ADDENDUM TO THE PAPER "HYPERBOLIC PLANE-GEOMETRY REVISITED"

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Abstract. In the paper "Hyperbolic plane-geometry revisited" [8] we stated several formulas without proof. The purpose of this note is to give electronic source for the omitting proof.

1. Introduction

1.1. **Well-known formulas on hyperbolic trigonometry.** In this paper, we use the following notations. The points A, B, C denote the vertices of a triangle. The lengths of the edges opposite to these vertices are a, b, c, respectively. The angles at A, B, C are denoted by α, β, γ, respectively. If the triangle has a right angle, it is always at C. The symbol δ denotes half of the area of the triangle; more precisely, we have $2\delta = \pi - (\alpha + \beta + \gamma)$.

- **Connections between the trigonometric and hyperbolic trigonometric functions:**

$$\sinh a = \frac{1}{i}\sin(ia), \quad \cosh a = \cos(ia), \quad \tanh a = \frac{1}{i}\tan(ia)$$

- **Law of sines:**

$$\sinh a : \sinh b : \sinh c = \sin\alpha : \sin\beta : \sin\gamma \tag{1}$$

- **Law of cosines:**

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos\gamma \tag{2}$$

- **Law of cosines on the angles:**

$$\cos\gamma = -\cos\alpha\cos\beta + \sin\alpha\sin\beta\cosh c \tag{3}$$

- **The area of the triangle:**

$$T := 2\delta = \pi - (\alpha + \beta + \gamma) \text{ or } \tan\frac{T}{2} = \left(\tanh\frac{a_1}{2} + \tanh\frac{a_1}{2}\right)\tanh\frac{m_a}{2} \tag{4}$$

where m_a is the height of the triangle corresponding to A and a_1, a_2 are the signed lengths of the segments into which the foot point of the height divide the side BC.

- **Heron's formula:**

$$\tan\frac{T}{4} = \sqrt{\tanh\frac{s}{2}\tanh\frac{s-a}{2}\tanh\frac{s-b}{2}\tanh\frac{s-c}{2}} \tag{5}$$

- **Formulas on Lambert's quadrangle:** The vertices of the quadrangle are A, B, C, D and the lengths of the edges are $AB = a, BC = b, CD = c$ and $DA = d$, respectively. The only angle which is not right-angle is $BCD\measuredangle = \varphi$. Then, for the sides, we have:

$$\tanh b = \tanh d \cosh a, \quad \tanh c = \tanh a \cosh d,$$

and

$$\sinh b = \sinh d \cosh c, \quad \sinh c = \sinh a \cosh b,$$

moreover, for the angles, we have:

$$\cos\varphi = \tanh b \tanh c = \sinh a \sinh d \quad \sin\varphi = \frac{\cosh d}{\cosh b} = \frac{\cosh a}{\cosh c},$$

and

$$\tan\varphi = \frac{1}{\tanh a \sinh b} = \frac{1}{\tanh d \sinh c}.$$

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2. The distance of the points and on the lengths of the segments

2.1. **The extracted hyperbolic theorem of sines.**

Statement 1 (Statement 2.1. in [8])**.** *Denote by a, b, c, d, e the edge lengths of the successive sides of a pentagon with five right angles on the hyperbolic plane. Then we have the following formulas:*

$$\cosh d = \sinh a \sinh b \quad \sinh c = \frac{\cosh a}{\sqrt{\sinh^2 a \sinh^2 b - 1}} \quad \sinh e = \frac{\cosh b}{\sqrt{\sinh^2 a \sinh^2 b - 1}}.$$

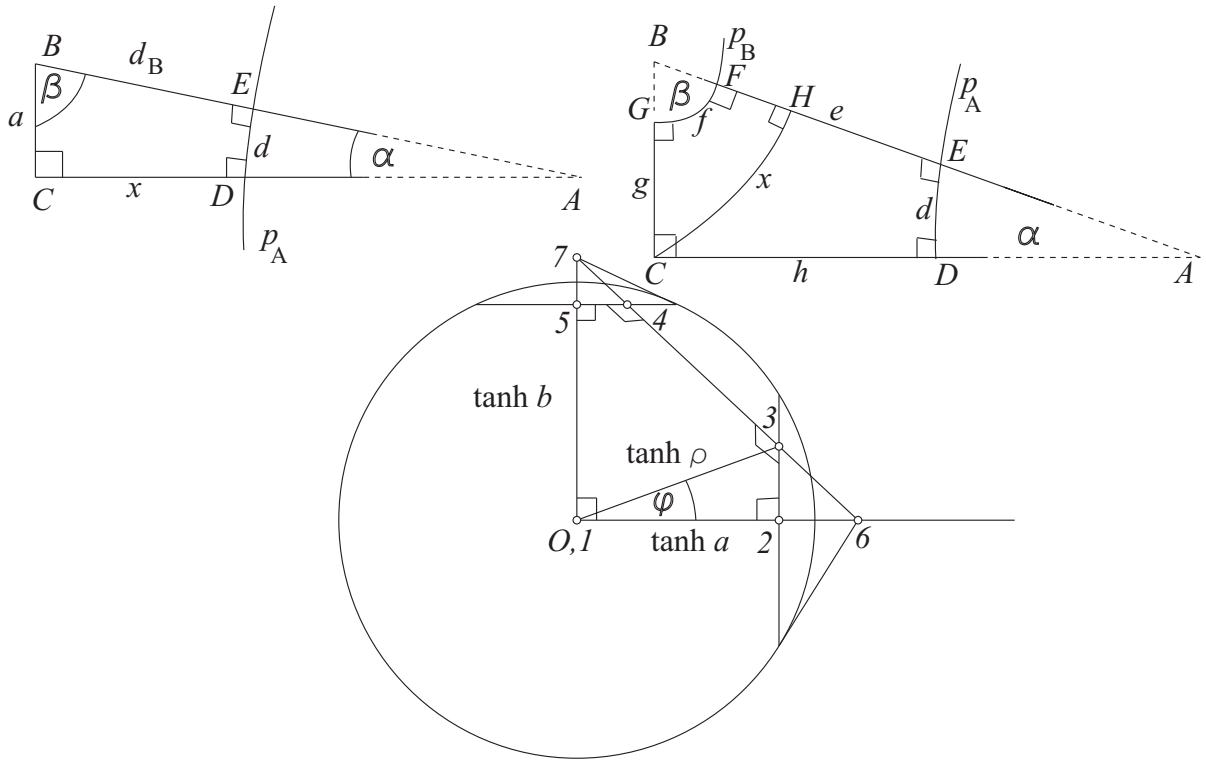

Figure 1. Hyperbolic theorem of sines with non-real vertices

We prove the statement using Weierstrass homogeneous coordinates of the hyperbolic plane. Before the proof we recall the formula of (usual) distance of points with respect to such homogeneous coordinates. Consider the hyperboloid model of the hyperbolic plane H embedded into a 3-dimensional pseudo-Euclidean space with indefinite inner product with signature $(-,-,+)$. The points of the plane can be considered as the unit sphere of this space containing those elements which scalar square is equal to 1 and last coordinates are positives, respectively. It can be seen that the distance between two points $X = (x, y, z)^T$ and $X' = (x', y', z')^T$ holds the following formula:

$$\cosh |XX'| = -xx' - yy' + zz'.$$

Consider now the projection of H into the plane $z = 1$ from the origin. Then we get a projective (Cayley-Klein) model of H with the usual metric.

Proof. Assume that a pentagon 12345 with five right angles lies in this model as in Fig. 1 (bottom) the vertex 1 is the origin and the edges 12 and 51 lies on the first two axes of the coordinate system. Now we have to determine the length of the edge 34 using as parameter the respective lengths a and b of the edges 12 and 51. To this we can determine the coordinates of the points III, IV of H which mapped into the points 3,4, respectively. Consider the point X and its image 3. We have to determine first the Euclidean distance $\rho := |03|$ and the angle $\varphi := (2O3)_{\measuredangle}$ and then the coordinates of X are $\sinh \rho \cos \phi, \sinh \rho \sin \varphi, \cosh \rho$, respectively. If the hyperbolic length of 12 and 51 are a and b, respectively, then their Euclidean distances are $\tanh a$ and $\tanh b$, respectively. Obvious that the line 34 intersects the axes in such points 6 and 7, whose distances from the origin are $1/\tanh a$ and $1/\tanh b$, respectively.

From this we get that

$$\cosh\rho = \frac{\cosh^2 a\tanh b}{\sqrt{\cosh^2 a\tanh^2 b - 1}} \quad \sinh\rho = \frac{\sqrt{\sinh^2 a\cosh^2 a\tanh^2 b + 1}}{\sqrt{\cosh^2 a\tanh^2 b - 1}}$$

and

$$\cos\varphi = \frac{\sqrt{\sinh^2 a\cosh^2 a\tanh^2 b}}{\sqrt{\sinh^2 a\cosh^2 a\tanh^2 b + 1}} \quad \sin\varphi = \frac{1}{\sqrt{\sinh^2 a\cosh^2 a\tanh^2 b + 1}}.$$

From these quantities we get

$$x = \frac{\sinh a\cosh a\tanh b}{\sqrt{\cosh^2 a\tanh^2 b - 1}}, \quad y = \frac{1}{\sqrt{\cosh^2 a\tanh^2 b - 1}}, \quad z = \frac{\cosh^2 a\tanh b}{\sqrt{\cosh^2 a\tanh^2 b - 1}},$$

and similarly for the pre-image X' of the point 4 we get

$$x' = \frac{1}{\sqrt{\cosh^2 b\tanh^2 a - 1}} \quad y' = \frac{\sinh b\cosh b\tanh a}{\sqrt{\cosh^2 b\tanh^2 a - 1}} \quad z' = \frac{\cosh^2 b\tanh a}{\sqrt{\cosh^2 b\tanh^2 a - 1}}.$$

Finally the inner product of these vectors gives the first required formula

$$\cosh d = \cosh|XX'| = \sinh a\sinh b.$$

The other two formulas of the statement are simple consequences of this first one. □

3. Power, inversion and centres of similitude

Lemma 1 (Lemma 3.1. in [8]). *The product* $\tanh(PA)/2\cdot\tanh(PB)/2$ *is constant if* P *is a fixed (but arbitrary) point (real, at infinity or ideal),* P, A, B *are collinear and* A, B *are on a cycle of the hyperbolic plane (meaning that in the fixed projective model of the real projective plane it has a proper part).*

Proof. To prove this we have to consider three cases with respect to the type of the cycle with the necessary subcases with respect to the possible types of the points P, A, B.

