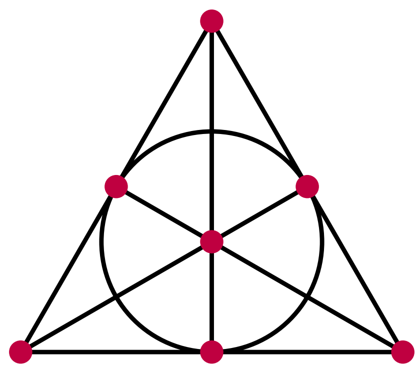
The codewords of the Hamming code

0	0	0	0	0	0	0
1	1	1	0	0	0	0
0	0	1	1	1	0	0
1	0	0	0	1	1	0
1	0	0	1	0	0	1
0	1	0	0	1	0	1
0	0	1	0	0	1	1
0	1	0	1	0	1	0
0	0	0	1	1	1	1
1	1	0	0	0	1	1
0	1	1	1	0	0	1
0	1	1	0	1	1	0
1	0	1	1	0	1	0
1	1	0	1	1	0	0
1	0	1	0	1	0	1
1	1	1	1	1	1	1

- $1 + 7 + 7 + 1 = 16$ bit sequences of length 7.
- YELLOW: All 0's and all 1's.
- RED: The matrix M of the Fano plane
- BLUE: The complementer matrix of M.

Claim

Any two codewords of the Hamming code differ in at least 3 positions.



The Hamming code: computer memory error correction

0	0	0	0	0	0	0	0
14	1	1	1	0	0	0	0
3	0	0	1	1	1	0	0
8	1	0	0	0	1	1	0
9	1	0	0	1	0	0	1
4	0	1	0	0	1	0	1
2	0	0	1	0	0	1	1
5	0	1	0	1	0	1	0
1	0	0	0	1	1	1	1
12	1	1	0	0	0	1	1
7	0	1	1	1	0	0	1
6	0	1	1	0	1	1	0
11	1	0	1	1	0	1	0
13	1	1	0	1	1	0	0
10	1	0	1	0	1	0	1
15	1	1	1	1	1	1	1

Claim 1

The first four bits of the codewords contain all 0/1 vectors of length 4 precisely once.

- GREEN: Information bits
- SÁRGA: Parity check bits

Claim 2

The Hamming code can detect 2 errors and correct 1 error.

Claim 3

The Hamming code is linear over $\mathbb{F}_2$

Outline

Basic concepts

Definition: Error correction codes over a finite alphabet

Let Q be a finite set and n a positive integer. Any subset C of the Cartesian product Q^n is called a code of length n over the alphabet Q.

- The elements of C are called codewords.
- The encoding map is a $1-1$ correspondence between the set of messages $\mathcal{M}$ and C.
- The channel noise is a random map from C to Q^n, uniform on each component.
- The decoding map is a 2-step procedure.
- Step 1 (hard): a function from Q^n to $C \cup \{?\}$.
- Step 2 (easy): the inverse of the encoding function, mapping $C \cup \{?\}$ to $\mathcal{M} \cup \{?\}$.
- Output „?” means uncorrectable transmission error (erasure).

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Hamming distance and nearest neighbor decoding

Definition

- For two tuples $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n)$ the Hamming distance
$$d_H(\boldsymbol{x}, \boldsymbol{y}) = |\{i \mid x_i \neq y_i\}|$$
is the number of position where $\boldsymbol{x}, \boldsymbol{y}$ differ.
- The minimum distance of the code $C \subseteq Q^n$ is
$$d(C) = \min\{d_H(\boldsymbol{x}, \boldsymbol{y}) \mid \boldsymbol{x}, \boldsymbol{y} \in C, \boldsymbol{x} \neq \boldsymbol{y}\}.$$
- The map $D : Q^n \to C \cup \{?\}$ is a nearest neighbor decoding, if $D(\boldsymbol{x})$ is one of the nearest codewords to $\boldsymbol{x}$ w.r.t. the Hamming distance.

Theorem

The Hamming distance defines a metric in the geometric sense. Any code can detect $d(C) - 1$ and correct $\lfloor \frac{d(C)-1}{2} \rfloor$ errors per codewords.

Codes with good parameters

Definition: Information rate, error correction rate

- The number of information symbols per codeword is approx. $\log |C|$.
- The information rate of C is $R = \frac{\log |C|}{n}$.
- The error correction rate of C is $\delta = \frac{d(C)}{n}$.

Remarks.

