Stochastic Models — Third HW problem set

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The number of dots • is the value of an exercise. **Hand in solutions for 12 points by June 8 Thu 4:15 pm.** If you have seriously tried to solve some problem, but got stuck, I will be happy to help. Also, if your final solution to a problem has some mistake but has some potential to work, then I will give it back and you can try and correct the mistake.

Recall the following definition. Let G_n be a sequence of finite graphs. Pick a uniform random root ρ_n from $V(G_n)$, and take the ball $B_{G_n,\rho_n}(r)$ around it in the graph metric, with some fixed radius $r \in \mathbb{Z}_+$. We get a distribution $\mu_{n,r}$ on finite rooted graphs. We say that the sequence $\{G_n\}$ converges in the **Benjamini-Schramm sense** (also called **local weak convergence**) to a random rooted graph (G, ρ), if, for every r, the distributions $\mu_{n,r}$ converge weakly as $n \to \infty$ to the distribution of $B_{G,\rho}(r)$. The simplest case is when the limit is a transitive infinite graph G: the measures $\mu_{n,r}$ converge to the Dirac measure on a single graph, the r-ball of G. The following exercise generalizes the examples of boxes in $\mathbb{Z}^d$ and balls in the d-regular tree $\mathbb{T}_d$ that we saw on class:

▷ **Exercise 1.••** Show that a transitive graph G has a sequence G_n of *subgraphs* converging to it in the Benjamini-Schramm (local weak) sense if and only if it is amenable.

▷ **Exercise 2.•** Assuming Exercise 21 from HW2, show that our random d-regular bipartite multi-graph $M_{n,n,d}$ converges to the d-regular tree $\mathbb{T}_d$ in the Benjamini-Schramm sense. (Here the randomness for the measure $\mu_{n,r}$ comes from two sources: we take a random root ρ_n in the random graph G_n, and want to show convergence in this joint probability space.)

▷ **Exercise 3.••** Let G_n be the Erdős-Rényi random graph $G(n, \lambda/n)$, with any $\lambda \in \mathbb{R}_+$ fixed. Show that the Benjamini-Schramm limit of G_n is the $\mathsf{PGW}(\lambda)$ tree, the Galton-Watson tree with offpsring $\mathsf{Poisson}(\lambda)$, rooted as normally. (As in the previous exercise, convergence in the joint probability space of choosing the random graph and the random root.)

▷ **Exercise 4.•** Show that the functions $v_k(t) := \frac{k^{k-1}}{k!} e^{-kt} t^{k-1}$ for $k = 1, 2, \dots$ indeed satisfy the Smoluchowski coagulation equations

$$\frac{d}{dt} v_k(t) = -k v_k(t) + \frac{k}{2} \sum_{\ell=1}^{k-1} v_\ell(t) v_{k-\ell}(t)\,.$$

▷ **Exercise 5.•** Can it happen for some iid sequence $X_1, X_2, \dots$ that $(X_1 + \cdots + X_n)/a_n$ almost surely converges, for some sequence $a_n \to \infty$ (e.g., $a_n = n$ or $a_n = \sqrt{n}$) to a random variable that is not an almost sure constant? (Hint: think of Kolmogorov's 0/1 law.)

▷ **Exercise 6.••** Give an example of a random sequence $(M_n)_{n=0}^\infty$ such that $\mathbf{E}[\, M_{n+1} \mid M_n \,] = M_n$ for all $n \geq 0$, but which is not a martingale (in its natural filtration $\mathcal{F}_n = \sigma\{M_0, \dots, M_n\}$).

▷ **Exercise 7.** Consider a Galton-Watson tree with offspring distribution ξ, with $\mathbf{E}\xi = \mu$. Let Z_n be the size of the nth level, with $Z_0 = 1$, the root.

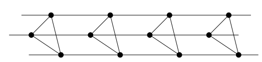
(a)• Find $\mathbf{E}[\, Z_n \,]$, and using this show that $\mu < 1$ implies that the GW tree is finite almost surely.

(b)$^{\bullet\bullet}$ Extending the previous part, show that Z_n/μ^n is a martingale, and using a Martingale Convergence Theorem (non-negative martingales converge almost surely to some almost surely finite variable), show that $\mu = 1$ with $\mathbf{P}[\,\xi = 1\,] \neq 1$ also implies that the GW tree is finite almost surely.

