

On the energy spectrum of Yang–Mills instantons over asymptotically locally flat spaces

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Abstract

In this paper we prove that over an asymptotically locally flat (ALF) Riemannian four-manifold the energy of an “admissible” $SU(2)$ Yang–Mills is always integer. This result sharpens the previously known energy identity for such Yang–Mills instantons over ALF geometries. Furthermore we demonstrate that this statement continues to hold for the larger gauge group $U(2)$.

Finally we make the observation that there might be a natural relationship between 4 dimensional Yang–Mills theory over an ALF space and 2 dimensional conformal field theory. This would provide a further support for the existence of a similar correspondence investigated by several authors recently.

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1 Introduction

Asymptotically locally flat (ALF) spaces are non-compact complete Riemannian 4-manifolds including mathematically as well as physically important spaces such as the Riemannian Schwarzschild and Kerr geometries [15] and interesting hyper-Kähler examples like the flat $\mathbb{R}^3 \times S^1$, the A_k ALF (or multi-Taub–NUT or ALF Gibbons–Hawking) geometries [22], the Atiyah–Hitchin manifold [3] and its cousins the so-called D_k ALF spaces [13].

Recently there has been some effort to understand Yang–Mills instantons over these spaces from both mathematical (e.g. [8, 9, 17, 18, 19, 20, 21, 29]) and physical (e.g. [6, 7, 10, 11, 12, 25, 27, 28, 34, 37]) sides. A central question is whether or not their moduli spaces are finite dimensional manifolds (with mild isolated singularities). Experienced with the compact case an expected condition is the discreteness of their energy spectrum. For instance in [20] natural asymptotical analytical conditions on these solutions have been imposed in this spirit: the energy spectrum of these “admissible” Yang–Mills instantons is characterized by Chern–Simons

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invariants of the infinitely distant boundary of the ALF space. Hence the energy spectrum is discrete but in principle can contain fractional values corresponding to smooth Yang–Mills instantons with non-trivial holonomy at infinity. Over an A_k ALE (or multi-Eguchi–Hanson or ALE Gibbons–Hawking) space smooth irreducible $SU(2)$ Yang–Mills instanton solutions forming nice moduli spaces indeed can have fractional energy (cf. [5] or [19, Theorem 4.1]). On the contrary over ALF spaces—although finite energy solutions are relatively easy to find—no “admissible” Yang–Mills instantons of fractional energy are known to exist (e.g. [17, 19, 21]). Meanwhile other known finite energy solutions (e.g. [6, 7, 8, 9, 12, 14, 26]) with integer or fractional energy may fit into larger continuous energy families, cf. [29].

The paper is organized as follows. In Sect. 2 we argue why “admissibility” (cf. Definition 2.1 here or [21, Definition 2.1]) is a quite weak and good condition to impose on $SU(2)$ Yang–Mills instantons in the ALF scenario: we shall see shortly that it is a sharp but easily satisfied condition moreover can be used in *all* known classes of ALF spaces (both the Kähler and non-Kähler examples) to rule out $SU(2)$ Yang–Mills instantons with continuous energy. “Admissibility” rests on two important technicalities: a geometrically motivated smooth compactification of an ALF space discovered by Hausel–Hunsicker–Mazzeo [23] and a codimension 2 singularity removal theorem of Sibner–Sibner [31] and Råde [30].

In Sect. 3 we come to our main result and demonstrate that “admissibility” excludes not only continuous but even fractional energy Yang–Mills instantons i.e., *over an ALF space the energy of a smooth “admissible” $SU(2)$ Yang–Mills instanton is always integer* although its associated holonomy at infinity is not necessarily trivial (cf. Theorem 3.2 here).

A straightforward generalization in Sect. 4 shows that our result remains valid for the larger gauge group $SU(2) \times U(1)$ and—under some topological conditions for the underlying vector bundle—also for $U(2)$ although when deriving these results some care is needed due to the existence of non-topological $U(1)$ Yang–Mills instantons (cf. Theorem 4.1 here).

Finally in Sect. 5 we speculate if the existence of the aforementioned compactification of an ALF space with its pleasant properties in Yang–Mills theory might indicate an intrinsic AGT-like relationship at the quantum level between $4d$ YM theory and $2d$ CFT as it has been suggested recently by several authors from a different angle, cf. e.g. [1, 2, 4, 33].

2 An overview of Yang–Mills theory over ALF spaces

So let (M, g) be an ALF space as defined in [20, 23]. Topologically, an ALF space (with a single end) admits a decomposition $M = K \cup W$ where K is a compact interior space and W is an end or neck homeomorphic to $N \times \mathbb{R}^+$ where

$$\pi : N \longrightarrow B_{+\infty} \tag{1}$$

is a connected, compact, oriented three-manifold fibered over a connected compact surface $B_{+\infty}$ with circle fibers $F \cong S^1$. Consider $W \cong N \times \mathbb{R}^+$ and denote by $r \in [0, +\infty)$ the radial coordinate parameterizing \mathbb{R}^+ . Regarding the complete Riemannian metric g there exists a diffeomorphism $\phi : N \times \mathbb{R}^+ \rightarrow W$ such that

$$\phi^*(g|_W) = dr^2 + r^2(\pi^*g_{B_{+\infty}})' + h'_F$$

where $g_{B_{+\infty}}$ is a smooth metric on $B_{+\infty}$, h_F is a symmetric 2-tensor on N which restricts to a metric along the fibers $F \cong S^1$ and $(\pi^*g_{B_{+\infty}})'$ as well as h'_F are smooth non-vanishing $O(1)$

extensions of $\pi^*g_{B_{+\infty}}$ and h_F over W , respectively. Furthermore, we impose that the curvature R_g of g decays like

$$|\phi^*(R_g|_W)| \sim O(r^{-3}).$$

Here R_g is regarded as a map $R_g : C^\infty(\Lambda^2 M) \rightarrow C^\infty(\Lambda^2 M)$ and its pointwise norm is calculated accordingly in an orthonormal frame. Hence the Pontryagin number of our ALF spaces is finite.