- Mathematicians look for codes with high information and error correcting rates.
- The Singleton bound restricts $R + \delta \leq 1 + \frac{1}{n}$.
- Engineers compare codes using their BER curves.
- In fact, the package error ratio p^* of the code is a function of bit error ratio p of the channel.
- For the Hamming code of length 7, we have

$$p^* = 1 - (1-p)^7 - 7p(1-p)^6 \approx 21p^2 + o(3).$$

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Good news on binary linear codes

Nothing is *easier* than to produce good binary linear codes:

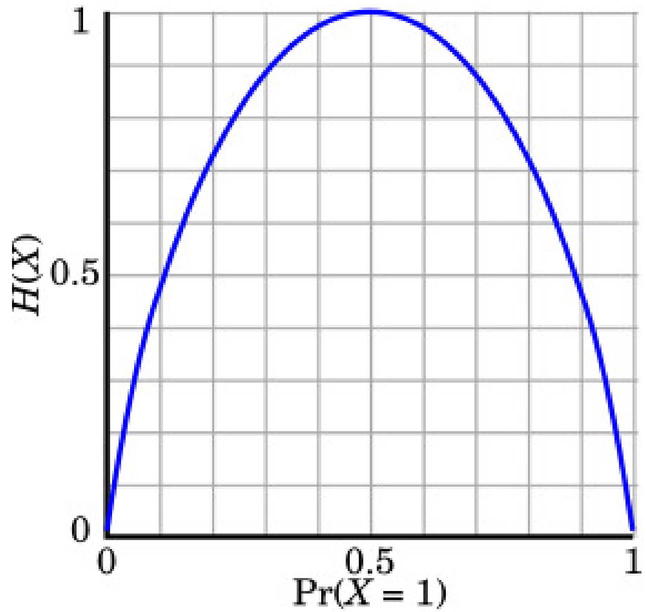
Theorem (Shannon's Noisy-Channel Coding Theorem 1948)

Define the binary entropy function

$$H(p) = -p\log_2 p - (1-p)\log_2(1-p).$$

Fix constants $0 < p < 1/2$, $0 < R < 1 - H(p)$ *and* $\varepsilon > 0$. *Then:*

- *for n sufficiently big,*
- *the „**random**" binary linear code*
- *of length n and rate R satisfies*
- $p^* \leq \varepsilon$.



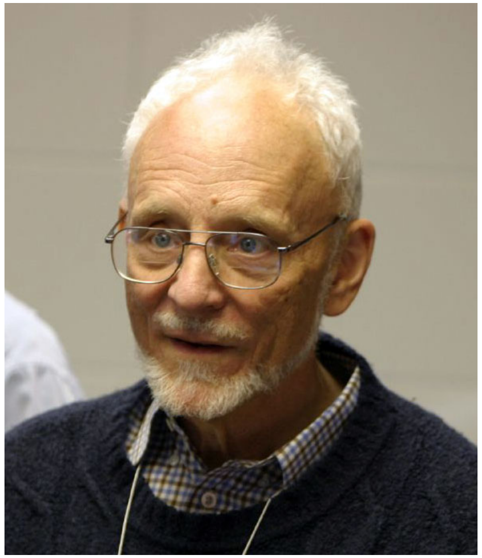
Bad news on binary linear codes

It is almost *hopeless* to make use of random binary linear codes:

Theorem (Berlekamp, McEliece, van Tilborg, 1978)

The following problem is ***NP-complete:*** *Given a $k \times n$ binary matrix A, a binary vector $\mathbf{y}$ and an integer $w > 0$. Let C be the subspace spanned by the rows of A. Is there an element $\mathbf{c} \in C$ such that $d_H(\mathbf{c}, \mathbf{y}) \leq w$?*

Robert McEliece
(1942-2019)



Elwyn Berlekamp
(1940-2019)

Henk van Tilborg
(1947-)

Aspects of decoding of linear codes

- Let $C \leq \mathbb{F}_2^n$ be a binary linear code of length n and dimension k.
- Let $\boldsymbol{x} \in C$ be the sent codeword and $\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{e}$ the received word with error $\boldsymbol{e}$.
- With **hard-decision decoding** we have $\boldsymbol{y}, \boldsymbol{e} \in \mathbb{F}_2^n$.
- Efficient decoding algorithms when C has some algebraic and/or combinatorial structure: *Golay code, Reed-Solomon code, LDPC codes.*
- With **soft-decision decoding** we have $\boldsymbol{y} \in [0,1]^n$.
- Easiest example for the *repetition code:*

$$\text{decode to} \begin{cases} \mathbf{1} & \text{if} \quad \sum y_i \geq 0.5 \\ \mathbf{0} & \text{if} \quad \sum y_i < 0.5 \end{cases}$$

