On the connection between the projection and the extension of a parallelotope. *

Á.G.Horváth
Department of Geometry,
Budapest University of Technology and Economics (BME),
H-1521 Budapest,
Hungary

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Abstract

This paper consists of a result concerning the connection between the parallel projection $P_{\mathbf{v},H}$ of a parallelotope P along the direction $\mathbf{v}$ (into a transversal hyperplane H) and the extension $P+S(\mathbf{v})$ meaning the Minkowski sum of P and the segment $S(\mathbf{v})=\{\lambda\mathbf{v}|-1\leq\lambda\leq 1\}$. The author defines a sublattice $L_{\mathbf{v}}$ of the lattice of translations of P associated to the direction $\mathbf{v}$ and proves that the extention $P+S(\mathbf{v})$ is a parallelotope if and only if the parallel projection $P_{\mathbf{v},H}$ is a parallelotope with respect to the lattice of translations $L_{\mathbf{v},H}$ which is the projection of the lattice $L_{\mathbf{v}}$ along the direction $\mathbf{v}$ into the hyperplane H.

1 Introduction

One of the classical problems of discrete geometry is the problem of classification of parallelotopes. The class of convex polytopes called by parallelotope was introduced independently by Minkowski (as extremal bodies) [5] and Voronoi (as parallelohedra)[9],[10], respectively. There are several books containing the classical results on parallelotopes (see e.g. [3]), but there

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is an interesting result of B.A.Venkov on the projection of parallelotopes which is not widely known yet [7]. This nice paper contains the concept of nonzero thickness of a parallelotope in the direction of a k-subspace of the n-dimensional space and proved a theorem saying that, if P is an n-dimensional parallelotope of nonzero thickness along the subspace X^k, then a projection along X^k is a parallelotope, too. In the last years V.Grishukhin gave a characterization of nonzero thickness in a direction of a line ([2]). He proved that a parallelotope of non-zero-width in the direction of a line is the Minkowski sum (denoted by + in this paper) of a parallelotope of zero width in this direction with a segment of this line. He also proved a necessary and sufficient condition guaranteeing sum of a parallelotope with a segment is a parallelotope, too. Method for which from a parallelotope using the Minkowski addition we define another one, we will call "the method of extension". In [1] the authors gave a new enumeration of four-dimensional parallelotopes showing that any such a parallelotope is either a zonotope or the Minkowski sum of a zonotope with the regular 24-cell.

In this paper we connect the above mentioned two nice results on the parallel projection and the extension, respectively. We introduce the concept of a lattice called Venkov lattice and prove that the necessary and sufficient condition for the extension along a line of a parallelotope being a parallelotope is that its parallel projection along this direction into a transversal hyperplane let also be a parallelotope with respect to the parallel projection of the Venkov lattice.

2 The definition of the Venkov lattice

A parallelotope is a convex polytope whose translated copies tile the space. The history of parallelotopes is contained in [3]. In the famous paper [6] Venkov characterize the parallelotopes with the following three properties:

- (i) P is centrally symmetric
- (ii) each facet of P is centrally symmetric
- (iii) projection along any $n-2$-dimensional face of P is either a parallelogram or a centrally symmetric hexagon.

This theorem was rediscovered by McMullen in 1980.(See [4].) The concept of belts is very important and can be defined in the following way. Consider a face $F(n-2)$ of dimension $n-2$ and one of the facets containing it. Reflecting $F(n-2)$ in the center of the facet we get another $n-2$-face

$F'(n-2)$ belonging to another facet of P. Reflecting now $F'(n-2)$ we get the face $F''(n-2)$, and so on till we get back to $F(n-2)$. The chain of $n-2$ faces and $(n-1)$ facets is the belt associated to the face $F(n-2)$ having either 4 or 6 facets by Venkov's theorem. These two types of belts are called 2-belt or 3-belt, respectively.

In this paper we need Venkov's results and notions about a polytope of **nonzero thickness** in the direction of a k-dimensional subspace X^k. This is a polytope whose intersection with any affine k-space parallel to X^k is either k-dimensional or empty. Of course each facet $F(n-1)$ can be associated to a lattice vector which represents the translation $\mathbf{t}$ of P into those parallelotope $P+\mathbf{t}$ which also contains $F(n-1)$. We call this lattice vector the **relevant** of $F(n-1)$. We also use the concept of a **facet vector** of a facet meaning a vector orthogonal to the hyperplane of the examined facet of P. Venkov in [7] proved that if the parallelotope is of nonzero thickness along X^k then a projection along X^k is a parallelotope (of dimension $(n-k)$) with respect to the projection of the lattice generated by the projections of the relevants of those facets of P which are parallel to X^k.