(A): In the case of a circle we have more cases.

- P is a real point A, B are real points. In this case the center O of the circle is real and we can consider the real line through O and perpendicular to the line AB. The intersection of these lines is the real point C. Consider the triangles ACO and PCO, respectively. These have a common side OC and a respective right angle at C. For the pair of points choose such segments from the pair of possible segments, that the relation $AB = AC \cup CB$ be valid (see Fig. 2). From the Pythagorean Theorem we have $\cosh AC/\cosh CP = \cosh OA/\cosh PO$.

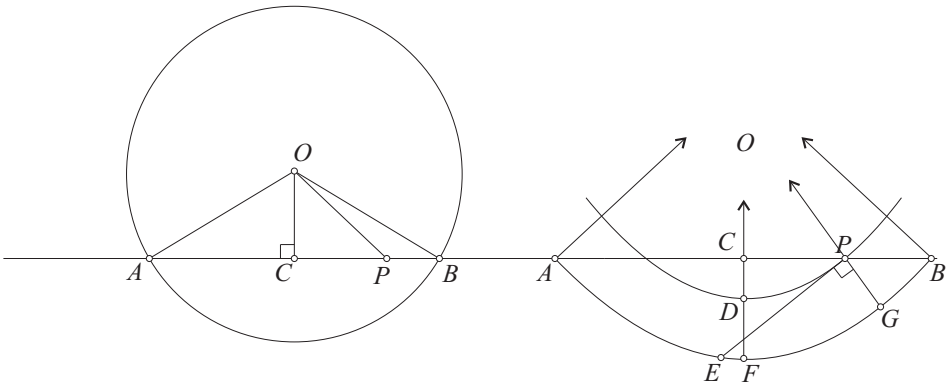

Figure 2. Power of a point into a cycle

Hence

$$\tanh\frac{AP}{2}\tanh\frac{BP}{2} = \tanh\frac{AC+CP}{2}\tanh\frac{BC-PC}{2} = \tanh\frac{AC+CP}{2}\tanh\frac{(AC-CP)}{2} =$$
$$= \frac{\sinh\frac{AC+CP}{2}}{\cosh\frac{AC+CP}{2}}\frac{\sinh\frac{AC-CP}{2}}{\cosh\frac{AC-CP}{2}} = \frac{\cosh AC-\cosh CP}{\cosh AC+\cosh CP} = \frac{\cosh OA-\cosh PO}{\cosh OA+\cosh PO} =$$
$$= \tanh\frac{OA+PO}{2}\tanh\frac{OA-PO}{2} = \text{constant} = c.$$

We note that the absolute value of c is less or equal to 1 and the sign of c depends only on the fact that P is a point in the interior or a point of the exterior of the given circle. Additionally it is equal to zero if and only if either $P = A$ or $P = B$, holds.

- P is an infinite point A, B are real points. According to our agreements on the length of a segment and using of the symbols $\pm\infty$ the required product is either 1 or -1.
- Finally if P is an ideal point and A, B are real points, then using the enumeration above originating from the extracted Pythagorean Theorem we get that

$$c = \tanh\frac{OA+PO}{2}\tanh\frac{OA-PO}{2} = \tanh\frac{OA+d+(\pi/2)i}{2}\tanh\frac{OA-d-(\pi/2)i}{2} =$$
$$= \frac{\cosh OA - \cosh(d+(\pi/2)i)}{\cosh OA + \cosh(d+(\pi/2)i)} = \frac{\cosh OA + \sinh d}{\cosh OA - \sinh d},$$

showing that the absolute value of c is greater than 1, and the sign of c depends on the ratio of the radius of the circle and the distance d (between the polar of P and the center of the circle).

(B): In the case of paracycle the point O is at infinite. In Fig.6 we can see that if P is real then there is an unique paracycle through P with the same pencil of parallel lines. Now if $C \neq P$ we have the following calculation:

$$\tanh\frac{AP}{2}\tanh\frac{BP}{2} = \tanh\frac{AC+CP}{2}\tanh\frac{BC-PC}{2} = \tanh\frac{AC+CP}{2}\tanh\frac{(AC-CP)}{2} =$$
$$= \frac{\sinh\frac{AC+CP}{2}}{\cosh\frac{AC+CP}{2}}\frac{\sinh\frac{AC-CP}{2}}{\cosh\frac{AC-CP}{2}} = \frac{\cosh AC - \cosh CP}{\cosh AC + \cosh CP} = \frac{\frac{\cosh AC}{\cosh CP} - 1}{\frac{\cosh AC}{\cosh CP} + 1}.$$

But using the equality on the diameter and height of a segment of a paracycle (*see also* eg. [7]) we get

$$\frac{\cosh AC}{\cosh CP} = \frac{e^{CF}}{e^{CD}} = e^{CF-CD} = e^{PG} = \cosh EP$$

showing that it is independent from the position of the secant AB. For $C = P$ this value is ± 1 and it is the result in that case, too, if P is at infinity. The absolute value of c is less then 1 for real P and greater than 1 for ideal P.

(C): In the case of hypercycle we have again more cases. First we assume that A, B and P are real points, respectively. O is an ideal point and C is the halving point of the segment AB ($AB = AC \cup CB = AP \cup PB$ as on Fig. 3). Let FG be the basic line of the hypercycle with

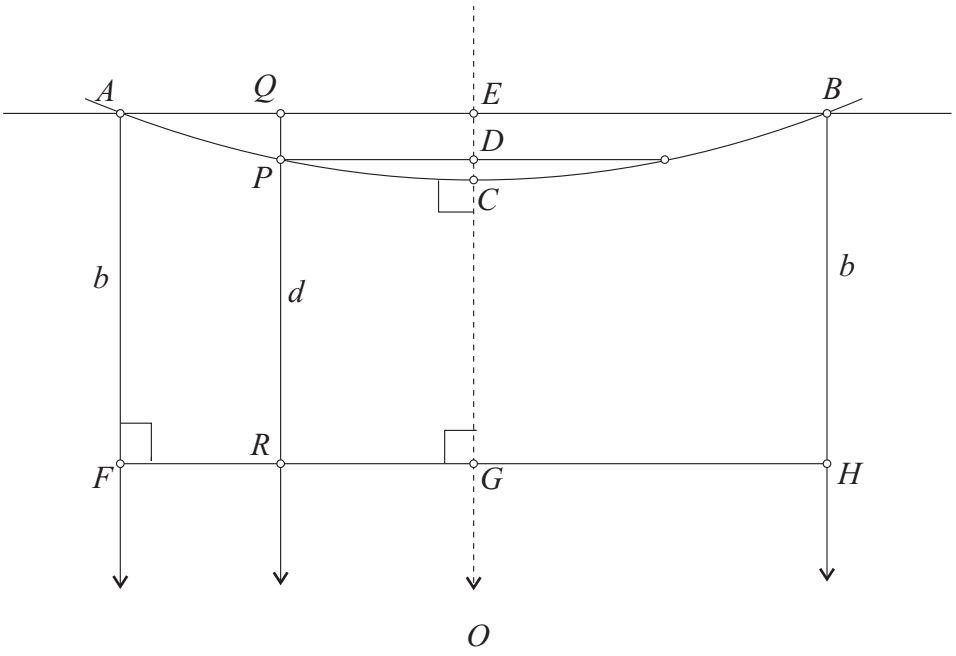

FIGURE 3. Power of a point into a hypercycle

distance b. Then all of the radiuses are orthogonal to FG. The minimal distance of a point of

the segment AB from the line FG attained at the radius through E (and C). As in the case of paracycles we get that

$$\tanh\frac{AP}{2}\tanh\frac{BP}{2}=\frac{\frac{\cosh AC}{\cosh CP}-1}{\frac{\cosh AC}{\cosh CP}+1},$$

and from the quadrangle $AFGC$ with three right-angle we get that

$$\frac{\cosh AC}{\cosh CP}=\frac{\sinh AF}{\sinh GC}:\frac{\sinh PR}{\sinh GC}=\frac{\sinh b}{\sinh d},$$

where d is the distance of the point P from the basic line of the hypercycle. Thus the latter term is independent from the choice of the points A, B on the hypercycle implying that the examined value has the same property. Denote by c this constant. Of course $b \gtrless d$ implies that $c \gtrless 0$ and the absolute value of c is less than 1. If A, B are real points and P at infinity then $c = \pm 1$. The result in the case when A, B, P are distinct, non-ideal points and at least one among is at infinity can be gotten analogousy.