- Further examples: *Viterbi, turbo code.*

Outline

Linear codes over finite fields

Definition: Finite field $\mathbb{F}_q$ of order q

- Let p be a prime, n a positive integer and $q = p^n$ a prime power.
- There is a (unique) algebraic structure $\mathbb{F}_q$ of order q, endowed with four operations

$$x + y, \quad x - y, \quad x \cdot y, \quad x/y.$$

- The operation satisfy the usual arithmetic axioms.

Definition: Linear code

Let C be a linear subspace of $\mathbb{F}_q^n$. Then C is a linear code of length n over the alphabet $\mathbb{F}_q$.

- If $k = \dim C$ then $|C| = q^k$ and $R = k/n$.
- C is may be given by generators (generator matrix) or by a system of linear equations (parity check matrix).
- Encoding function is **matrix calculus:** fast and easy $\mathbb{F}_q^k \to \mathbb{F}_q^n$.

Generalized Reed-Solomon codes

- Let q be a prime power, n, k nonnegative integers such that $1 \le k \le n \le q$.
- Let $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_n\}$ be n distinct elements of $\mathbb{F}_q$, $\boldsymbol{v} = (v_1, \ldots, v_n)$ a nonzero vector of $\mathbb{F}_q^n$ with $v_i \neq 0$ for all i.

Definition

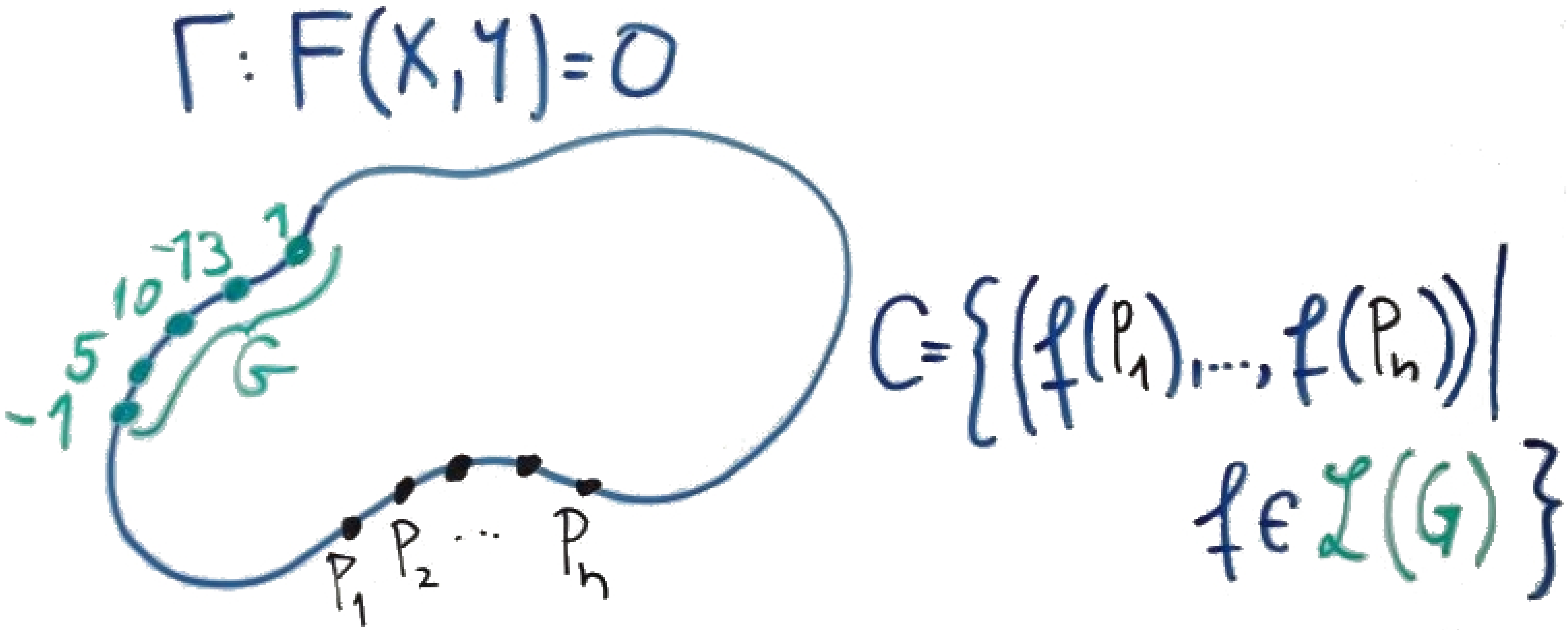
The Generalized Reed-Solomon code, denoted by $\mathbf{GRS}_k(\alpha, \boldsymbol{v})$ consists of all vectors

$$(v_1 f(\alpha_1), v_2 f(\alpha_2), \ldots, v_n f(\alpha_n)),$$

where $f(z)$ is a polynomial over $\mathbb{F}_q$ of degree less than k.