For any real number $0 < R < +\infty$ let $\overline{M}_R \subset M$ be the truncated manifold with boundary containing all the points of $K \subset M$ as well as those $x \in W \subset M$ for which $r(x) \leq R$.

Definition 2.1. *Let (M, g) be an ALF four-manifold. Take an arbitrary finite energy $SU(2)$ -connection ∇_A on a (necessarily trivial) rank 2 complex $SU(2)$ vector bundle E_0 over M . This connection is said to be admissible if it satisfies two conditions ([20, Definitions 2.1 and 2.2]):*

- (i) *The first is called the weak holonomy condition and says that to ∇_A there exists a smooth flat $SU(2)$ -connection $\nabla_\Gamma|_W$ on $E_0|_W$ along the end $W \subset M$ and a constant $0 < c(g) < +\infty$, independent of $R > 0$, such that there exists a gauge on $M \setminus \overline{M}_R$ satisfying*

$$\|A - \Gamma\|_{L^2_{1,\Gamma}(M \setminus \overline{M}_R)} \leq c \|F_A\|_{L^2(M \setminus \overline{M}_R)};$$

- (ii) *The second condition requires ∇_A to decay rapidly at infinity i.e.,*

$$\lim_{R \rightarrow +\infty} \sqrt{R} \|F_A\|_{L^2(M \setminus \overline{M}_R)} = 0.$$

Note that admissibility could have been defined for an arbitrary Lie group G . However for $G = SU(2)$ this definition is natural and admissible self-dual connections are the “good objects” to consider in the ALF scenario as we argued in [20] because:

- (i) We demonstrated in [20, Theorem 2.3] that if N in (1) is an arbitrary circle bundle over $B_{+\infty} \not\cong S^2, \mathbb{R}P^2$ or a trivial circle bundle over $B_{+\infty} \cong S^2, \mathbb{R}P^2$ then for a rapidly decaying connection the weak holonomy condition is satisfied. Hence in spite of its analytical shape it is in fact a mild topological condition only and is essentially always valid (except for instance in the important case of the multi-Taub–NUT geometries). On the other hand the rapid decay is indeed a non-trivial analytical condition and is somewhat stronger than assuming simply finite energy;
- (ii) The energy of any admissible $SU(2)$ Yang–Mills instanton ∇_A belongs to a discrete set characterized by Chern–Simons invariants $\tau_N(\Gamma_{+\infty})$ of the infinitely distant boundary N . More precisely we know ([20, Theorem 2.2] or Theorem 3.1 here) that

$$e = \frac{1}{8\pi^2} \|F_A\|_{L^2(M)}^2 \equiv \tau_N(\Gamma_{+\infty}) \pmod{\mathbb{Z}} \quad (2)$$

where $\nabla_\Gamma|_W = d + \Gamma$ is a flat connection (associated to ∇_A by the weak holonomy condition (i) of Definition 2.1) in a smooth gauge in which the limit of its restriction $\Gamma_{+\infty} := \lim_{r \rightarrow +\infty} \Gamma|_{N \times \{r\}}$ exists (such gauge indeed exists, cf. [20, Section 2]);

- (iii) The framed moduli spaces $\mathcal{M}(e, \Gamma)$ of irreducible admissible $SU(2)$ Yang–Mills instantons over a Ricci-flat ALF space are smooth possibly empty manifolds and [20, Theorem 3.2]

$$\dim \mathcal{M}(e, \Gamma) = 8(e + \tau_N(\Theta_{+\infty}) - \tau_N(\Gamma_{+\infty})) - 3b^-(X) \quad (3)$$

where $\nabla_{\Theta}|_W = d + \Theta$ is the trivial flat connection in a gauge in which both the limit $\Theta_{+\infty} := \lim_{r \rightarrow +\infty} \Theta|_{N \times \{r\}}$ and $\Gamma_{+\infty}$ exists moreover X is the Hausel–Hunsicker–Mazzeo compactification ([23], also defined here soon) of (M, g) with its induced orientation.¹

Remark. 1. Several explicit examples demonstrate the relevance of both conditions in Definition 2.1. For instance dropping only the weak holonomy condition (i) of Definition 2.1 there exist rapidly decaying smooth reducible $SU(2)$ Yang–Mills anti-instantons over the multi-Taub–NUT spaces with arbitrary positive energy [20, Sections 2 and 4] (these will be reviewed in the Remark of Sect. 3 here). Likewise, dropping only the rapid decay condition (ii) of Definition 2.1 there exist smooth irreducible $SU(2)$ Yang–Mills instantons over the Riemannian Schwarzschild space [26] or over $\mathbb{R}^3 \times S^1$ with its flat metric [14, 29] with continuous energy spectrum. All of these solutions are pathological in some sense: for instance they do not form nice moduli spaces.

2. On the contrary, the simplest *unframed* moduli space $\widehat{\mathcal{M}}(1, \Theta)$ over the multi-Taub–NUT space consisting of all admissible $SU(2)$ Yang–Mills anti-instantons with $e = 1$ and having trivial holonomy at infinity i.e., satisfying $\nabla_{\Gamma}|_W = \nabla_{\Theta}|_W$ is a usual moduli space: by (3) it is five dimensional and looks like a singular disk fibration over \mathbb{R}^3 with usual conical singularities corresponding to the reducible solutions at the NUTs. It has been constructed explicitly by the aid of the classical conformal rescaling method in [21, Theorem 4.2]. Moreover all higher integer energy moduli spaces $\mathcal{M}(k, \Theta)$ hence $\widehat{\mathcal{M}}(k, \Theta)$ are non-empty [21, Theorem 4.3].

Before proceeding further we take the opportunity and explain why Definition 2.1 is a natural one to impose in the ALF context. In this way also the two technical tools which make the ALF scenario so special among the non-compact geometries and will be used throughout the paper can be introduced.

The first tool is the *Hausel–Hunsicker–Mazzeo compactification* of an ALF space [23]. Take an ALF space (M, g) as before. Compactify M by simply shrinking all the circle fibers in the fibration (1). Let us denote this space by X . It is easy to see that X is a connected compact smooth 4-manifold without boundary and inherits an orientation from (M, g) . Topologically

$$X = M \cup B_{+\infty} \quad (4)$$

and $B_{+\infty}$ represents a smoothly embedded two codimensional surface in X .