Finally, we have to consider all cases when at least one point is ideal (and by our assumption at least one from A and B is real). Of course, from the definitions of the length of a general segment we can use complex numbers as in (A) to prove our statement. For instance, assume that P and O are ideal points such that the line PO is also ideal and A, B are a real points (see Fig. 4). The examined expression is

$$c=\tanh\frac{AP}{2}\tanh\frac{BP}{2}=\frac{\cosh AC-\cosh CP}{\cosh AC+\cosh CP}=\frac{\sinh AF-\sinh PR}{\sinh AF+\sinh PR}=\frac{\sinh b-\sinh i\varphi}{\sinh b+\sinh i\varphi}=$$

$$=\frac{i\sinh b+\sin\varphi}{i\sinh b-\sin\varphi}.$$

where φ is the angle of the respective polars of P and R. This proves the statement, again. The

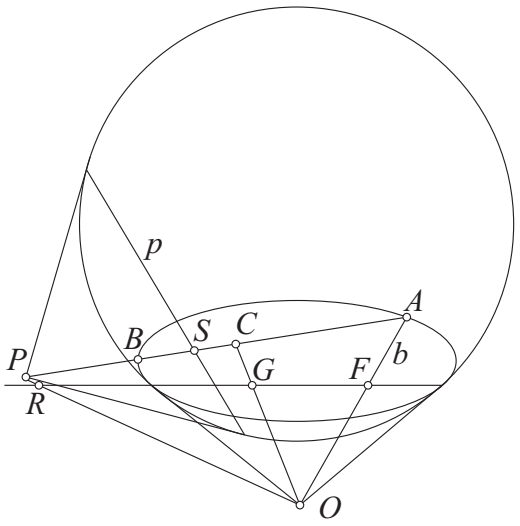

FIGURE 4. Power with ideal point P.

remaining cases can be proved analogously and we omit their proofs.

□

Lemma 2 (Lemma 3.4. in [8])**.** *Two points S, S' which divide the segments OO' and $O'O$, joining the centers of the two cycles in the hyperbolic ratio of the hyperbolic sines of the radii r, r' are the centers of similitude of the cycles. By formula, if*

$$\sinh OS:\sinh SO'=\sinh O'S':\sinh S'O=\sinh r:\sinh r'$$

then the points S, S' are the centers of similitude of the given cycles.

Proof. Consider a line through the point S which intersects the cycles in M and M'. Consider also the triangles OMS and $O'M'S$, respectively. Since $OSM\measuredangle = O'SM'\measuredangle$ from our assumption (using the general hyperbolic theorem of sines) follows the other equality $OMS\measuredangle = O'M'S\measuredangle$. This implies that a

tangent from S to one of the cycles is also a tangent to the other one. This means that S (and analogously S') is a center of similitude of the cycles. □

We also have the following

Lemma 3 (Lemma 3.5. in [8]). *If the secant through a centre of similitude S meets the cycles in the corresponding points M, M' then* $\tanh\frac{1}{2}SM$ *and* $\tanh\frac{1}{2}SM'$ *are in a given ratio.*

Proof. First we have to prove the hyperbolic analogy of the formula known as "Napier's analogy" (see in [4]) in spherical trigonometry. Consider the identity

$$\tanh\frac{a+b}{2}\coth\frac{c}{2}=\frac{\tanh\frac{a}{2}\coth\frac{c}{2}+\tanh\frac{b}{2}\coth\frac{c}{2}}{1+\tanh\frac{a}{2}\tanh\frac{b}{2}}$$

and substitute to this equality the equalities

$$\tanh\frac{a}{2}\coth\frac{c}{2}=\frac{\sin(\alpha+\delta)}{\sin(\gamma+\delta)}\quad \tanh\frac{b}{2}\coth\frac{c}{2}=\frac{\sin(\beta+\delta)}{\sin(\gamma+\delta)},$$

where 2δ is the defect of the triangle defined by $2\delta=\pi-(\alpha+\beta+\gamma)$. (This equality can be shown in the following way. Add to the hyperbolic theorem of cosine for angle $\cos\alpha=-\cos\beta\cos\gamma+\sin\beta\sin\gamma\cosh a$ the identity $\cos(\beta+\gamma)=\cos\beta\cos\gamma-\sin\beta\sin\gamma$ and use the formulas on the half of a distance then we get $\sinh\frac{a}{2}=\sqrt{(\sin\delta\sin(\alpha+\delta))/(\sin\beta\sin C)}$. Similarly, we get that

$$\cosh\frac{a}{2}=\sqrt{(\sin(\beta+\delta)\sin(\gamma+\delta))/(\sin\beta\sin\gamma)}$$

and the required equality follows.) Then we get

$$\tanh\frac{a+b}{2}\coth\frac{c}{2}=\frac{\sin(\alpha+\delta)+\sin(\beta+\delta)}{\sin(\gamma+\delta)+\sin\delta}=\frac{\cos\frac{\alpha-\beta}{2}}{\cos\frac{\alpha+\beta}{2}},$$

or equivalently

$$\tanh\frac{a+b}{2}=\frac{\cos\frac{\alpha-\beta}{2}}{\cos\frac{\alpha+\beta}{2}}\tanh\frac{c}{2}.$$

Using this formula we have that

$$\tanh\frac{1}{2}SM:\tanh\frac{1}{2}SM'=\tanh\frac{1}{2}(SO+r):\tanh\frac{1}{2}(SO'+r')=\text{const.}$$

□

Since using the extended concepts two points always determine a line and two lines always determine a point, all concepts defined on the sphere also can be used on the hyperbolic plane. Thus we use the concepts of "axis of similitude", "inverse and homothetic pair of points", "homothetic to and inverse of a curve γ with respect to a fixed point S (which "can be real point, a point at infinity, or an ideal point, respectively") as in the case of the sphere. More precisely we have:

Lemma 4 (Lemma 3.6. in [8]). *The six centers of similitude of three cycles taken in pairs lie three by three on four lines, called* axes of similitude *of the cycles.*

Proof. If A, B, C their centers and a, b, c the corresponding radii of the cycles, A', B', C' the internal centers of similitude, and A'', B'', C'' the externals; then we have by definitions (see [16] p.70 or [15])

$$(ABC'') := \sinh AC'' : \sinh C''B = \sinh a : \sinh b,$$

and similarly

$$(BCA'') = \sinh b : \sinh c,\quad (CAB'') = \sinh c : \sinh a.$$

Hence

$$(ABC'')(BCA'')(CAB'') = 1.$$

Now the convers of the Menelaos-theorem is also valid (see [15] p.169) implying that the points A'', B'', C'' are collinear. Similarly, it may be shown that any two internal centers and an external center lie on a line. □

4. Applications of the theory

4.0.1. *Construction of Gergonne.* Gergonne's construction (see e.g. [5] and see in Fig. 5) solve the following problem in the Euclidean plane:

Construct a circle touching three given circles of the Euclidean plane.

A nice construction is the following:

- Draw the point P of power of the given circles c_1, c_2, c_3 and an axis of similitude of certain three centres of similitude.
- Join the poles P_1, P_2, P_3 of this axis of similitude with respect to the circles c_1, c_2, c_3 with the point P by straight lines. Then the lines PP_i cut the circles c_i in two points Q_{i1} and Q_{i2}.
- A suitable choice $Q_{1j(1)}, Q_{2j(2)}, Q_{3j(3)}$ will give the touching points of some sought circle and c_1, c_2, c_3. More exactly, there are two such choices $Q_{1j(1)}, Q_{2j(2)}, Q_{3j(3)}$ and $Q_{1k(1)}, Q_{2k(2)}, Q_{3k(3)}$, satisfying $j(i) \neq k(i)$ for $1 \leq i \leq 3$, where $|PP_{ij(i)}| \leq |PP_{ik(i)}|$.

By the results of the preceding section we can say this construction on the hyperbolic plane too. We note that in the paper [6] this construction was proved by the conformal model. In this section we can give a proof without using any models.

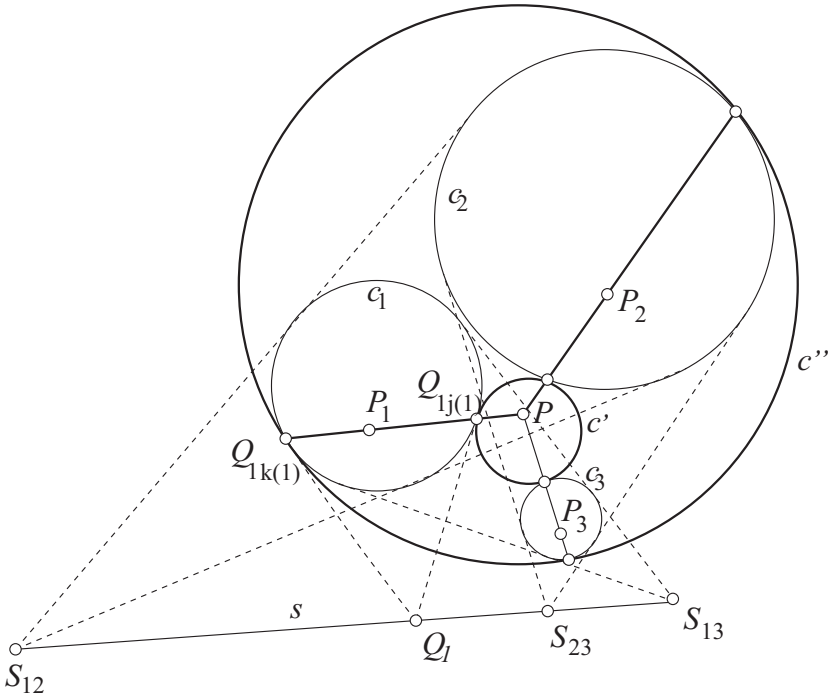

Figure 5. The construction of Gergonne

In Fig.5 the axis of similitude contains the three outer centers of similitude, in which case, choosing for $Q_{ij(i)}$ the intersection points closer to P, we obtain the common outward touching cycle, and for choosing the farther intersection points to P we obtain the common touching cycle that contains c_1, c_2, c_3. We denoted these circles in Fig.9 by c' and c'', respectively.