- A rich class of codes with an **efficient decoding** up to $(n-k)/2$ errors.
- Used in QR codes with $q = 256$.

Algebraic-geometric codes and curves over finite fields



- An algebraic plane curve Γ is given by a polynomial $F(X, Y) = 0$ over the finite field $\mathbb{F}_q$.
- **Hard:** points, divisors G, functions, evaluation, Riemann-Roch space $\mathscr{L}(G)$.
- **Advantage to RS:** More than q points, longer codes.

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Motivation

- In this section, we present a public key cryptosystem that was proposed by McEliece in 1978.
- Its **security** is based on the hardness of binary decoding.
- In the last decades, this system was **not used** because **(1)** the keys are large, **(2)** the encrypted messages are long, and **(3)** there are not many safe binary codes beside binary BCH and Goppa codes.
- However, this system is one of the few which resists the quantum attack by Peter Shor (1994).
- The **recent progress** in the construction of **quantum computers** indicates that in 30 years, the recently used cryptosystems (RSA, ECC, etc.) will have to be replaced.

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Public key (asymmetric) cryptography

- In a **public key (or asymmetric) cryptosystem,** each user X has two keys,
- a private key $K_D(X)$ and a public key $K_E(X)$.
- If Bob wants to send a message m to Alice, he encrypts it to m' using Alice's public key $K_E(\text{Alice})$.
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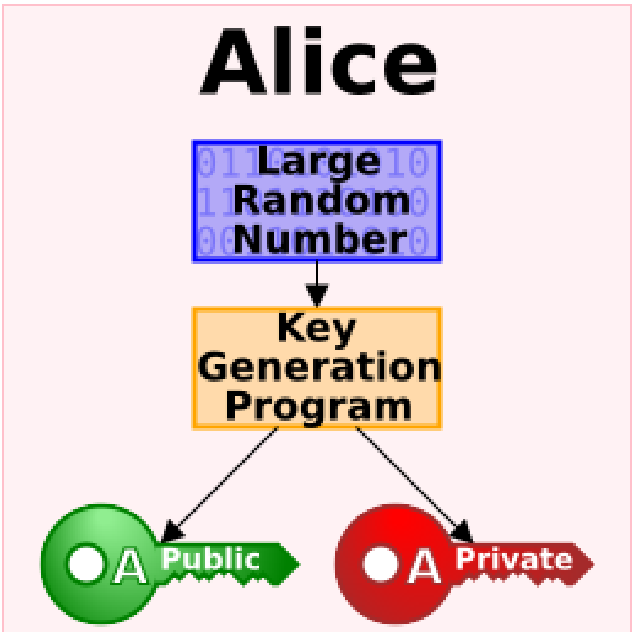