As we mentioned in the introduction Grishukhin in [2] proved that the parallelotope P has nonzero thickness along the line determined by the vector $\mathbf{v}$ if and only if it can be given in the form $P = P_0 + S(\mathbf{v})$ where the polytope P_0 is also a parallelotope having zero width in the direction $\mathbf{v}$. (Here the segment $S(\mathbf{v}) = \{\lambda\mathbf{v} | -1 \leq \lambda \leq 1\}$ is the same as we define it in the abstract of this paper.) In his paper Grishukhin also proved that the assertions that "$P+S(\mathbf{v})$ is a parallelotope" and "the vector $\mathbf{v}$ is orthogonal to at least one facet vector of each 3-belt of P", are equivalent. Therefore using Venkov's theorem we get that if P_0 and $P = P_0 + S(\mathbf{v})$ are parallelotopes then the parallel projection $\{P_0\}_{\mathbf{v},H}$ along the line of $\mathbf{v}$ into a transversal hyperplane H of P is also a parallelotope. (The projections of P_0 and P is the same polytope of H.) On the other hand the converse statement are not true as we can easily see. Consider a regular hexagon. Let the direction of the projection be orthogonal to one of its main diagonals. The projection of the hexagon to a (transversal to the direction of the projection) line is a segment which is a parallelotope but the corresponding extension of the hexagon is an octagon which is not a parallelotope. The problem is that there is no connection between the lattice of translations of the projected parallelotope and the original lattice. In this way we have to scrutinize these two lattices. First we define the relevants of an $n-2$-dimensional face of P that determines a 2-belt on the boundary of P.

Definition 1 *Let P be a parallelotope and $F(n-2)$ one of its faces of* *dimension $n-2$. Then there exist two facets of P, say $F'(n-1)$ and $F''(n-1)$ with relevants $\mathbf{t}'$ and $\mathbf{t}''$, respectively, which contain this face. If $F(n-2)$ determine a 2-belt then we call the lattice vector $\mathbf{t} = \mathbf{t}' + \mathbf{t}''$* **the relevant** *of it. (This is a lattice vector of the lattice of translations associated to $F(n-2)$.)*

We note that if the face $F(n-2)$ determines a 3-belt then it is impossible to define such a relevant of $F(n-2)$. Now we define the collection of the faces of P which play an important role with respect to a projection along a line determined by a fixed vector $\mathbf{v}$. Let us denote the **shadow boundary of** P in the direction of $\mathbf{v}$ by $sh_{\mathbf{v}}P$. Of course this set collects those boundary points $\mathbf{x}$ of P for which the line $\{\mathbf{x}+\lambda\mathbf{v}|\lambda \in R\}$ is a support line of P. (There is no point of the line $\{\mathbf{x}+\lambda\mathbf{v} \mid \lambda \in R\}$ belonging to the interior of P.) It is known that the shadow boundary of a convex polytope is a collection of closed faces of dimension at least $n-2$.

Now we are ready for the definition of the lattice associated to the direction $\mathbf{v}$ of the projection.

Definition 2 *Let P be a parallelotope with lattice L and $\mathbf{v}$ be a given direction of the space. We denote by $L_{\mathbf{v}}$ the sublattice of L spanned by those relevants of P whose corresponding faces are maximal ones (with respect to the face-lattice of the parallelotope) of faces belonging to the shadow boundary $sh_{\mathbf{v}}P$ of direction $\mathbf{v}$. We shall say that this lattice is the* **Venkov lattice** *associated to the direction $\mathbf{v}$.*

We remark that there are two types of faces of dimension $n-2$ belonging to $sh_{\mathbf{v}}P$. A face of the first type does not belong to any facet of $sh_{\mathbf{v}}P$ while a face of the second one belongs to a facet of $sh_{\mathbf{v}}P$. Since we require maximality we have to consider only the faces of the first type which are transversal to the direction of $\mathbf{v}$.

We also note that in the case when the parallelotope has nonzero thickness in the direction of $\mathbf{v}$, $L_{\mathbf{v}}$ is the $(n-1)$-dimensional lattice (defined by Venkov) generated by the relevants of those $(n-1)$-dimensional faces which are parallel to the line of $\mathbf{v}$. On the other hand such a lattice was investigated in the paper of Végh ([8]) corresponding to Dirichlet-Voronoi cells.