The second tool is a *codimension 2 singularity removal theorem of Sibner–Sibner* [31] and *Råde* [30]. Let us fix some notation which is in agreement with that of [20]. Take the truncated manifold $\overline{M}_R \subset M$ already used in Definition 2.1 and put $V_R^\times := M \setminus \overline{M}_R \subseteq W$ for the remaining open tail of the original space M . Note that $V_R^\times \cong N \times (R, +\infty)$. Consider the fibration (1) of the boundary and let $U \subset B_{+\infty}$ be a coordinate patch of the base space. We obtain a corresponding domain $U_R^\times \subset V_R^\times$ what we call an *elementary neighbourhood* [20]. It follows that

¹Recall that the *framed moduli space* $\mathcal{M}(e, \Gamma)$ consists of pairs $([\nabla_A], \Gamma)$ where $[\nabla_A]$'s are the $L_{2,\Gamma}^2$ gauge equivalence classes of irreducible admissible $SU(2)$ self-dual connections of energy $e \geq 0$ and asymptotics given by $\nabla_{\Gamma}|_W$ while Γ is a fixed smooth gauge at infinity. Forgetting about the fixed framing Γ at infinity we obtain from $\mathcal{M}(e, \Gamma)$ the *unframed moduli space* $\widehat{\mathcal{M}}(e, \Gamma)$ consisting of the gauge equivalence classes $[\nabla_A]$ only. We therefore find $\dim \widehat{\mathcal{M}}(e, \Gamma) = \dim \mathcal{M}(e, \Gamma) - \dim SU(2) = \dim \mathcal{M}(e, \Gamma) - 3$.

$\pi^{-1}(U) \cong B^2 \times S^1$ consequently $U_R^\times \cong B^2 \times S^1 \times (R, +\infty) \cong B^2 \times (B^2)^\times$ i.e., it is a semi-infinite-cylinder-bundle over a disk B^2 hence $\pi_1(U_R^\times) \cong \mathbb{Z}$. Assume now that a smooth flat local SU(2)-connection $\nabla_{\Gamma_m}|_{U_R^\times}$ is given. There exists a canonical gauge $\nabla_{\Gamma_m}|_{U_R^\times} = d + \Gamma_m$ where

$$\Gamma_m = \begin{pmatrix} im & 0 \\ 0 & -im \end{pmatrix} d\tau. \quad (5)$$

Here $\tau \in [0, 2\pi)$ parameterizes a circle in U_R^\times generating $\pi_1(U_R^\times) \cong \mathbb{Z}$ and $m \in [0, 1)$ is the *local holonomy* of the flat local connection. The restriction of the globally defined trivial connection $\nabla_{\emptyset}|_W$ to U_R^\times corresponds to $m = 0$ and also may exist other global flat connections on W with this property. However in general the flat local connection $\nabla_{\Gamma_m}|_{U_R^\times}$ does *not* extend to a flat global connection $\nabla_\Gamma|_W$ on the neck.

A tubular neighbourhood $B_{+\infty} \subset V_R \subset X$ is a trivial B^2 -bundle over $B_{+\infty}$ and looks like $V_R \cong N \times (R, +\infty]/\sim$ where \sim means that $N \times \{+\infty\}$ is pinched into $B_{+\infty}$. Consider a finite open covering $B_{+\infty} = \cup_\alpha U_\alpha$ of the base space in (1) and take the associated elementary neighbourhoods $U_{R,\alpha}$ whose collection with $U_{R,\alpha} \cong B^2 \times S^1 \times (R, +\infty]/\sim \cong B^2 \times B^2$ gives a finite covering for V_R . If a finite energy SU(2) connection ∇_A is given on (M, g) then the rapid decay condition in Definition 2.1 makes sure that $|F_A|_{g'} \rightarrow 0$ a.e. as $R \rightarrow +\infty$ i.e., it will be a finite energy connection on $(X \setminus B_{+\infty}, g')$ as well where g' is a regularized metric on X which belongs to the conformal class of g on the complement of V_R with some large $0 < R < +\infty$ (the original metric does not extend to X , even conformally). Then the singularity removal theorem we recall now ensures us that there exist constants $m \in [0, 1)$ independent of α and $0 < c(g', \alpha) < +\infty$ as well as a local L^2_{1,Γ_m} gauge on $U_{R,\alpha}^\times$ in which $\nabla_A|_{U_{R,\alpha}^\times} = d + A|_{U_{R,\alpha}^\times}$ and $\nabla_{\Gamma_m}|_{U_{R,\alpha}^\times} = d + \Gamma_m$ satisfy the inequality

$$\|A|_{U_{R,\alpha}^\times} - \Gamma_m\|_{L^2_{1,\Gamma_m}(U_{R,\alpha}^\times)} \leq c(g', \alpha) \|F_A\|_{L^2(U_{R,\alpha})}.$$

We recognize this as the local version of the weak holonomy condition in Definition 2.1. The Sibner–Sibner–Råde theorem also says that ∇_A extends over the singularity $B_{+\infty}$ if and only if $m = 0$. Assume now that the flat local connections $\nabla_{\Gamma_m}|_{U_{R,\alpha}^\times}$ have a (possibly not unique) extension over the whole V_R^\times and patch together into a smooth flat connection $\nabla_\Gamma|_{V_R^\times}$. It is easy that the only obstruction against this is the situation if $\nabla_{\Gamma_m}|_{U_{R,\alpha}^\times}$ with $m \neq 0$ is in the kernel of $i_* : \pi_1(U_{R,\alpha}^\times) \rightarrow \pi_1(V_R^\times)$ induced by $i : U_{R,\alpha}^\times \subset V_R^\times$. In [20, Theorem 2.3] this is converted into an easily decidable mild topological condition on the fibration (1). In other words flat local connections essentially always can be extended over the whole neck. Then there exists a gauge on $E_0|_{V_R^\times}$ in which $\nabla_A|_{V_R^\times} = d + A$ with A at least in $L^2_{1,\Gamma}$ and $\nabla_\Gamma|_{V_R^\times} = d + \Gamma$ with Γ being smooth such that one can find smooth gauge transformations independent of R satisfying $\gamma_\alpha^{-1} \Gamma_m \gamma_\alpha + \gamma_\alpha^{-1} d \gamma_\alpha = \Gamma|_{U_{R,\alpha}^\times}$ where Γ_m is the canonical local gauge (5). Then easily follows that the local estimates above patch together and give part (i) of Definition 2.1. We are convinced now that the admissibility condition is essentially a consequence of finite energy.