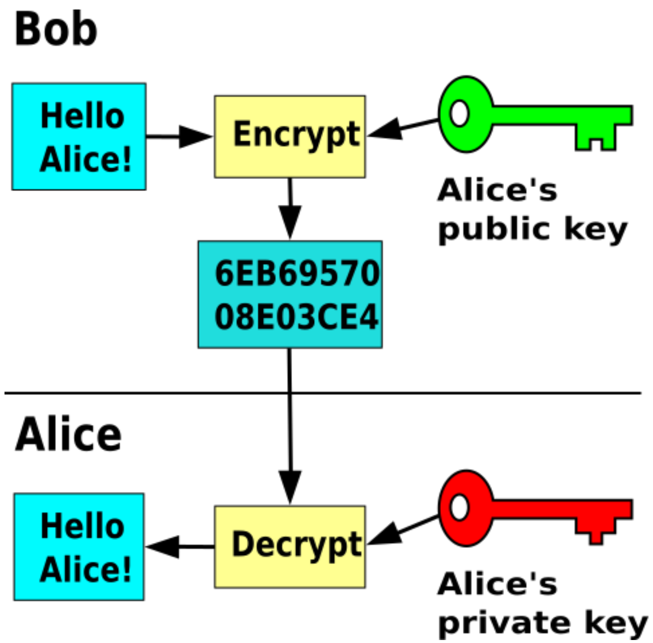

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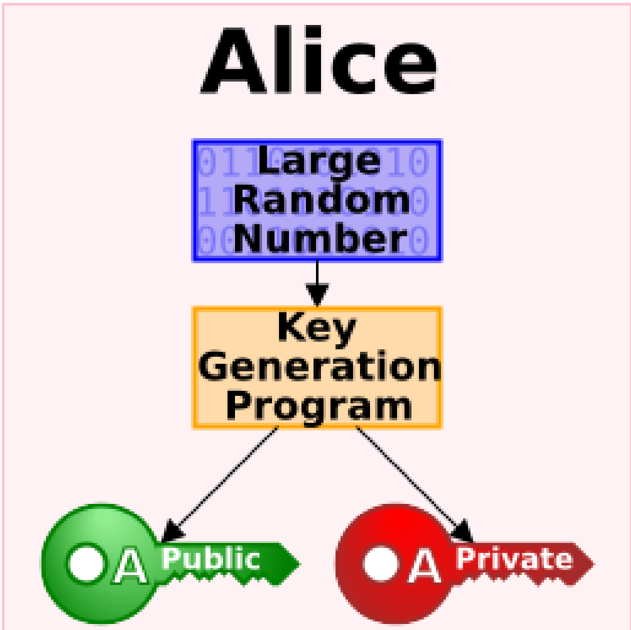

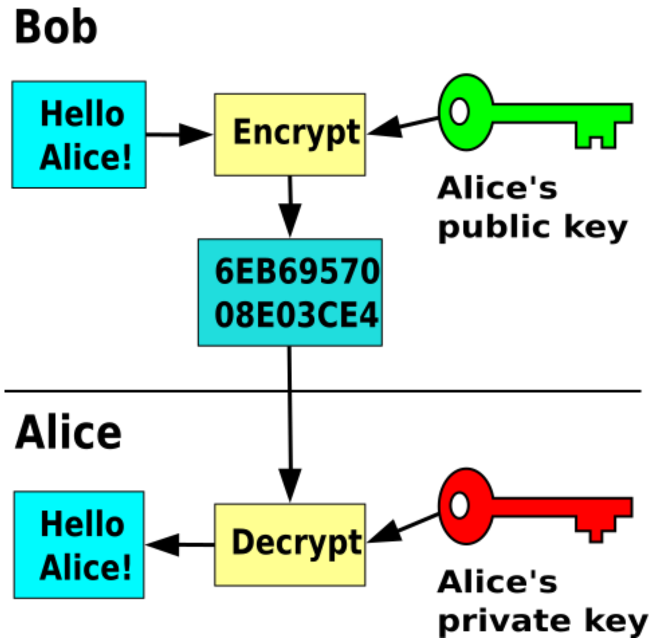

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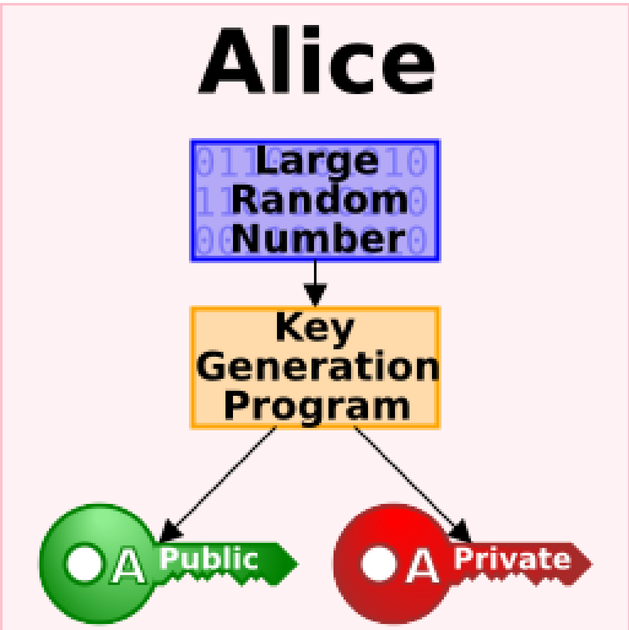