We observe that the dimension of the lattice defined above, strongly depends on the direction $\mathbf{v}$. To see this we consider the regular hexagonal prism with respect to different directions. In Fig.1 and Fig.2 the direction is orthogonal to the plane of the paper and the dimensions of the corresponding lattices are 1, 3 and 2, respectively, as it can be seen easily.

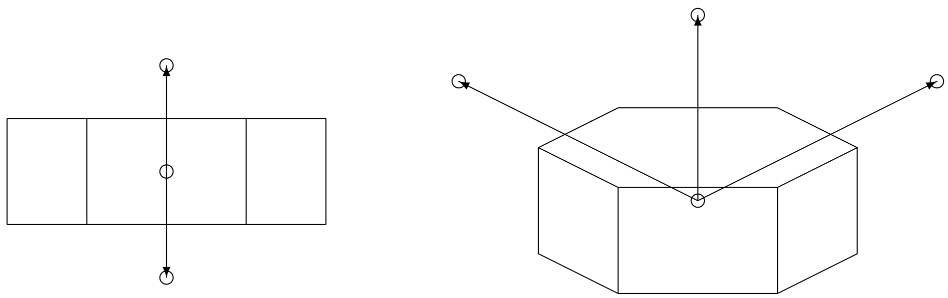

Figure 1: Venkov lattices of dimension 1 and 3, respectively.

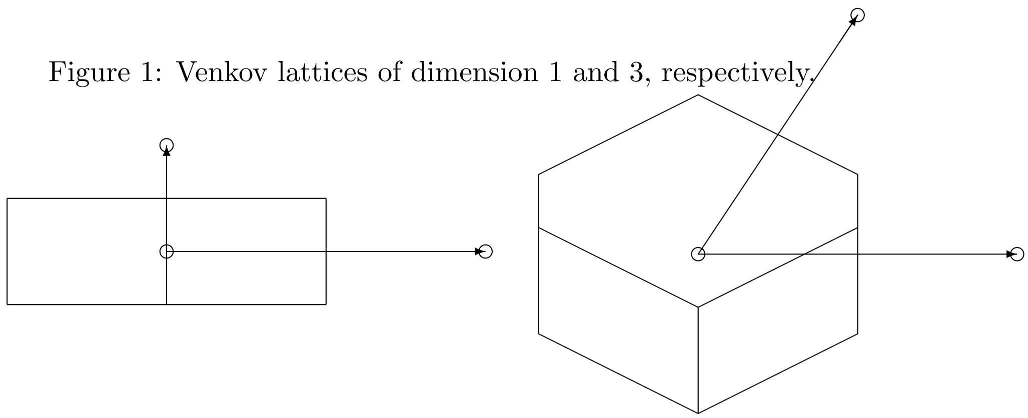

Figure 2: Venkov lattices of dimension 2.

3 The theorem

Now we are ready to give the connection between the projection and extension of a parallelotope.

Theorem 1 *Let P_0 be a parallelotope, $\mathbf{v}$ be a direction and H be a hyperplane transversal to $\mathbf{v}$. Then the following statements are equivalent.*

1. *The polytope $P_0 + S(\mathbf{v})$ is a parallelotope*
2. *The projection $\{P_0\}_{\mathbf{v},H}$ of P_0 along the line of $\mathbf{v}$ onto the hyperplane H is a parallelotope with respect to the projection $\{L_{\mathbf{v}}\}_{\mathbf{v},H}$ of the Venkov lattice $L_{\mathbf{v}}$.*

Proof: If the extension $P_0 + S(\mathbf{v})$ is a parallelotope then it has nonzero thickness in the direction of $\mathbf{v}$. So its Venkov lattice is generated by those relevants of $P_0 + S(\mathbf{v})$ whose facets associated to these relevants are parallel to the direction $\mathbf{v}$. In this way the second statement follows from the theorem of Venkov.

The converse statement is not so easy.