(c)$^{\bullet\bullet}$ If $\mu > 1$ and $\mathbf{E}[\,\xi^2\,] < \infty$, show that $\mathbf{E}\big[\,Z_n^2\,\big] \leq C(\mathbf{E}Z_n)^2$ with a constant $C < \infty$ that does not depend on n. (Hint: use the conditional variance formula $\mathbf{D}^2[Z_n] = \mathbf{E}\big[\,\mathbf{D}^2[Z_n \mid Z_{n-1}]\big] + \mathbf{D}^2\big[\mathbf{E}[\,Z_n \mid Z_{n-1}\,]\big]$.) Using this and the **Second Moment Method**, namely, if $X \geq 0$ a.s., then $\mathbf{P}[\,X > 0\,] \geq (\mathbf{E}X)^2/\mathbf{E}[\,X^2\,]$ (you can look this up, e.g., in PGG Section 12.3), deduce that the GW tree is infinite with positive probability.

(d)$^{\bullet}$ Extend the previous part to the case $\mathbf{E}\xi = \infty$ or $\mathbf{D}\xi = \infty$ by a truncation $\xi \mathbf{1}_{\xi < K}$ for K large enough.

▷ **Exercise 8.**$^{\bullet}$ What is the critical bond percolation density for the infinite triangular ladder?



▷ **Exercise 9.**$^{\bullet}$ We saw in class for the binary tree $\mathbb{T}$ that $p_c(\mathbb{T}) = 1/2$. Using this, show that the 3-regular tree has $p_c(\mathbb{T}_3) = 1/2$, as well.

▷ **Exercise 10.** Consider site percolation on $\mathbb{Z}^2$; i.e., instead of deleting or keeping the edges (bonds), we are keeping or deleting the vertices. Show that $1/3 \leq p_c(\mathbb{Z}^2, \text{site}) \leq 5/6$.

For Bernoulli bond percolation on any connected infinite graph G, any $o \in V(G)$, define

$$p_T := \inf\big\{p : \mathbf{E}_p[\,|\mathscr{C}_o|\,] = \infty\big\},$$

where $\mathscr{C}_o$ denotes the cluster of vertex o. T is for the honour of Temperley. As for the critical density p_c defined in class, one can show that this does not depend on o. Obviously, $p_T \leq p_c$ for any graph.

▷ **Exercise 11.** Consider Bernoulli bond percolation on the canopy tree Λ (the Benjamini-Schramm limit of the balls $B_n(o)$ in the 3-regular tree $\mathbb{T}_3$).

(a)$^{\bullet}$ Show that $p_c(\Lambda) = 1$.

(b)$^{\bullet\bullet}$ Find $p_T(\Lambda)$.

As in class, the **Ising model** on a finite graph $G(V, E)$ is the random spin configuration $\sigma : V \longrightarrow \{\pm 1\}$ defined as follows. Given an external magnetic field $h \in \mathbb{R}$, the Hamiltonian is

$$H_h(\sigma) := -h \sum_{x \in V(G)} \sigma(x) - \sum_{(x,y) \in E(G)} \sigma(x)\sigma(y)\,,$$

and then the measure, at inverse temperature $\beta = 1/T \geq 0$, is

$$\mathbf{P}_{\beta,h}[\sigma] := \frac{\exp(-\beta H_h(\sigma))}{Z_{\beta,h}}\,, \quad \text{where} \quad Z_{\beta,h} := \sum_{\sigma} \exp(-\beta H_h(\sigma))\,.$$

▷ **Exercise 12.** The partition function $Z_{\beta,h}$ contains a lot of information about the model:

(a)$^{\bullet}$ Show that the **expected total energy** is

$$\mathbf{E}_{\beta,h}[\,H\,] = -\frac{\partial}{\partial \beta} \ln Z_{\beta,h}\,, \text{ with variance } \mathrm{Var}_{\beta,h}[H] = -\frac{\partial}{\partial \beta} \mathbf{E}_{\beta,h}[\,H\,]\,.$$