Choosing, e.g., for c_1, c_3 and c_2, c_3 the inner centers of similitude, and then for c_1, c_2 the outer center of similitude, we obtain another axis of similitude (by permuting the indices we obtain still two more similar cases). Then defining the points P_i and $P_{ij(i)}$ analogously like above, if $PQ_{1j(1)} \leq PQ_{1k(1)}$, $PQ_{2j(2)} \leq PQ_{2k(2)}$, and $PQ_{3j(3)} \geq PQ_{3k(3)}$, then the circle $Q_{1j(1)}Q_{2j(2)}Q_{3j(3)}$ touches c_1, c_2, c_3, contains c_3 and touches c_1, c_2 externally, while the circle $Q_{1k(1)}Q_{2k(2)}Q_{3k(3)}$ touches c_1, c_2, c_3, contains c_1, c_2, and touches c_3 externally.

Summing up: there are eight cycles touching each of c_1, c_2, c_3.

An Euclidean proof of the pertinence of this construction on circles can be rewritten also by hyperbolic terminology.

Gergonne's construction. Consider the cycles c' and c'' touching c_1, c_2 and c_3, in any of the four above described cases; in Fig. 5 c' touches each of c_1, c_2, c_3 externally, and c'' touches each of c_1, c_2, c_3 internally. Then the line joining the touching points $Q_{ij(i)}$ and $Q_{ik(i)}$ passes through one of the centers of similitude P of c' and c''. Thus P is the point of power of c_1, c_2 and c_3. On the other hand, two of the three given cycles (say c_1 and c_2) give a touching pair with respect to c' and c'', hence its outer center of similitude S_{12} has the same power with respect to c' and c''. So the three outer centers of similitude S_{12}, S_{13} and S_{23} are on the axis of power of c' and c''. (It is also (by definition) an axis of similitude with respect to c_1, c_2 and c_3, say s. For c', c'' being another pair of touching circles, in the other three cases, the respective changes have to be made in the choice.) Since the pole Q_i (with respect to the cycle c_i) of the line joining $Q_{ij(i)}$ and $Q_{ik(i)}$ is the intersection point of the common tangents of c' and c_i at $Q_{ij(i)}$, and c'' and c_i at $Q_{ik(i)}$, respectively, it is also on s. By the theorem of pole-polar we get that the pole P_i of s with respect to c_i lies on the line $Q_{ij(i)}Q_{ik(i)}$. This proves the construction. □

4.1. **Applications for triangle centers.**

4.1.1. *Staudtian and angular Staudtian of a hyperbolic triangle:* Let

$$n = n(ABC) := \sqrt{\sinh s \sinh(s-a)\sinh(s-b)\sinh(s-c)},$$

then we have

$$\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2} = \frac{n^2}{\sinh s \sinh a \sinh b \sinh c}. \tag{6}$$

The proof of this equality is the following. From (2) we get

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos\gamma = \cosh(a-b) + \sinh a \sinh b(1-\cos\gamma),$$

implying first that

$$\sin^2\frac{\gamma}{2} = \frac{1-\cos\gamma}{2} = \frac{-\cosh(a-b)+\cosh c}{2\sinh a \sinh b} = \frac{\sinh\frac{a-b+c}{2}\sinh\frac{-a+b+c}{2}}{\sinh a \sinh b} = \\ = \frac{\sinh(s-a)\sinh(s-b)}{\sinh a \sinh b},$$

and the statement follows immediately. Similarly we also have that

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos\gamma = \cosh(a-b) - \sinh a \sinh b(1+\cos\gamma),$$

implying that

$$\cos^2\frac{\gamma}{2} = \frac{1+\cos\gamma}{2} = \frac{1}{2}\frac{\cosh(a+b)-\cosh c}{\sinh a \sinh b} = \frac{\sinh s \sinh(s-c)}{\sinh a \sinh b}.$$

$$N = N(ABC) := \sqrt{\sin\delta \sin(\delta+\alpha)\sin(\delta+\beta)\sin(\delta+\gamma)}.$$

On the angular Staudtian we have analogous formulas as on the Staudtian. Use now the law of cosines on the angles. Then we have

$$\cos\gamma = -\cos\alpha\cos\beta + \sin\alpha\sin\beta\cosh c$$

and adding to this equation the addition formula of the cosine function we get that

$$\sin\alpha\sin\beta(\cosh c - 1) = \cos\gamma + \cos(\alpha+\beta) = 2\cos\frac{\alpha+\beta+\gamma}{2}\cos\frac{\alpha+\beta-\gamma}{2}.$$

From this we get that

$$\sinh\frac{c}{2} = \sqrt{\frac{\sin\delta\sin(\delta+\gamma)}{\sin\alpha\sin\beta}}. \tag{7}$$

Analogously we get that

$$\sin\alpha\sin\beta(\cosh c + 1) = \cos\gamma + \cos(\alpha-\beta) = 2\cos\frac{\alpha-\beta+\gamma}{2}\cos\frac{-\alpha+\beta+\gamma}{2},$$

implying that

$$\cosh\frac{c}{2} = \sqrt{\frac{\sin(\delta+\beta)\sin(\delta+\alpha)}{\sin\alpha\sin\beta}}. \tag{8}$$

From these we get

$$\cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2}=\frac{N^2}{\sin\alpha\sin\beta\sin\gamma\sin\delta}. \tag{9}$$

Finally we also have that

$$\sinh a=\frac{2N}{\sin\beta\sin\gamma},\quad \sinh b=\frac{2N}{\sin\alpha\sin\gamma},\quad \sinh c=\frac{2N}{\sin\alpha\sin\beta}, \tag{10}$$

and from the first equality of (13) we get that

$$N=\frac{1}{2}\sinh a\sin\beta\sin\gamma=\frac{1}{2}\sinh h_C\sin\gamma, \tag{11}$$

where h_C is the height of the triangle corresponding to the vertex C. The connection between the two Staudtians gives by the formula

$$2n^2=N\sinh a\sinh b\sinh c. \tag{12}$$

In fact, from (7) and (13) we get that

$$\sin\alpha\sinh a=\frac{4nN}{\sin\beta\sin\gamma\sinh b\sinh c}$$

implying that

$$\sin\alpha\sin\beta\sin\gamma\sinh a\sinh b\sinh c=4nN.$$

On the other hand from (7) we get immediately that

$$\sin\alpha\sin\beta\sin\gamma=\frac{8n^3}{\sinh^2 a\sinh^2 b\sinh^2 c}$$

and thus

$$2n^2=\sinh a\sinh b\sinh cN,$$

as we stated. The connection between the two types of the Staudtian can be understood if we dived to the first equality of (7) by the analogous one in (19). Then we have

$$\frac{\sin\alpha}{\sinh a}=\frac{n}{N}\frac{\sin\beta}{\sinh b}\frac{\sin\gamma}{\sinh c}$$

which using the hyperbolic theorem of sines leads to the equality

$$\frac{N}{n}=\frac{\sin\alpha}{\sinh a}. \tag{13}$$

4.1.2. *On the centroid (or median point) of a triangle.*

Theorem 1 (Theorem 4.3. in [8])**.** *We have the following formulas connected with the centroid:*

- **Property of equal Staudtians.**

$$n_A(M)=n_B(M)=n_C(M) \tag{14}$$

- **The ratio of section (AM_AM) depends on the vertex.**

$$\frac{\sinh AM}{\sinh MM_A}=2\cosh\frac{a}{2},\quad \frac{\sinh BM}{\sinh MM_B}=2\cosh\frac{b}{2},\quad \frac{\sinh CM}{\sinh MM_C}=2\cosh\frac{c}{2} \tag{15}$$

- **The ratio of section (AMM_A) is independent from the vertex.**

$$\frac{\sinh AM_A}{\sinh MM_A}=\frac{\sinh BM_B}{\sinh MM_B}=\frac{\sinh CM_C}{\sinh MM_C}=\frac{n}{n_A(M)}. \tag{16}$$

- **The "center of gravity" property of M.** *If y is any line of the plane then we have*

$$\sinh d'_M=\frac{\sinh d'_A+\sinh d'_B+\sinh d'_C}{\sqrt{1+2(1+\cosh a+\cosh b+\cosh c)}}. \tag{17}$$

where d'_A, d'_B, d'_C, d'_M mean the signed distances of the points A,B,C,M to the line y, respectively.