Firstly we prove that from the second condition it follows that the dimension of the lattice $L_{\mathbf{v}}$ is $(n-1)$, too. Since the projection of $L_{\mathbf{v}}$ along $\mathbf{v}$ is an $(n-1)$-dimensional lattice in the hyperplane H, the dimension of $L_{\mathbf{v}}$ is greater or equal to $(n-1)$. Assuming that it is equal to n clearly follows the existence of a vector $\mathbf{v}^0 \in L_{\mathbf{v}}$, parallel to $\mathbf{v}$. (Of course we may assume that this vector is a primitive one, meaning that it is the shortest one in its direction in $L_{\mathbf{v}}$.)

Consider the unbounded prism $H_{\mathbf{v}} := P_0 + \{\lambda\mathbf{v} \mid \lambda \in R\}$ where R is the set of real numbers. The facets of $H_{\mathbf{v}}$ form the $(n-1)$-dimensional prisms of the form $F(i) + \{\lambda\mathbf{v} \mid \lambda \in R\}$, where $F(i)$ is an i-dimensional face of P_0 belonging to the shadow boundary $sh_{\mathbf{v}}P_0$. (The possible values of i are $n-2$ and $(n-1)$, respectively.) Denote by $\mathcal{G} := \{\mathbf{g}_i \mid i = 1, \ldots, \sigma\}$ the set of the generators of $L_{\mathbf{v}}$, which are the relevants (in the sence of Definition 1) of a face of $sh_{\mathbf{v}}P_0$. Let $\mathcal{H}_{\mathbf{v}}$ be the set of prisms whose elements are the translated copies of $H_{\mathbf{v}}$ by the vectors of $L_{\mathbf{v}}$. On the basis of our assumption it is obvious that $\mathcal{H}_{\mathbf{v}}$ is a tiling of the n-dimensional space with respect to the lattice of translations $\{L_{\mathbf{v}}\}_{\mathbf{v},H}$. One of the prisms of this tiling contains infinitely many copies of tiles of the packing $P + L_{\mathbf{v}}$, these are of the form $P + \mathbf{z} + k\mathbf{v}^0$ with a fixed vector $\mathbf{z} \in L_{\mathbf{v}}$ and an integer k. We now prove that each of the prisms contains only one parallelotope of the tiling $P + L_{\mathbf{v}}$, contradicting the existence of $\mathbf{v}^0$.

Define the neighbours of the parallelotope P as the set of those parallelotopes that arise from P by a translation with a vector of $\mathcal{G}$. The neighbours of the neighbours are those parallelotopes of $P + L_{\mathbf{v}}$ which are of the form $P^{(i,j)} = P + \mathbf{g}_i + \mathbf{g}_j$ with two vectors of $\mathcal{G}$, and so on. Of course each of the elements of the tiling of prisms can be corresponded to at least one multi-index $(i_1, \ldots, i_l)$ $(i_s \in \{1, \ldots, \sigma\})$ meaning a translation of P by the vector $\mathbf{g}_1 + \cdots + \mathbf{g}_l$. We have to prove that there is no such an index $(i_1, \cdots, i_l)$ for which $P^{(i_1, \cdots, i_l)} \subset H_{\mathbf{v}}$ and $P \neq P^{(i_1, \cdots, i_l)}$. In fact, from this statement it follows that in every prism we have only one parallelotope as we stated. Consider now a multi-index $(i_1, \cdots, i_l)$ defining a closed contour of the polyhedral complex of the prisms $\mathcal{H}_{\mathbf{v}}$. The consecutive elements of this contour are $H_{\mathbf{v}}, H_{\mathbf{v}} + \mathbf{g}_{i_1}, \ldots, H_{\mathbf{v}} + (\mathbf{g}_{i_1} + \cdots + \mathbf{g}_{i_l}) = H_{\mathbf{v}}$, respectively. Since the polyhedron (or body) of this complex is R^n which is a simply connected topological space, the contour can be given as the sum of such a (little) cycle of prisms in which the prisms of a given cycle contain a common edge. We have to prove only that on such a cycle our correspondence (by the concept of neighbours) of the prisms to parallelotopes is welldefined. Consider now such a cycle with its common edge (being a line parallel to $\mathbf{v}$) $e := H_{\mathbf{v}} \cap (H_{\mathbf{v}} + \mathbf{g}_{i_1}) \cap \cdots \cap (H_{\mathbf{v}} + (\mathbf{g}_{i_1} + \cdots + \mathbf{g}_{i_k})) = H_{\mathbf{v}}$. By the definition of neighbour of parallelotopes the intersection $e \cap P$ can be either a vertex or an edge of P belonging to the pairwise defined intersection faces $P^{(i_1,\dots,i_r)} \cap P^{(i_1,\dots,i_{(r+1)})}$ meaning that it is a face of all the parallelotopes belonging to this cycle. Since of two parallelotopes in $H_{\mathbf{v}}$ cannot have a common edge in e and if they have a common vertex in e than the complete intersection $P \cap e$ is an edge, we get that $P^{(i_1,\dots,i_l)} = P$ as we stated. Thus the concept of neighbours give a bijection between the prisms and parallelotopes, meaning that the dimension of $L_{\mathbf{v}}$ is equal to $(n-1)$.