3 An improved energy identity for SU(2)

After getting some feeling of the admissibility assumption in the case of $G = \text{SU}(2)$ Yang–Mills theory over an ALF space, in this section we demonstrate that there exist no admissible SU(2) Yang–Mills instantons of fractional energy over any ALF space i.e., the energy identity (2) is superfluous. For the sake of completeness first we reproduce [20, Theorem 2.2] here.

Taking into account that the self-duality equations are conformally invariant, we can rescale our metric without affecting self-duality. Hence rescale the original ALF metric g with a positive function $f : M \rightarrow \mathbb{R}^+$ satisfying $f|_W \sim O(r^{-2})$ and write $\tilde{g} := f^2g$. In what follows *this* rescaled metric \tilde{g} will be used everywhere to calculate various Sobolev norms.

Using the notation of Sect. 2 let $\overline{M}_R \subset M$ be the truncated manifold-with-boundary with $r(x) \leq R$ whose boundary is $\partial\overline{M}_R \cong N \times \{R\}$. Given an $SU(2)$ -connection $\nabla_A = d + A$ in some gauge on the trivial bundle E_0 the Chern–Simons functional evaluated on its restriction to $\partial\overline{M}_R$ is

$$\tau_{\partial\overline{M}_R}(A_R) := -\frac{1}{8\pi^2} \int_{\partial\overline{M}_R} \text{tr} \left(dA_R \wedge A_R + \frac{2}{3} A_R \wedge A_R \wedge A_R \right)$$

in the induced gauge $A_R := A|_{\partial\overline{M}_R}$ on the boundary.

First—motivated by [35]—we prove two continuity results for the Chern–Simons functional in three dimensions which are interesting on their own right (also cf. [20, Lemma 2.1]). Only the first one will be used in this paper.

Lemma 3.1. *Take two $SU(2)$ -connections ∇_{A_R} and ∇_{B_R} on the trivial bundle $E_0|_{\partial\overline{M}_R}$. For some fixed $0 < R < +\infty$ on the compact Riemannian three-manifold $(\partial\overline{M}_R, \tilde{g}|_{\partial\overline{M}_R})$ consider Sobolev norms $\|\cdot\|_{L^p_k}$ with respect to the metric $\tilde{g}|_{\partial\overline{M}_R}$ and the connection ∇_{B_R} for instance.*

(i) *Assume that there exists a gauge in which $\nabla_{A_R} = d + A_R$ and $\nabla_{B_R} = d + B_R$ satisfy $A_R, B_R \in L^2_1(\partial\overline{M}_R; \Lambda^1(\partial\overline{M}_R) \otimes \mathfrak{su}(2))$. Then there exists an estimate*

$$|\tau_{\partial\overline{M}_R}(A_R) - \tau_{\partial\overline{M}_R}(B_R)| \leq \left(\|F_{A_R}\|_{L^2(\partial\overline{M}_R)} + \|F_{B_R}\|_{L^2(\partial\overline{M}_R)} \right) \|A_R - B_R\|_{L^2_1(\partial\overline{M}_R)} + c_1^3 \|A_R - B_R\|_{L^2_1(\partial\overline{M}_R)}^3$$

i.e., the Chern–Simons functional is continuous in the L^2_{1,B_R} -norm in this sense. Here the constant $0 < c_1(B_R, R) < +\infty$ is the constant of the Sobolev embedding $L^2_1 \subset L^3$.

(ii) *Assume that there exists a gauge in which $\nabla_{A_R} = d + A_R$ and $\nabla_{B_R} = d + B_R$ satisfy $A_R, B_R \in L^{\frac{3}{2}}_1(\partial\overline{M}_R; \Lambda^1(\partial\overline{M}_R) \otimes \mathfrak{su}(2))$. Then there exists a sharper estimate*

$$|\tau_{\partial\overline{M}_R}(A_R) - \tau_{\partial\overline{M}_R}(B_R)| \leq c_2 \|A_R - B_R\|_{L^{\frac{3}{2}}_1(\partial\overline{M}_R)} + c_3 \|A_R - B_R\|_{L^{\frac{3}{2}}_1(\partial\overline{M}_R)}^3$$

i.e., the Chern–Simons functional is continuous even in the stronger $L^{\frac{3}{2}}_{1,B_R}$ -norm in this sense. Here the constants are

$$c_2(B_R, R) := c_4 + (c_4 + c_4^3) \|B_R\|_{L^{\frac{3}{2}}_1(\partial\overline{M}_R)} + c_4^3 \|B_R\|_{L^{\frac{3}{2}}_1(\partial\overline{M}_R)}^2$$

$$c_3(B_R, R) := c_4 + c_4^3 + c_4^3 \|B_R\|_{L^{\frac{3}{2}}_1(\partial\overline{M}_R)}$$

with $0 < c_4(B_R, R) < +\infty$ being the constant of the sharp Sobolev embedding $L^{\frac{3}{2}}_1 \subset L^3$.