- **The "minimality" property of M.** *If Y is any point of the plane then we have*

$$\cosh YM=\frac{\cosh YA+\cosh YB+\cosh YC}{\frac{n}{n_A(M)}}=\frac{\cosh YA+\cosh YB+\cosh YC}{\sqrt{1+2(1+\cosh a+\cosh b+\cosh c)}}. \tag{18}$$

Proof. The property (17) is a simple consequence of (9). Thus the centroid is the unit point with respect to the triangular coordinate system. Let the feet of the perpendiculars from M and the altitudes are X_A, X_B, X_C, and H_A,H_B, H_C, respectively. (19) follows from (17) since

$$\frac{\sinh AM_A}{\sinh MM_A} = \frac{\sinh AH_A}{\sinh MX_A} = \frac{n}{n_A(M)} = \frac{n}{n_B(M)} = \frac{\sinh BM_B}{\sinh MM_B}.$$

From (1) we get

$$\frac{\sinh MM_A}{\sinh MC} = \frac{\sin M_ACM\measuredangle}{\sin CM_AA\measuredangle} \text{ and } \frac{\sinh AM}{\sinh MC} = \frac{\sin ACM\measuredangle}{\sin CAM_A\measuredangle}$$

implying

$$\frac{\sinh AM}{\sinh MM_A} = \frac{\sin ACM\measuredangle \sin CM_AA\measuredangle}{\sin M_ACM\measuredangle \sin CAM_A\measuredangle} = \frac{\sin ACM\measuredangle}{\sin M_ACM\measuredangle}\frac{\sinh b}{\sinh \frac{a}{2}}.$$

On the other hand the equalities

$$\frac{\sin ACM\measuredangle}{\sin CM_CA\measuredangle} = \frac{\sinh \frac{c}{2}}{\sinh b} \text{ and } \frac{\sin BCM\measuredangle}{\sin BM_CA\measuredangle} = \frac{\sinh \frac{c}{2}}{\sinh a}$$

imply the equalities

$$\frac{\sin ACM\measuredangle}{\sin M_ACM\measuredangle} = \frac{\sin ACM\measuredangle}{\sin BCM\measuredangle} = \frac{\sinh a}{\sinh b}.$$

Hence we get

$$\frac{\sinh AM}{\sinh MM_A} = \frac{\sinh a}{\sinh b}\frac{\sinh b}{\sinh \frac{a}{2}} = 2\cosh\frac{a}{2}$$

proving (18). To prove (21), observe that in the triangle ABC holds the equality

$$\cosh a + \cosh b = 2\cosh\frac{c}{2}\cosh CM_C. \tag{19}$$

In fact, the law of cosines (2) with respect to the triangles ACM_C and BCM_C gives the equalities

$$\cosh a = \cosh\frac{c}{2}\cosh MM_C - \sinh\frac{c}{2}\sinh MM_C\cos CM_CB\measuredangle$$

and

$$\cosh b = \cosh\frac{c}{2}\cosh MM_C - \sinh\frac{c}{2}\sinh MM_C\cos CM_CA\measuredangle =$$
$$= \cosh\frac{c}{2}\cosh MM_C + \sinh\frac{c}{2}\sinh MM_C\cos CM_CB\measuredangle.$$

Adding these equalities we give the required one. Hence we have

$$\cosh YA + \cosh YB = 2\cosh\frac{c}{2}\cosh YM_C.$$

Consider now the triangles YCM and YM_CM, respectively. Using the law of cosines as in the previous formula we have that

$$\cosh YC = \cosh MY\cosh MC - \sinh MY\sinh MC\cos YMC\measuredangle$$

and

$$\cosh YM_C = \cosh MY\cosh M_CM + \sinh MY\sinh M_CM\cos YMC\measuredangle.$$

From these equations we get

$$\sinh M_CM\cosh YC + \sinh MC\cosh YM_C =$$
$$= \cosh YM(\cosh MC\sinh M_CM + \cosh M_CM\sinh MC) = \cosh YM\sinh M_CC.$$

Now

$$\cosh YA + \cosh YB = 2\cosh\frac{c}{2}\left(\frac{\cosh YM\sinh M_CC}{\sinh MC} - \frac{\sinh M_CM\cosh YC}{\sinh MC}\right) =$$
$$= \frac{\sinh MC}{\sinh M_CM}\left(\frac{\cosh YM\sinh M_CC}{\sinh MC} - \frac{\sinh M_CM\cosh YC}{\sinh MC}\right) = \cosh YM\frac{\sinh M_CC}{\sinh M_CM} - \cosh YC,$$

proves the first equality of (21). The second equality in (21) can be gotten from the equations

$$\frac{\sinh CM_C}{\sinh MM_C} = \frac{n}{n_A(M)}, \quad \frac{\sinh(CM_C - MM_C)}{\sinh MM_C} = 2\cosh\frac{c}{2}, \quad \cosh a + \cosh b = 2\cosh\frac{c}{2}\cosh CM_C,$$

eliminating CM_C and MM_C between these equations. We leave the calculation to the reader.

Finally, consider the minimality property (21) in the case when Y is an ideal point and A, B, C are real ones, respectively. Now M is also a real point and we have to consider the polar of Y which is a real line

y. Denote by the real (and positive) geometric distances of the points A, B, C, M to y is d_A, d_B, d_C, d_M, respectively. (21) says that

$$\cosh\left(d_M + \varepsilon_M i\frac{\pi}{2}\right) = \frac{\cosh\left(d_A + \varepsilon_A i\frac{\pi}{2}\right) + \cosh\left(d_B + \varepsilon_B i\frac{\pi}{2}\right) + \cosh\left(d_C + \varepsilon_C i\frac{\pi}{2}\right)}{\sqrt{1+2(1+\cosh a+\cosh b+\cosh c)}},$$

where ε_M is a sign depending on the positions of Y, M and $Y_M := y \cap YM$ on its line YM. It is $+$ if the segment $MY_M \subset MY$ and $-$ if this relation does not hold. (Similar definition are valid for $\varepsilon_A, \varepsilon_B$ and ε_C, respectively.) It is clear that these signs give the same value if the corresponding points lie on the same half-plane of the line y. Thus if we fixed the sign of one of the points (which distinct to zero) then the other signs have to be determined uniquely, too. Hence we have the equality

$$\varepsilon_M \sinh d_M = \frac{\varepsilon_A \sinh d_A + \varepsilon_B \sinh d_B + \varepsilon_C \sinh d_C}{\sqrt{1+2(1+\cosh a+\cosh b+\cosh c)}}$$

or equivalently

$$\sinh d'_M = \frac{\sinh d'_A + \sinh d'_B + \sinh d'_C}{\sqrt{1+2(1+\cosh a+\cosh b+\cosh c)}}$$

as we stated in (20). □

4.1.3. *On the center of the circumscribed cycle.*

Theorem 2 (Theorem 4.6. in [8])**.** *The following formulas are valid on the circumradiuses R, R_A, R_B and R_C, respectively.*

- **Formulas by the angular Staudtian of the triangle are:**

$$\tanh R = \frac{\sin\delta}{N}, \quad \tanh R_A = \frac{\sin(\delta+\alpha)}{N}, \quad \tanh R_B = \frac{\sin(\delta+\beta)}{N}, \quad \tanh R_C = \frac{\sin(\delta+\gamma)}{N} \tag{20}$$

- **Formulas by the lengths of the edges are:**

$$\begin{aligned}
\tanh R &= \tanh\frac{a}{2}\tanh\frac{b}{2}\tanh\frac{c}{2}\cos\frac{\alpha+\beta+\gamma}{2} = \frac{2\sinh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}}{n} \\
\tanh R_A &= \tanh\frac{a}{2}\coth\frac{b}{2}\coth\frac{c}{2}\cos\frac{-\alpha+\beta+\gamma}{2} = \frac{2\sinh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2}}{n} \\
\tanh R_B &= \coth\frac{a}{2}\tanh\frac{b}{2}\coth\frac{c}{2}\cos\frac{\alpha-\beta+\gamma}{2} = \frac{2\cosh\frac{a}{2}\sinh\frac{b}{2}\cosh\frac{c}{2}}{n} \\
\tanh R_C &= \coth\frac{a}{2}\coth\frac{b}{2}\tanh\frac{c}{2}\cos\frac{\alpha+\beta-\gamma}{2} = \frac{2\cosh\frac{a}{2}\cosh\frac{b}{2}\sinh\frac{c}{2}}{n}
\end{aligned} \tag{21}$$

- **The ratio of the triangular coordinates of the circumcenter O is:**

$$n_A(0) : n_B(O) : n_C(O) = \cos(\delta+\alpha)\sinh a : \cos(\delta+\beta)\sinh b : \cos(\delta+\gamma)\sinh c \tag{22}$$

Proof. Assume that the radius CO divides the angle γ at C into the angles γ_1 and γ_2, respectively (see Fig. 6). Then we have $OCA\measuredangle = OAC\measuredangle = \gamma_1$, $OCB\measuredangle = OBC\measuredangle = \gamma_2$, hence $OAB\measuredangle = \alpha - \gamma_1$ and $OBA\measuredangle = \beta - \gamma_2$. Since $OAB\measuredangle = OBA\measuredangle$ we get that $OAB\measuredangle = \frac{1}{2}(\alpha+\beta-\gamma) = \pi/2 - (\delta+\gamma)$.

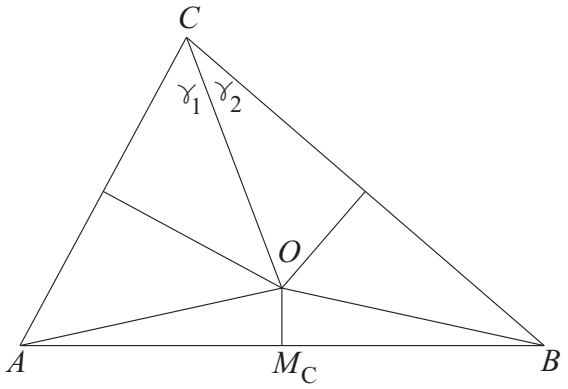

FIGURE 6. The circumcenter.