Secondly we check that the condition on 3-belts in Grishukhin's theorem (See: Theorem 1 in [2]) holds: that is; we prove that, among the facet vectors of a 3-belt, there exists at least one which is orthogonal to the direction of $\mathbf{v}$. Let $\pm F_i(n-1)$ be the respective facets of a 3-belt for $i = 1, 2, 3$. The associated relevants are $\mathbf{t}_i$; changing signs if necessary, we can assume that $\mathbf{t}_1 + \mathbf{t}_2 + \mathbf{t}_3 = 0$ holds. (Clearly we have for each i that $F_i(n-1) = P_0 \cap P_0 + \mathbf{t}_i$.) Assuming that there is no one among the six facets $\pm F_i(n-1)$ belonging to the shadow boundary of P_0, three facets have to belong to the lower and the others to the upper cap of P_0, respectively.

The concept of lower and upper caps was defined in the paper [2] in the following way. Let I_{P_0} be that set of indices for which P_0 defined by the equality $P_0 = \{\mathbf{x} \in R^n : \mathbf{p}_i^T \mathbf{x} \geq b_i, i \in I_{P_0}\}$. If $\delta \in \{\pm 1, 0\}$ then for the vector $\mathbf{v}$ let $I_\delta(\mathbf{v})$ be the set of indices containing those elements of I_{P_0} for which $sgn(\mathbf{v}^T \mathbf{p}_i) = \delta$ (the vectors $\mathbf{p}_i$ are the facet vectors). The upper cap and lower cap of P_0 with respect to the direction $\mathbf{v}$ are the sets of facets $\{F_i(n-1) : i \in I_{+1}(\mathbf{v})\}$ and $\{F_i(n-1) : i \in I_{-1}(\mathbf{v})\}$, respectively. It is clear that the intersection of the (set theoretical) union of the facets of the two caps is the union of the faces, of dimension $n-2$ of the shadow boundary, transversal to $\mathbf{v}$.

If we assume that $F_1(n-1), -F_2(n-1)$ and $F_3(n-1)$ are in the upper cap then by the definition of belts we know that for example the intersection $G_{-1,3} = F_3 \cap (-F_1)$ has dimension $n-2$ meaning that this face belongs to the shadow boundary. If now the direction $\mathbf{v}$ is parallel to $G_{-1,3}$ then it is also parallel to the facets $-F_1$, F_3, respectively, hence these facets do not belong to the upper and lower caps, respectively. This is a contradiction implying that the direction $\mathbf{v}$ should be transversal to $G_{-1,3}$. Consider now the projections $\{P_0\}_{\mathbf{v},H}$ and $\{G_{-1,3}\}_{\mathbf{v},H}$ of P_0 and $G_{-1,3}$ (along $\mathbf{v}$ on H), respectively. The projection $\{G_{-1,3}\}_{\mathbf{v},H}$ of dimension $n-2$ is a facet of $\{P_0\}_{\mathbf{v},H}$ implying (by the tiling property) the existence of a vector $\mathbf{t}$ in the Venkov lattice for which $\{P_0\}_{\mathbf{v},H} \cap \{P_0 + \mathbf{t}\}_{\mathbf{v},H} = \{G_{-1,3}\}_{\mathbf{v},H}$. (This step use the fact proved in the first part of this proof that there is transversal to $\mathbf{v}$ a hyperplane containing the Venkov lattice, so the above $\mathbf{t}$ is unique and thus is a generator of it.) This later equality implies that the common face of the parallelotopes P_0 and $P_0 + \mathbf{t}$ is $G_{-1,3}$ which contradicts the assumption that this belt is a 3-belt. This means that at least two opposite facets of this 3-belt belong to the shadow boundary of P_0, and hence the condition of Grishukhin holds. Thus the extension $P_0 + \mathbf{v}$ is a parallelotope as we stated.

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