(b)$^{\bullet}$ The **average free energy** or **pressure** is defined by $f(\beta, h) := (\beta|V|)^{-1} \ln Z_{\beta,h}$. Show that for the **average total magnetization** $M(\sigma) := |V|^{-1} \sum_{x \in V} \sigma(x)$, we have

$$m(\beta, h) := \mathbf{E}_{\beta,h}[\,M\,] = \frac{\partial}{\partial h} f(\beta, h)\,.$$

(c)$^\bullet$ The **susceptibility** of the total magnetization to a change in the external magnetic field is

$$\chi(\beta,h) := \frac{1}{\beta}\frac{\partial}{\partial h} m(\beta,h) = \frac{1}{\beta}\frac{\partial^2}{\partial h^2} f(\beta,h)\,.$$

Relate this quantity to $\mathrm{Var}_{\beta,h}[M]$. Deduce that $f(\beta,h)$ is convex in h.

▷ **Exercise 13.** Consider the Ising model on an interval, $\{\sigma_i : i = -n,\dots,n-1,n\}$, with no boundary condition, at any inverse temperature $\beta \in [0,\infty)$.

(a)$^\bullet$ Show that $\{\sigma_i : i = -n,\dots,n-1,n\}$ is the trajectory of a stationary irreducible Markov chain on $\{-,+\}$.

(b)$^{\bullet\bullet}$ Show that $S_n := \sum_{i=-n}^{n} \sigma_i$ has $\mathrm{Var}[S_n] \sim C_\beta n$, as $n \to \infty$, for some $C_\beta \in (0,\infty)$. That is, in dimension 1, there is no long range order for any $\beta < \infty$.

Here are two quite canonical random spanning tree models on finite graphs:

▷ **Exercise 14.** On any finite graph $G(V,E)$, assign iid random edge weights $\xi = (\xi_e)_{e\in E}$ to the edges, from an atomless non-negative valued distribution. Consider the spanning tree of G that minimizes the sum of the edge weights — this is the Minimal Spanning Tree MST_ξ.

(a)$^\bullet$ Show that one can construct this tree by removing from every cycle of G the edge with the largest label.

(b)$^\bullet$ Conclude that the distribution of MST_ξ does not depend on the distribution of the ξ_e's. Hence we can denote this random tree just by MST, the Minimal Spanning Tree of the graph.

(c)$^{\bullet\bullet}$ Consider the uniform distribution on all the spanning trees of G — this is the Uniform Spanning Tree UST. Give a finite graph on which $\mathsf{MST} \neq \mathsf{UST}$ with positive probability.

The **Fortuin-Kasteleyn random cluster measure** $\mathsf{FK}(p,q)$ on a finite graph G, with $p \in [0,1]$ and $q > 0$, is the invariant bond percolation model given by, for any $\omega \subset E(G)$,

$$\mathbf{P}_{\mathsf{FK}(p,q)}[\omega] := \frac{p^{|\omega|}\,(1-p)^{|E\setminus\omega|}\,q^{k(\omega)}}{Z_{\mathsf{FK}(p,q)}} \qquad \text{with} \quad Z_{\mathsf{FK}(p,q)} := \sum_{\omega\subseteq E} p^{|\omega|}\,(1-p)^{|E\setminus\omega|}\,q^{k(\omega)}\,,$$

where $k(\omega)$ is the number of clusters of ω.

▷ **Exercise 15.**$^{\bullet\bullet}$ Consider $\mathsf{FK}(p,q)$ on the $n\times n$ two-dimensional lattice torus $(\mathbb{Z}/n\mathbb{Z})^2$. Given a configuration ω, the dual configuration ω^* is defined on the dual torus: the dual vertices are the primal faces, and two are connected by a dual edge iff the edge between the primal faces is not present in ω. Show that for $p = p_{\text{self-dual}}(q) = \frac{\sqrt{q}}{1+\sqrt{q}}$, the dual configuration ω^* has the same distribution as ω.

▷ **Exercise 16.**$^\bullet$ For any finite tree, show that $\mathsf{FK}(p,q)$ is just Bernoulli bond percolation at some density $\tilde{p}(q)$, which you should identify.

▷ **Exercise 17.**$^\bullet$ For any finite graph, show that $\lim_{p\to 0+}\lim_{q\to 0+} \mathsf{FK}(p,q) = \mathsf{UST}$.