From this we get

$$\tanh\frac{c}{2}=\tanh R\cos(\pi/2-(\delta+\gamma))=\tanh R\sin(\delta+\gamma).$$

From (10) and (11) we get

$$\tanh\frac{c}{2}=\sqrt{\frac{\sin\delta\sin(\delta+\gamma)}{\sin(\delta+\beta)\sin(\delta+\alpha)}}$$

implying

$$\tanh R=\sqrt{\frac{\sin\delta}{\sin(\delta+\alpha)\sin(\delta+\beta)\sin(\delta+\gamma)}}.$$

From this the first equality in (23) immediately follows. Substituting $\alpha'=\alpha$, $\beta'=-\beta+\pi$, $\gamma'=-\gamma+\pi$ into the first equation of (23) and using that $\delta'=(\pi-(\alpha-\beta-\gamma+2\pi))/2=(-\alpha+\beta+\gamma-\pi)/2=-(\delta+\alpha)$ we get the formula of (23) on R_A:

$$\tanh R_A=\sqrt{\frac{-\sin(\delta+\alpha)}{\sin(-\delta)\sin(\pi-\delta-\beta-\alpha)\sin(\pi-\delta-\gamma-\alpha)}}=\sqrt{\frac{\sin(\delta+\alpha)}{\sin\delta\sin(\delta+\gamma)\sin(\delta+\beta)}}=\frac{\sin(\delta+\alpha)}{N}.$$

Analogously as of (16) or (17) we have the formulas

$$\sinh\frac{a}{2}=\sqrt{\frac{\sin\delta\sin(\delta+\alpha)}{\sin\gamma\sin\beta}}\ \text{ and }\ \sinh\frac{b}{2}=\sqrt{\frac{\sin\delta\sin(\delta+\beta)}{\sin\alpha\sin\gamma}},$$

and

$$\cosh\frac{a}{2}=\sqrt{\frac{\sin(\delta+\beta)\sin(\delta+\gamma)}{\sin\gamma\sin\beta}}\ \text{ and }\ \cosh\frac{b}{2}=\sqrt{\frac{\sin(\delta+\gamma)\sin(\delta+\alpha)}{\sin\alpha\sin\gamma}}.$$

Thus we have

$$\frac{\sinh\frac{a}{2}}{\cosh\frac{b}{2}\cosh\frac{c}{2}}=\sqrt{\frac{\sin^2\alpha\sin\delta}{\sin(\delta+\gamma)\sin(\delta+\alpha)\sin(\delta+\beta)}}=\sin\alpha\tanh R$$

giving the formula

$$\tanh R=\frac{\sinh\frac{a}{2}}{\sin\alpha\cosh\frac{b}{2}\cosh\frac{c}{2}}. \tag{23}$$

Similarly we get

$$\sinh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}=\sqrt{\frac{\sin^3\delta\sin(\delta+\alpha)\sin(\delta+\beta)\sin(\delta+\gamma)}{\sin^2\alpha\sin^2\beta\sin^2\gamma}}=\frac{\sin^2\delta\coth R}{\sin\alpha\sin\beta\sin\gamma},$$

and with the same manner we have

$$\cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2}=\sqrt{\frac{\sin^2(\delta+\alpha)\sin^2(\delta+\beta)\sin^2(\delta+\gamma)}{\sin^2\alpha\sin^2\beta\sin^2\gamma}}=\frac{\sin\delta\coth^2 R}{\sin\alpha\sin\beta\sin\gamma}.$$

Dividing by the two equalities we get the first equality of the first row in (24):

$$\tanh R=\tanh\frac{a}{2}\tanh\frac{b}{2}\tanh\frac{c}{2}\sin\delta.$$

Using (7) and (14) we also have that

$$\sin\alpha\sin\beta\sin\gamma=\frac{8n^3}{\sinh^2 a\sinh^2 b\sinh^2 c}=\frac{8n^3N^2}{4n^4}=\frac{2N^2}{n} \tag{24}$$

giving immediately the second equality of the first row in (24)

$$\sinh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}=\frac{\sin^2\delta\coth R}{\sin\alpha\sin\beta\sin\gamma}=\frac{n\sin^2\delta\coth R}{2N^2}=\frac{n\tanh R}{2}.$$

Substituting the complementary lengths and (the same) angles (if it is necessary) to these equations we get the results of the remaining rows in (24).

By (8) we have that

$$n(AOB)=\frac{1}{2}\sin\frac{\alpha+\beta-\gamma}{2}\sinh R\sinh c$$

and

$$n(BOC)=\frac{1}{2}\sin\frac{-\alpha+\beta+\gamma}{2}\sinh R\sinh a.$$

Hence

$$n_A(0) : n_B(O) : n_C(O) = \sin\frac{-\alpha+\beta+\gamma}{2}\sinh a : \sin\frac{\alpha-\beta+\gamma}{2}\sinh b : \sin\frac{\alpha+\beta-\gamma}{2}\sinh c,$$

as we stated in (25). □

4.1.4. *On the center of the inscribed and escribed cycles.* We are aware that the bisectors of the interior angles of a hyperbolic triangle are concurrent at a point I, called the incenter, which is equidistant from the sides of the triangle. The radius of the *incircle* or *inscribed circle*, whose center is at the incenter and touches the sides, shall be designated by r. Similarly the bisector of any interior angle and those of the exterior angles at the other vertices, are concurrent at point outside the triangle; these three points are called *excenters*, and the corresponding tangent cycles *excycles* or *escribed cycles*. The excenter lying on AI is denoted ba I_A, and the radius of the escribed cycle with center at I_A is r_A. We denote by X_A, X_B, X_C the points where the interior bisectors meets BC, AC, AB, respectively. Similarly Y_A, Y_B and Y_C denote the intersection of the exterior bisector at A, B and C with BC, AC and AB, respectively. We

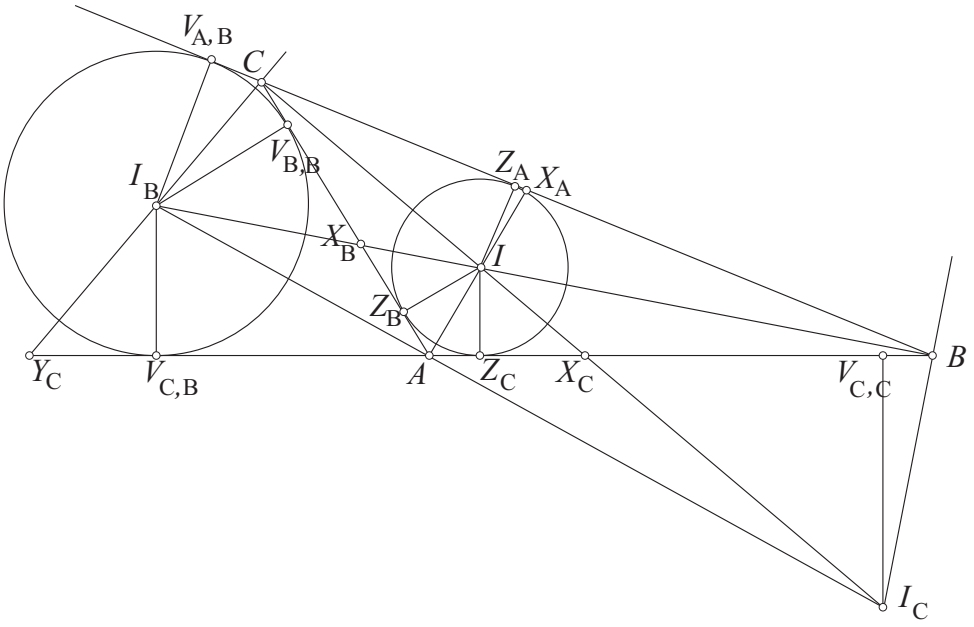

FIGURE 7. Incircles and excycles.

note that the excenters and the points of intersection of the sides with the bisectors of the corresponding exterior angle could be points at infinity or also could be ideal points. Let denote the touching points of the incircle Z_A, Z_B and Z_C on the lines BC, AC and AB, respectively and the touching points of the excycles with center I_A, I_B and I_C are the triples $\{V_{A,A}, V_{B,A}, V_{C,A}\}$, $\{V_{A,B}, V_{B,B}, V_{C,B}\}$ and $\{V_{A,C}, V_{B,C}, V_{C,C}\}$, respectively (see in Fig. 13).