Proof. Both inequalities rest on the identity

$$\begin{aligned} & \tau_{\partial\overline{M}_R}(A_R) - \tau_{\partial\overline{M}_R}(B_R) = \\ & -\frac{1}{8\pi^2} \int_{\partial\overline{M}_R} \text{tr} \left((F_{A_R} + F_{B_R}) \wedge (A_R - B_R) - \frac{1}{3} (A_R - B_R) \wedge (A_R - B_R) \wedge (A_R - B_R) \right). \quad (6) \end{aligned}$$

(i) By the aid of Hölder’s inequalities $1 = \frac{1}{2} + \frac{1}{2}$ and $1 = \frac{1}{3} + \frac{1}{3} + \frac{1}{3}$ and the Sobolev embedding $L^3 \subset L^2_1$ with $0 < c_1(B_R, R) < +\infty$ it easily follows that the absolute value of the quadratic term can be estimated from above simply by

$$\begin{aligned} & \left(\|F_{A_R}\|_{L^2(\partial\overline{M}_R)} + \|F_{B_R}\|_{L^2(\partial\overline{M}_R)} \right) \|A_R - B_R\|_{L^2(\partial\overline{M}_R)} \\ & \leq \left(\|F_{A_R}\|_{L^2(\partial\overline{M}_R)} + \|F_{B_R}\|_{L^2(\partial\overline{M}_R)} \right) \|A_R - B_R\|_{L^2_1(\partial\overline{M}_R)} \end{aligned}$$

while the absolute value of the cubic term has an estimate from above like

$$\|A_R - B_R\|_{L^3(\partial\overline{M}_R)}^3 \leq c_1^3 \|A_R - B_R\|_{L^2_1(\partial\overline{M}_R)}^3$$

which proves the first part.

(ii) Inserting $F_{A_R} = dA_R + A_R \wedge A_R$ and $F_{B_R} = dB_R + B_R \wedge B_R$ into the identity (6) we write $\tau_{\partial\overline{M}_R}(A_R) - \tau_{\partial\overline{M}_R}(B_R)$ as the sum of the following three terms:

$$-\frac{1}{8\pi^2} \int_{\partial\overline{M}_R} \text{tr}(d(A_R - B_R) \wedge (A_R - B_R))$$

and

$$-\frac{1}{8\pi^2} \cdot \frac{2}{3} \int_{\partial\overline{M}_R} \text{tr}((A_R - B_R) \wedge (A_R - B_R) \wedge (A_R - B_R))$$

and

$$-\frac{1}{8\pi^2} \int_{\partial\overline{M}_R} \text{tr}((2 dB_R + A_R \wedge B_R + B_R \wedge A_R) \wedge (A_R - B_R)).$$

Making use of Hölder’s inequalities with $1 = \frac{2}{3} + \frac{1}{3}$ and $1 = \frac{1}{3} + \frac{1}{3} + \frac{1}{3}$, the sharp Sobolev embedding $L^{\frac{3}{2}}_1 \subset L^3$ with a constant $0 < c_4(B_R, R) < +\infty$ and the elementary inequality $x^2 \leq x + x^3$ we proceed as follows. The absolute value of the first term can be estimated from above by

$$\begin{aligned} \|d(A_R - B_R)\|_{L^{\frac{3}{2}}(\partial\overline{M}_R)} \|A_R - B_R\|_{L^3(\partial\overline{M}_R)} & \leq c_4 \|A_R - B_R\|_{L^{\frac{3}{2}}_1(\partial\overline{M}_R)} \|A_R - B_R\|_{L^{\frac{3}{2}}(\partial\overline{M}_R)} \\ & \leq c_4 \left(\|A_R - B_R\|_{L^{\frac{3}{2}}_1(\partial\overline{M}_R)} + \|A_R - B_R\|_{L^{\frac{3}{2}}_1(\partial\overline{M}_R)}^3 \right). \end{aligned}$$

The absolute value of the second term can be estimated from above by

$$\|A_R - B_R\|_{L^3(\partial\overline{M}_R)}^3 \leq c_4^3 \|A_R - B_R\|_{L^{\frac{3}{2}}_1(\partial\overline{M}_R)}^3.$$

Finally we adjust the third term via Stokes’ theorem and some algebra into the shape

$$-\frac{1}{8\pi^2} \cdot 2 \int_{\partial\overline{M}_R} \text{tr}(B_R \wedge d(A_R - B_R) + B_R \wedge (A_R - B_R) \wedge (A_R - B_R) + B_R \wedge B_R \wedge (A_R - B_R)).$$

Then we can estimate the absolute value of the third term from above by

$$\begin{aligned}
& \|B_R\|_{L^3(\partial\overline{M}_R)} \|d(A_R - B_R)\|_{L^{\frac{3}{2}}(\partial\overline{M}_R)} + \|B_R\|_{L^3(\partial\overline{M}_R)} \|A_R - B_R\|_{L^3(\partial\overline{M}_R)}^2 \\
& + \|B_R\|_{L^3(\partial\overline{M}_R)}^2 \|A_R - B_R\|_{L^3(\partial\overline{M}_R)} \\
\leq & \left((c_4 + c_4^3) \|B_R\|_{L^{\frac{3}{2}}(\partial\overline{M}_R)} + c_4^3 \|B_R\|_{L^{\frac{3}{2}}(\partial\overline{M}_R)}^2 \right) \|A_R - B_R\|_{L^{\frac{3}{2}}(\partial\overline{M}_R)} \\
& + c_4^3 \|B_R\|_{L^{\frac{3}{2}}(\partial\overline{M}_R)} \|A_R - B_R\|_{L^{\frac{3}{2}}(\partial\overline{M}_R)}^3.
\end{aligned}$$

Putting together these estimates we obtain the inequality of the second part of the lemma. \diamond

Remark. We also record here that in the case of flat connections (6) shows that the inequalities of the lemma cut down to

$$|\tau_{\partial\overline{M}_R}(\Gamma'_R) - \tau_{\partial\overline{M}_R}(\Gamma''_R)| \leq \|\Gamma'_R - \Gamma''_R\|_{L^3(\partial\overline{M}_R)}^3 \leq c_1^3 \|\Gamma'_R - \Gamma''_R\|_{L^2(\partial\overline{M}_R)}^3$$

and a similar one for the $L^{\frac{3}{2}}$ -norm.

Now we are in a position to reprove [20, Theorem 2.2] following the steps of [20, Section 2]. Take an admissible $SU(2)$ -connection ∇_A on E_0 and the corresponding flat connection $\nabla_\Gamma|_{V_R^\times}$ to which it converges. Suppose that we are in the gauge on $E_0|_{V_R^\times}$ in which $\nabla_A|_{V_R^\times} = d + A$ and $\nabla_\Gamma|_{V_R^\times} = d + \Gamma$ and the corresponding connection 1-forms satisfy the inequality in part (i) of Definition 2.1. Let A_R and Γ_R be their restrictions to $\partial\overline{M}_R$.