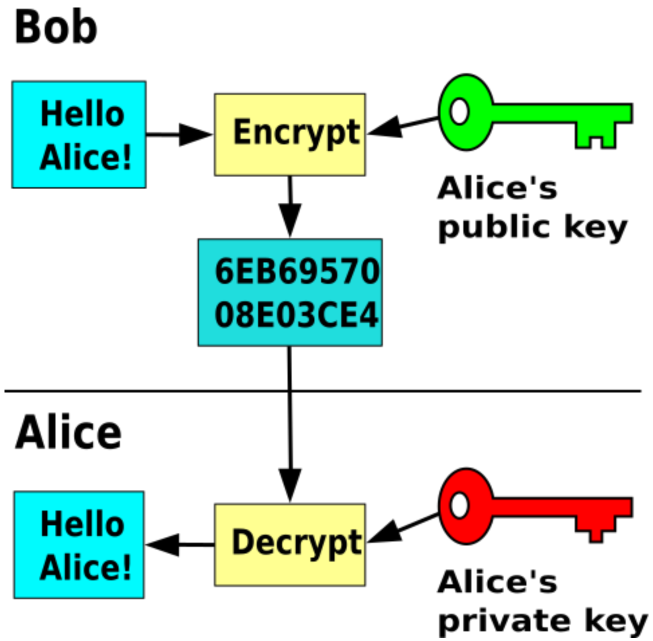

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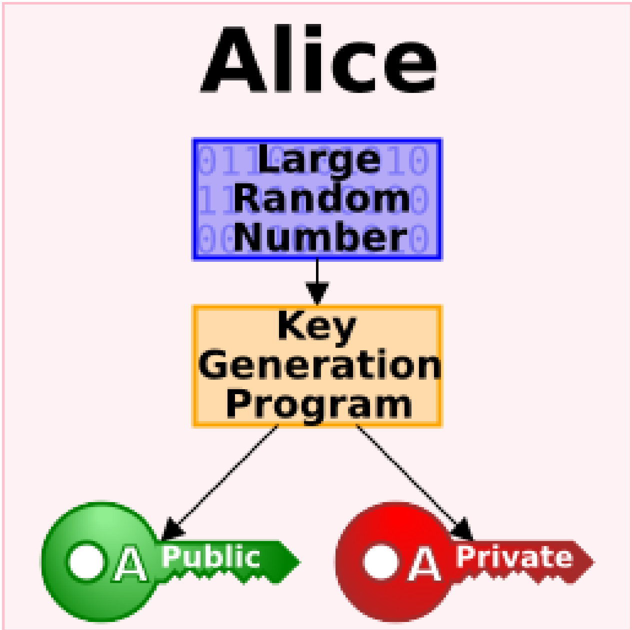

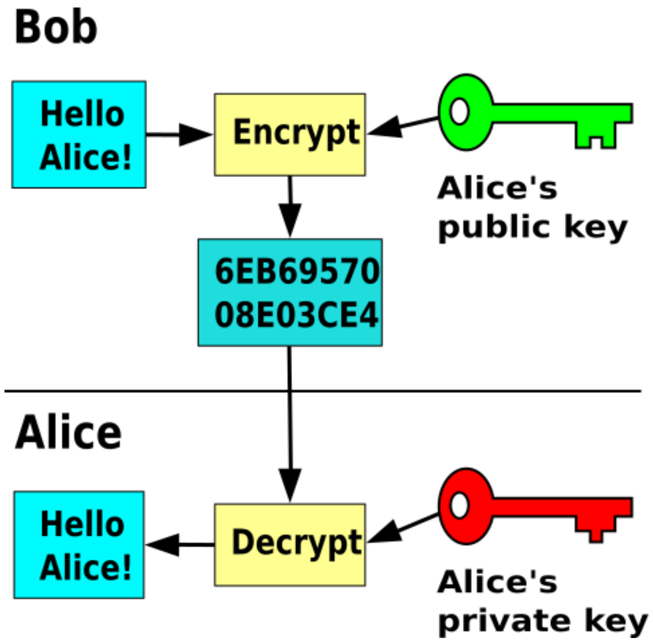

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- The **McEliece Cryptosystem** is based on a **binary linear code** C of length n and dimension k, which has a **fast algorithm** correcting up to t errors per code word. Let G denote the $n \times k$ generator matrix of C.
- **Creation of Alice's keys** She picks a random $k \times k$ invertible matrix S and a random $n \times n$ permutation matrix P. Her *private key* is the pair (S, P) and her *public key* is the $n \times k$ matrix $G' = SGP$.
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- Make probabilistic decoding algorithms into deterministic ones.
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