Theorem 3 (Theorem 4.10. in [8])**.** *On the radiuses r, r_A, r_B or r_C we have the following formulas .*

- **Formulas by Staudtian are:**

$$\tanh r = \frac{n}{\sinh s}, \quad \tanh r_A = \frac{n}{\sinh(s-a)}, \quad \tanh r_B = \frac{n}{\sinh(s-b)}, \quad \tanh r_C = \frac{n}{\sinh(s-c)} \tag{25}$$

- **Formulas by angular Staudtian are**

$$\tanh r = \frac{N}{2\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}, \tag{26}$$

$$\coth r = \frac{\sin(\delta+\alpha)+\sin(\delta+\beta)+\sin(\delta+\gamma)+\sin\delta}{2N} \tag{27}$$

$$\coth r_A = \frac{-\sin(\delta+\alpha)+\sin(\delta+\beta)+\sin(\delta+\gamma)-\sin\delta}{2N} \tag{28}$$

$$\coth r_B = \frac{\sin(\delta+\alpha)-\sin(\delta+\beta)+\sin(\delta+\gamma)-\sin\delta}{2N}$$

$$\coth r_C = \frac{\sin(\delta+\alpha)+\sin(\delta+\beta)-\sin(\delta+\gamma)-\sin\delta}{2N}$$

- **Connections among the circumradiuses and inradiuses are:**

$$\begin{aligned} \tanh R+\tanh R_A &= \coth r_B+\coth r_C \\ \tanh R_B+\tanh R_C &= \coth r+\coth r_A \\ \tanh R+\coth r &= \frac{1}{2}\left(\tanh R+\tanh R_A+\tanh R_B+\tanh R_C\right) \end{aligned} \tag{29}$$

- **Triangular coordinates of the incenter and excenters are:**

$$n_A(I):n_B(I):n_C(I) = \sinh a:\sinh b:\sinh c \tag{30}$$

$$\begin{aligned} n_A(I_A):n_B(I_A):n_C(I_A) &= -\sinh a:\sinh b:\sinh c \\ n_A(I_B):n_B(I_B):n_C(I_B) &= \sinh a:-\sinh b:\sinh c \\ n_A(I_C):n_B(I_C):n_C(I_C) &= \sinh a:\sinh b:-\sinh c \end{aligned} \tag{31}$$

Proof. The triangular coordinates of I by (8) holds

$$n_A(I):n_B(I):n_C(I) = \sinh a:\sinh b:\sinh c$$

proving (35). To (36) we observe that the excircle with center I_B can be considered as the incircle of those triangle of the vertex set $\{A,B,C\}$ which edge-segment AC is equal to that of the corresponding edge-segment of the triangle ABC while the other two edge-segments are complementary to those of ABC. (In spherical geometry the above two triangle is called *colunar* because of their union is a lune.) We also have that the sign of the measure of the radius in one of the cases is the negative as the sign of the corresponding case of the incircle because of the side separates the two centers. Thus

$$n_A(I_B):n_B(I_B):n_C(I_C) = \sinh(-a+\pi i):-\sinh b:\sinh(-c+\pi i) = \sinh a:-\sinh b:\sinh c,$$

implying (36).

The equalities in (30) follows from the observation that we have $CZ_A = CZ_B = s-c$, $BZ_A = BZ_C = s-b$ and $AZ_B = AZ_C = s-a$, respectively, and thus

$$\tan\frac{\gamma}{2} = \frac{\tanh r}{\sinh(s-c)}.$$

In fact, $\sin\frac{\gamma}{2}$ and $\cos\frac{\gamma}{2}$ was calculated before (7) and from these quantities we get that

$$\tan\frac{\gamma}{2} = \sqrt{\frac{\sinh(s-a)\sinh(s-b)}{\sinh s\sinh(s-c)}} \tag{32}$$

Implying the first equality in (30). The other equalities follow from that the circumscribed triangles of the excycles have two sides with the property that its measure is the measure of the corresponding side of ABC subtracting from πi. More precisely the lengths of the sides of the circumscribed triangle of the excycle corresponding to the excenter I_B are $a' = -a+\pi i$, $b' = b$, and $c' = -c+\pi i$, respectively. The corresponding half-perimeter is $s' = (a'+b'+c')/2 = (-a+b-c)/2+\pi i$. This implies that

$$\tanh r_B = \sqrt{\frac{\sinh(s'-a')\sinh(s'-b')\sinh(s'-c')}{\sinh s'}} =$$

$$= \sqrt{\frac{\sinh(s-c)\sinh(-s+\pi i)\sinh(s-a)}{\sinh(-s+b+\pi i)}} = \frac{n}{\sinh(s-b)},$$

as we stated in (30).

Since we proved before (7) that

$$\cos\frac{\alpha}{2} = \sqrt{\frac{\sinh s\sinh(s-a)}{\sinh c\sinh b}}, \quad \cos\frac{\beta}{2} = \sqrt{\frac{\sinh s\sinh(s-b)}{\sinh a\sinh c}}, \quad \cos\frac{\gamma}{2} = \sqrt{\frac{\sinh s\sinh(s-c)}{\sinh a\sinh b}}, \tag{33}$$

then we have by (15) and (30) that

$$\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}=\sqrt{\frac{\sinh^3 s\sinh(s-a)\sinh(s-b)\sinh(s-c)}{\sinh^2 a\sinh^2 b\sinh^2 c}}=$$
$$=\frac{n\sinh s}{\sinh a\sinh b\sinh c}=\frac{N\sinh a}{2n}=\frac{N}{2\tanh r}$$

and (31) follows, too.

To prove (32) consider the equalities

$$\sin(\delta+\alpha)+\sin(\delta+\beta)=\cos\frac{-(\alpha-\beta)+\gamma}{2}+\cos\frac{(\alpha-\beta)+\gamma}{2}=2\cos\frac{\alpha-\beta}{2}\cos\frac{\gamma}{2}=$$
$$=2\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}+2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\gamma}{2},$$

and

$$\sin(\delta+\gamma)-\sin\delta=\cos\frac{(\alpha+\beta)-\gamma}{2}-\cos\frac{(\alpha+\beta)+\gamma}{2}=2\cos\frac{\gamma}{2}\cos\frac{\alpha+\beta}{2}=$$
$$=2\cos\frac{\gamma}{2}\cos\frac{\alpha}{2}\cos\frac{\beta}{2}-2\cos\frac{\gamma}{2}\sin\frac{\alpha}{2}\sin\frac{\beta}{2}.$$

Thus we get the equality

$$4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}=\sin(\delta+\alpha)+\sin(\delta+\beta)+\sin(\delta+\gamma)+\sin(\delta)$$

implying (32). The equations in (33) follow from (32) substituting two times $(\pi-\phi)$ into ϕ ($\phi=\alpha,\beta$ or $\phi=\gamma$).

Finally, (23), (32) and (33)implies the equalities in (34). □

The following formulas connect the radiuses of the circles and the lengths of the edges of the triangle.

Theorem 4. *Let a,b,c,s,r_A,r_B,r_C,r,R be the values defined for a hyperbolic triangle above. Then we have the following formulas:*

(34) $$-\coth r_A-\coth r_B-\coth r_C+\coth r=2\tanh R$$

(35) $$\coth r_A\coth r_B+\coth r_A\coth r_C+\coth r_B\coth r_C=$$
$$=\frac{1}{\sinh s\sinh(s-a)}+\frac{1}{\sinh s\sinh(s-b)}+\frac{1}{\sinh s\sinh(s-c)}$$

(36) $$\tanh r_A\tanh r_B+\tanh r_A\tanh r_C+\tanh r_B\tanh r_C=$$
$$=\frac{1}{2}\left(\cosh(a+b)+\cosh(a+c)+\cosh(b+c)-\cosh a-\cosh b-\cosh c\right)$$

(37) $$\coth r_A+\coth r_B+\coth r_C=$$
$$=\frac{1}{\tanh r}\left(\cosh a+\cosh b+\cosh c-\coth s\left(\sinh a+\sinh b+\sinh c\right)\right)$$

(38) $$\tanh r_A+\tanh r_B+\tanh r_C=$$
$$=\frac{1}{2\tanh r}\left(\cosh a+\cosh b+\cosh c-\cosh(b-a)-\cosh(c-a)-\cosh(c-b)\right)$$

(39) $$2(\sinh a\sinh b+\sinh a\sinh c+\sinh b\sinh c)=$$
$$+\quad\tanh r\left(\tanh r_A+\tanh r_B+\tanh r_C\right)+\tanh r_A\tanh r_B+\tanh r_A\tanh r_C+\tanh r_B\tanh r_C$$

Proof. From (32),(33) and (23) we get that

$$-\coth r_A-\coth r_B-\coth r_C+\coth r=2\frac{\sin\delta}{N}=2\tanh R,$$

as we stated in (39).

To prove (40) consider the equalities in (30) from which

$$\coth r_A\coth r_B+\coth r_A\coth r_C+\coth r_B\coth r_C=$$
$$=\frac{\sinh(s-a)\sinh(s-b)+\sinh(s-a)\sinh(s-c)+\sinh(s-c)\sinh(s-b)}{n^2}=$$

$$\frac{1}{\sinh s\sinh(s-a)}+\frac{1}{\sinh s\sinh(s-b)}+\frac{1}{\sinh s\sinh(s-c)}$$

Similarly we also get (41):

$$\tanh r_A\tanh r_B+\tanh r_A\tanh r_C+\tanh r_B\tanh r_C=\sinh s\sinh(s-a)+\sinh s\sinh(s-b)+$$
$$+\sinh s\sinh(s-c)=\frac{1}{2}\left(\cosh(a+b)+\cosh(a+c)+\cosh(b+c)-\cosh a-\cosh b-\cosh c\right).$$

Since we have

$$-2\tanh R+\coth r=\coth r_A+\coth r_B+\coth r_C=\frac{\sinh(s-a)+\sinh(s-b)+\sinh(s-c)}{n}=$$
$$=\frac{(\sinh(s-a)+\sinh(s-b)+\sinh(s-c))}{\sinh s\tanh r}=\frac{\cosh a+\cosh b+\cosh c-\coth s\left(\sinh a+\sinh b+\sinh c\right)}{\tanh r}$$

(42) is given. Furthermore we also have

$$\tanh r_A+\tanh r_B+\tanh r_C=$$
$$=\frac{n\left(\sinh(s-a)\sinh(s-b)+\sinh(s-a)\sinh(s-c)+\sinh(s-b)\sinh(s-c)\right)}{\sinh(s-a)\sinh(s-b)\sinh(s-c)}=$$
$$=\frac{\sinh s}{n}\left(\sinh(s-a)\sinh(s-b)+\sinh(s-a)\sinh(s-c)+\sinh(s-b)\sinh(s-c)\right)=$$
$$=\frac{\left(\sinh(s-a)\sinh(s-b)+\sinh(s-a)\sinh(s-c)+\sinh(s-b)\sinh(s-c)\right)}{\tanh r}=$$
$$=\frac{1}{2\tanh r}\left(\cosh a+\cosh b+\cosh c-\cosh(b-a)-\cosh(c-a)-\cosh(c-b)\right)$$

implying (43). From (41) and (43) we get

$$\tanh r\left(\tanh r_A+\tanh r_B+\tanh r_C\right)+\tanh r_A\tanh r_B+\tanh r_A\tanh r_C+\tanh r_B\tanh r_C=$$
$$=\cosh(a+b)+\cosh(a+c)+\cosh(b+c)-\cosh(b-a)-\cosh(c-a)-\cosh(c-b)=$$
$$=2(\sinh a\sinh b+\sinh a\sinh c+\sinh b\sinh c)$$

which implies (44). □

The following theorem gives a connection among the distance of the incenter and circumcenter, the radiuses r, R and the side-lengths a, b, c .