Theorem 3.1. (cf. [20, Theorem 2.2]) *Let (M, g) be an ALF space with an end $W \cong N \times \mathbb{R}^+$. Let E_0 be an $SU(2)$ vector bundle over M , necessarily trivial, with an admissible $SU(2)$ Yang–Mills instanton ∇_A on it: a smooth, finite energy self-dual connection satisfying Definition 2.1. Then*

$$\frac{1}{8\pi^2} \|F_A\|_{L^2(M)}^2 \equiv \tau_N(\Gamma_{+\infty}) \pmod{\mathbb{Z}}$$

that is, its energy is congruent to a Chern–Simons invariant of the boundary given by the flat connection $\nabla_\Gamma|_W$ in part (i) of Definition 2.1.

Proof. There exists some fixed $R_0 \in (R, +\infty)$ and a constant $0 < c_5(\tilde{g}) < +\infty$ independent of R such that we can begin to estimate the difference $|\tau_{\partial\overline{M}_R}(A_R) - \tau_{\partial\overline{M}_R}(\Gamma_R)|^{\frac{2}{3}}$ like

$$\frac{1}{R} \left| \tau_{\partial\overline{M}_{R_0}}(A_{R_0}) - \tau_{\partial\overline{M}_{R_0}}(\Gamma_{R_0}) \right|^{\frac{2}{3}} \leq c_5 \int_R^{+\infty} \left| \tau_{\partial\overline{M}_r}(A_r) - \tau_{\partial\overline{M}_r}(\Gamma_r) \right|^{\frac{2}{3}} r^{-2} dr$$

and then can go on by the aid of the inequality of part (i) of Lemma 3.1 with $F_{\Gamma_r} = 0$ and the elementary inequality $(x + y)^{\frac{2}{3}} \leq x^{\frac{2}{3}} + y^{\frac{2}{3}}$ to get

$$\begin{aligned}
& \leq c_5 \int_R^{+\infty} \left(\|F_{A_r}\|_{L^2(\partial\overline{M}_r)} \|A_r - \Gamma_r\|_{L^2_{1,\Gamma_r}(\partial\overline{M}_r)} + c_1^3 \|A_r - \Gamma_r\|_{L^2_{1,\Gamma_r}(\partial\overline{M}_r)}^3 \right)^{\frac{2}{3}} r^{-2} dr \\
& \leq c_5 \int_R^{+\infty} \left(\|F_{A_r}\|_{L^2(\partial\overline{M}_r)}^{\frac{2}{3}} \|A_r - \Gamma_r\|_{L^2_{1,\Gamma_r}(\partial\overline{M}_r)}^{\frac{2}{3}} + c_1^2 \|A_r - \Gamma_r\|_{L^2_{1,\Gamma_r}(\partial\overline{M}_r)}^2 \right) r^{-2} dr
\end{aligned}$$

and then can make further steps by applying on the first term Hölder's inequalities with $1 = \frac{2}{3} + \frac{1}{3}$ and then with $1 = \frac{1}{2} + \frac{1}{2}$ on $(\mathbb{R}^+, r^{-2}dr)$ to obtain

$$\begin{aligned} &\leq \|F_A\|_{L^2(V_R^\times)}^{\frac{4}{3}} \left(\int_R^{+\infty} \|A_r - \Gamma_r\|_{L^2_{1,\Gamma_r}(\partial\overline{M}_r)}^2 r^{-2} dr \right)^{\frac{1}{3}} + c_5 \int_R^{+\infty} c_1^2 \|A_r - \Gamma_r\|_{L^2_{1,\Gamma_r}(\partial\overline{M}_r)}^2 r^{-2} dr \\ &\leq c_5 \|F_A\|_{L^2(V_R^\times)}^{\frac{4}{3}} \|A - \Gamma\|_{L^2_{1,\Gamma}(V_R^\times)}^{\frac{2}{3}} + c_6 \|A - \Gamma\|_{L^2_{1,\Gamma}(V_R^\times)}^2. \end{aligned}$$

Finally referring to the weak holonomy condition in Definition 2.1 for some $R_0 \in (R, +\infty)$ we come up with

$$\left| \tau_{\partial\overline{M}_{R_0}}(A_{R_0}) - \tau_{\partial\overline{M}_{R_0}}(\Gamma_{R_0}) \right| \leq c_7 \left(2R \|F_A\|_{L^2(V_R^\times)}^2 \right)^{\frac{3}{2}} = 2^{\frac{3}{2}} c_7 \left(\sqrt{R} \|F_A\|_{L^2(V_R^\times)} \right)^3. \quad (7)$$

Taking into account the shape of the constants in part (i) of Lemma 3.1 we can assume that the overall constant $0 < c_7(\tilde{g}, \Gamma) < +\infty$ is bounded. Hence we obtain from (7) by the aid of the rapid decay condition i.e., part (ii) of Definition 2.1 that

$$\lim_{R \rightarrow +\infty} |\tau_{\partial\overline{M}_R}(A_R) - \tau_{\partial\overline{M}_R}(\Gamma_R)| = 0$$

or writing $\Gamma_{+\infty} = \lim_{R \rightarrow +\infty} \Gamma_R$ and regarding this as a flat connection on the infinitely distant boundary N of (1) we conclude that

$$\lim_{R \rightarrow +\infty} \tau_{\partial\overline{M}_R}(A_R) = \tau_N(\Gamma_{+\infty})$$

which gives the result when applied to a self-dual admissible $SU(2)$ connection ∇_A over the ALF space (M, g) as claimed. \diamond

It follows from [24, Theorem 4.3] already at this point that the energy spectrum consists of rational numbers. Now we show that in fact only those flat connections appear on which the Chern–Simons functional takes integer values i.e., $\tau_N(\Gamma_{+\infty}) \in \mathbb{N}$ in the previous theorem.