Theorem 5 (Theorem 4.11. in [8])**.** *Let O and I the center of the circumsrcibed and inscribed circles, respectively. Then we have*

$$\cosh OI=2\cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2}\cosh r\cosh R+\cosh\frac{a+b+c}{2}\cosh(R-r). \tag{40}$$

Proof. Since

$$\cosh(s-a)\cosh r=\cosh AI \text{ and } IAO\measuredangle=\frac{\alpha}{2}-\frac{\alpha+\beta-\gamma}{2}=\frac{-\beta+\gamma}{2}$$

thus from (2) we get that

$$\cosh OI=\cosh AI\cosh R-\sinh AI\sinh R\cos\frac{-\beta+\gamma}{2}.$$

Hence holds the equality

$$\cosh OI=\cosh(s-a)\cosh r\cosh R-\sinh r\sinh R\frac{\cos\frac{-\beta+\gamma}{2}}{\sin\frac{\alpha}{2}}.$$

Analogously to the proof of (6) we get that

$$\frac{\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}{\sin\frac{\alpha}{2}}=\sqrt{\frac{\frac{\sinh s\sinh(s-b)}{\sinh a\sinh c}\frac{\sinh s\sinh(s-c)}{\sinh a\sinh b}}{\frac{\sinh(s-b)\sinh(s-c)}{\sinh b\sinh c}}}=\frac{\sinh s}{\sinh a}$$

and also we have

$$\frac{\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}{\sin\frac{\alpha}{2}}=\sqrt{\frac{\frac{\sinh(s-a)\sinh(s-c)}{\sinh a\sinh c}\frac{\sinh(s-a)\sinh(s-b)}{\sinh a\sinh b}}{\frac{\sinh(s-b)\sinh(s-c)}{\sinh b\sinh c}}}=\frac{\sinh(s-a)}{\sinh a}.$$

Summing up we get that

$$\cosh OI = \cosh(s-a)\cosh r\cosh R - \sinh r\sinh R\frac{\sinh s+\sinh(s-a)}{\sinh a} =$$
$$= \cosh(s-a)\cosh r\cosh R - 2\sinh r\sinh R\frac{\sinh\frac{b+c}{2}\cosh\frac{a}{2}}{\sinh a} =$$
$$= \cosh\frac{-a+b+c}{2}\cosh r\cosh R - \sinh r\sinh R\frac{\sinh\frac{b+c}{2}}{\sinh\frac{a}{2}},$$

and also the similar formula

$$\cosh OI = \cosh\frac{a-b+c}{2}\cosh r\cosh R - \sinh r\sinh R\frac{\sinh\frac{a+c}{2}}{\sinh\frac{b}{2}}$$

and

$$\cosh OI = \cosh\frac{a+b-c}{2}\cosh r\cosh R - \sinh r\sinh R\frac{\sinh\frac{a+b}{2}}{\sinh\frac{c}{2}}.$$

Adding now the latter three formulas we get that

$$3\cosh OI = \left(\cosh\frac{-a+b+c}{2}+\cosh\frac{a-b+c}{2}+\cosh\frac{a+b-c}{2}\right)\cosh r\cosh R -$$
$$-\sinh r\sinh R\left(\frac{\sinh\frac{b+c}{2}}{\sinh\frac{a}{2}}+\frac{\sinh\frac{a+c}{2}}{\sinh\frac{b}{2}}+\frac{\sinh\frac{a+b}{2}}{\sinh\frac{c}{2}}\right).$$

Since

$$\cosh\frac{-a+b+c}{2} = \left(\cosh\frac{b+c}{2}\cosh\frac{a}{2}-\sinh\frac{b+c}{2}\sinh\frac{a}{2}\right) =$$
$$= \cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2}+\cosh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}-\sinh\frac{a}{2}\sinh\frac{b}{2}\cosh\frac{c}{2}-\sinh\frac{a}{2}\cosh\frac{b}{2}\sinh\frac{c}{2},$$

thus

$$\cosh\frac{-a+b+c}{2}+\cosh\frac{a-b+c}{2}+\cosh\frac{a+b-c}{2} =$$
$$= 3\cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2}-\cosh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}-\sinh\frac{a}{2}\sinh\frac{b}{2}\cosh\frac{c}{2}-\sinh\frac{a}{2}\cosh\frac{b}{2}\sinh\frac{c}{2}.$$

We also have that

$$\frac{\sinh\frac{b+c}{2}}{\sinh\frac{a}{2}}+\frac{\sinh\frac{a+c}{2}}{\sinh\frac{b}{2}}+\frac{\sinh\frac{a+b}{2}}{\sinh\frac{c}{2}} = \frac{\sinh\frac{b+c}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}+\sinh\frac{a+c}{2}\sinh\frac{a}{2}\sinh\frac{c}{2}+\sinh\frac{a+b}{2}\sinh\frac{a}{2}\sinh\frac{b}{2}}{\sinh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}}$$

and since

$$\sinh\frac{b+c}{2}\sinh\frac{b}{2}\sinh\frac{c}{2} = \sinh\left(s-\frac{a}{2}\right)\sinh\frac{b}{2}\sinh\frac{c}{2} =$$
$$= \sinh s\cosh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}-\cosh s\sinh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}$$

we get that

$$\frac{\sinh\frac{b+c}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}+\sinh\frac{a+c}{2}\sinh\frac{a}{2}\sinh\frac{c}{2}+\sinh\frac{a+b}{2}\sinh\frac{a}{2}\sinh\frac{b}{2}}{\sinh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}} =$$
$$= \left(\sinh s\cosh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}+\sinh s\sinh\frac{a}{2}\cosh\frac{b}{2}\sinh\frac{c}{2}+\sinh s\sinh\frac{a}{2}\sinh\frac{b}{2}\cosh\frac{c}{2}-\right.$$
$$\left.-3\cosh s\sinh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}\right)\frac{1}{\sinh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}}.$$

Using (46) we get that

$$\frac{\sinh\frac{b+c}{2}}{\sinh\frac{a}{2}}+\frac{\sinh\frac{a+c}{2}}{\sinh\frac{b}{2}}+\frac{\sinh\frac{a+b}{2}}{\sinh\frac{c}{2}} =$$
$$= \frac{2\left(\cosh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}+\sinh\frac{a}{2}\cosh\frac{b}{2}\sinh\frac{c}{2}+\sinh\frac{a}{2}\sinh\frac{b}{2}\cosh\frac{c}{2}\right)}{\tanh r\tanh R}-3\cosh s.$$

Thus we have

$$3\cosh OI = 3\left(\cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2}-\cosh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}-\right.$$
$$\left.-\sinh\frac{a}{2}\sinh\frac{b}{2}\cosh\frac{c}{2}-\sinh\frac{a}{2}\cosh\frac{b}{2}\sinh\frac{c}{2}\right)\cosh r\cosh R+3\cosh s\sinh r\sinh R$$

implying that

$$\cosh OI = \left(2\cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2} - \cosh s\right)\cosh r\cosh R + \cosh s\sinh r\sinh R =$$
$$= 2\cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2}\cosh r\cosh R + \cosh s\cosh(R-r) =$$
$$= 2\cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2}\cosh r\cosh R + \cosh\frac{a+b+c}{2}\cosh(R-r),$$

as we stated in (45). □

Remark. The second order approximation of (45) leads to the equality

$$1+\frac{OI^2}{2} = 2\left(1+\frac{r^2}{2}\right)\left(1+\frac{R^2}{2}\right)\left(1+\frac{a^2}{8}\right)\left(1+\frac{b^2}{8}\right)\left(1+\frac{c^2}{8}\right) - \left(1+\frac{(a+b+c)^2}{8}\right)\left(1+\frac{(R-r)^2}{2}\right).$$

From this we get that

$$OI^2 = R^2 + r^2 + \frac{a^2+b^2+c^2}{4} - \frac{ab+bc+ca}{2} + 2Rr.$$

But for Euclidean triangles we have (see [1])

$$a^2+b^2+c^2 = 2s^2 - 2(4R+r)r \text{ and } ab+bc+ca = s^2+(4R+r)r,$$

the equality above leads to the Euler's formula:

$$OI^2 = R^2 - 2rR.$$

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