Theorem 3.2. *Let (M, g) be an ALF space with an end $W \cong N \times \mathbb{R}^+$. Let E_0 be an $SU(2)$ vector bundle over M , necessarily trivial, with an admissible $SU(2)$ Yang–Mills instanton ∇_A on it i.e., a smooth, finite energy self-dual connection satisfying Definition 2.1. Then*

$$\frac{1}{8\pi^2} \|F_A\|_{L^2(M)}^2 \in \mathbb{N}$$

that is, in addition to Theorem 3.1 its energy is always integer.

Regarding the asymptotical flat connection $\nabla_\Gamma|_W$ in part (i) of Definition 2.1 either ∇_A is flat and $\nabla_\Gamma|_W$ extends smoothly over the whole M such that $\nabla_A = \nabla_\Gamma$ (in this case if $\nabla_\Gamma \neq \nabla_\Theta$ then $\pi_1(M) \neq 1$) or ∇_A is non-flat and $\nabla_\Gamma|_W$ is a flat connection with trivial local holonomy $m = 0$ in (5) (in this case if $\nabla_\Gamma|_W \neq \nabla_\Theta|_W$ then $\pi_1(B_{+\infty}) \neq 1$ in (1) and (4)).

Proof. The proof is a refinement of that of [20, Lemma 3.2]. Given a fixed number $0 < R < +\infty$ consider a smooth gluing function $f_R : M \rightarrow [0, 1]$ such that $f_R|_{\overline{M}_{\frac{R}{2}}} = 0$ and $f_R|_{V_R^\times} = 1$. Take a self-dual admissible connection ∇_A . Making use of the global gauge on E_0 in which part (i) of Definition 2.1 holds with the associated flat connection $\nabla_\Gamma|_{V_R^\times}$ write $\nabla_A = d + A$ and

$\nabla_\Gamma|_{V_R^\times} = d + \Gamma$. For technical reasons we suppose that $\nabla_\Gamma|_{V_R^\times}$ is extended smoothly over the whole M as a not necessarily flat connection $\nabla_{f_R\Gamma}$. Take

$$\nabla_{B(R)} := (1 - f_R)\nabla_A + f_R\nabla_{A-\Gamma}$$

which is a smooth glued connection on E_0 with curvature

$$F_{B(R)} = (1 - f_R)F_A + f_RF_{A-\Gamma} - (df_R \wedge \Gamma + f_R(1 - f_R)\Gamma \wedge \Gamma) \quad (8)$$

where the third perturbation term is compactly supported on the annulus $\overline{V_{\frac{R}{2}}^\times} \setminus V_R^\times$.

Taking into account (8) and the weak holonomy condition in Definition 2.1 the energy of $\nabla_{B(R)}$ for a fixed $0 < R < +\infty$ looks like

$$\begin{aligned} \frac{1}{8\pi^2} \|F_{B(R)}\|_{L^2(M)}^2 &= \frac{1}{8\pi^2} \|F_{B(R)}\|_{L^2(M \setminus V_R^\times)}^2 + \frac{1}{8\pi^2} \|F_{B(R)}\|_{L^2(V_R^\times)}^2 = K + \frac{1}{8\pi^2} \|F_{A-\Gamma}\|_{L^2(V_R^\times)}^2 \\ &\leq K + \|A - \Gamma\|_{L^2_{1,\Gamma}(V_R^\times)}^2 + c_8^4 \|A - \Gamma\|_{L^2_{1,\Gamma}(V_R^\times)}^4 \\ &\leq K + c^2 \|F_A\|_{L^2(V_R^\times)}^2 + (cc_8)^4 \|F_A\|_{L^2(V_R^\times)}^4 \leq K + c^2 e + (cc_8)^4 e^2 \end{aligned}$$

with some constant $0 \leq K < +\infty$ bounding the energy in the interior and a Sobolev constant c_8 from the embedding $L^2_1 \subset L^4$. The energy of $\nabla_{B(R)}$ is therefore finite because the original ∇_A has finite energy. In the same fashion for sufficiently large R

$$\left(\sqrt{R} \|F_{B(R)}\|_{L^2(V_R^\times)} \right)^2 \leq c^2 \left(\sqrt{R} \|F_A\|_{L^2(V_R^\times)} \right)^2 + (cc_8)^4 \left(\sqrt{R} \|F_A\|_{L^2(V_R^\times)} \right)^4$$

consequently $\nabla_{B(R)}$ also satisfies the rapid decay condition of Definition 2.1 because it holds for ∇_A . Hence we make two observations: (i) the connection $\nabla_{B(R)}$ is a smooth finite energy $SU(2)$ connection not only on E_0 over (M, g) but also on $E_0|_{X \setminus B_{+\infty}}$ over the regularized Riemannian manifold (X, g') where X is the compactification (4); and (ii) the connection $\nabla_{B(R)}$ obviously has trivial local holonomy at infinity i.e., $m = 0$ in (5). Consequently as a crucial observation we can apply the Sibner–Sibner–Råde codimension 2 singularity removal theorem to conclude that $\nabla_{B(R)}$ smoothly extends to an $SU(2)$ connection on some complex vector bundle E over X . This implies by Chern–Weil theory that

$$C_2(E) = -\frac{1}{8\pi^2} \int_M \text{tr} (F_{B(R)} \wedge F_{B(R)}) = n \in \mathbb{Z}$$

is a finite fixed integer independent of R .

We begin to estimate the difference between the energy $e = \frac{1}{8\pi^2} \|F_A\|_{L^2(M)}^2$ and the integer n above as follows:

$$\begin{aligned} n - e &= -\frac{1}{8\pi^2} \int_M \text{tr} (F_{B(R)} \wedge F_{B(R)}) - \frac{1}{8\pi^2} \|F_A\|_{L^2(M)}^2 \\ &= \frac{1}{8\pi^2} \int_{M \setminus V_R^\times} (\text{tr} (F_A \wedge F_A) - \text{tr} (F_{B(R)} \wedge F_{B(R)})) + \frac{1}{8\pi^2} \int_{V_R^\times} (\text{tr} (F_A \wedge F_A) - \text{tr} (F_{B(R)} \wedge F_{B(R)})). \end{aligned}$$

On the truncated manifold $\overline{M}_R = M \setminus V_R^\times$ we know that

$$\begin{aligned} \operatorname{tr}(F_A \wedge F_A) - \operatorname{tr}(F_{B(R)} \wedge F_{B(R)}) &= \operatorname{tr}(F_{B(R)+f_R\Gamma} \wedge F_{B(R)+f_R\Gamma}) - \operatorname{tr}(F_{B(R)} \wedge F_{B(R)}) \\ &= \operatorname{d} \operatorname{tr} \left(\operatorname{d}(f_R\Gamma) \wedge (f_R\Gamma) + \frac{2}{3} (f_R\Gamma) \wedge (f_R\Gamma) \wedge (f_R\Gamma) \right) \end{aligned}$$

consequently by the aid of Stokes' theorem on the oriented manifold $M \setminus V_R^\times$ with boundary $\partial\overline{M}_R$ on which $(f_R\Gamma)|_{\partial\overline{M}_R} = \Gamma_R$ holds, the first term takes the shape

$$\frac{1}{8\pi^2} \int_{M \setminus V_R^\times} (\operatorname{tr}(F_A \wedge F_A) - \operatorname{tr}(F_{B(R)} \wedge F_{B(R)})) = \tau_{\partial\overline{M}_R}(\Gamma_R) = \tau_N(\Gamma_{+\infty}).$$

Therefore making use of (8) we obtain by the aid of Definition 2.1 that

$$\begin{aligned} |n - e - \tau_N(\Gamma_{+\infty})| &= \frac{1}{8\pi^2} \left| \int_{V_R^\times} (\operatorname{tr}(F_A \wedge F_A) - \operatorname{tr}(F_{A-\Gamma} \wedge F_{A-\Gamma})) \right| \\ &\leq \|F_A\|_{L^2(V_R^\times)}^2 + \|F_{A-\Gamma}\|_{L^2(V_R^\times)}^2 \\ &\leq (1 + c^2) \|F_A\|_{L^2(V_R^\times)}^2 + (cc_8)^4 \|F_A\|_{L^2(V_R^\times)}^4. \end{aligned}$$

From this it follows that the right hand side is as small as we please when $R \rightarrow +\infty$ since ∇_A has finite energy consequently

$$n - e - \tau_N(\Gamma_{+\infty}) = 0. \quad (9)$$

We know by regularity of Yang–Mills fields that for sufficiently large R we can suppose that the gauge we use is smooth. In addition from Definition 2.1 we deduce that for some $R_0 \in (R, +\infty)$ there exist estimates

$$\|F_A\|_{L^2(\partial\overline{M}_{R_0})} \leq \sqrt{R} c_5 \|F_A\|_{L^2(V_R^\times)}, \quad \|A - \Gamma\|_{L^2_{1,\Gamma}(\partial\overline{M}_{R_0})} \leq \sqrt{R} c_5 \|F_A\|_{L^2(V_R^\times)}.$$

Therefore F_A extends over $\partial\overline{M}_{+\infty} = B_{+\infty} \subset X$ and the extension pointwise satisfies $F_A|_{B_{+\infty}} = 0$ which shows that if ∇_A is extendible over $B_{+\infty}$ then it must be a flat connection on it. But $\nabla_{A-\Gamma}$ always extends and pointwise satisfies $(A - \Gamma)|_{B_{+\infty}} = 0$ and $\nabla_\Gamma(A - \Gamma)|_{B_{+\infty}} = 0$.

Two possibilities exist. The first is when ∇_A is not extendible. Then $A|_{V_R^\times} = \Gamma|_{V_R^\times}$ and self-duality yields $A = \Gamma$ i.e., $\nabla_\Gamma|_{V_R^\times}$ extends smoothly over the whole M as ∇_A hence $\nabla_A = \nabla_\Gamma$ is a flat connection. Its energy is therefore zero and Theorem 3.1 gives that $\tau_N(\Gamma_{+\infty}) \in \mathbb{Z}$ in (9).

The second case is when ∇_A is extendible hence $\nabla_\Gamma|_{V_R^\times}$ is a flat connection with trivial local holonomy i.e., $m = 0$ in (5). This means that $\nabla_\Gamma|_{V_R^\times}$ is the pullback of a flat connection via $\pi \times \operatorname{Id}_{(R,+\infty)} : V_R^\times \cong N \times (R, +\infty) \rightarrow B_{+\infty} \times (R, +\infty)$. Consequently the restriction $\Gamma_{+\infty}$ of $\Gamma|_{V_R^\times}$ to the infinitely distant boundary N in (1) is also a pullback connection 1-form via $\pi : N \rightarrow B_{+\infty}$. It is known that such a flat connection has vanishing Chern–Simons invariant (cf. [24, pp. 547–548 and Theorem 4.3]) i.e., $\tau_N(\Gamma_{+\infty})$ is an integer in (9).

We conclude that the admissible self-dual connection ∇_A has integer energy. \diamond

Remark. We cannot achieve in general that the local holonomy at infinity vanishes. For example there exists a family of smooth flat hence self-dual admissible connections on $\mathbb{R}^3 \times S^1$ parameterized by their holonomy $m \in [0, 1)$. Hence m is also their holonomy at the infinite.

4 The case of $U(2)$

In this section we demonstrate that Theorem 3.2 continues to hold for the slightly larger gauge groups $SU(2) \times U(1)$ and $U(2)$ but in the latter case with a topological condition on the underlying vector bundle.

However before doing this we make an important comment here. As we already mentioned, admissibility as it stands in Definition 2.1 can be formulated for an arbitrary (compact) Lie group G . Hence repeating the proof of the previous section we could obtain an analogue of Theorem 3.2 for arbitrary G . The reason we do not do this is that for Yang–Mills instantons with a general Lie group the admissibility would indeed be a very strong assumption hence the analogue of Theorem 3.2 would be a rather weak statement. This is because for general G the analogue of the powerful singularity removal theorem of Sibner–Sibner [31] and Råde [30] is not known consequently the weak holonomy condition would become a very strong requirement, cf. [20, Section 2].

Rather we restrict ourselves to those Lie groups which can be somehow “traced back” to $SU(2)$ in order to keep our results strong.

Lemma 4.1. *Let (M, g) be an ALF space and \tilde{E} be a rank 2 complex $SU(2) \times U(1)$ vector bundle on M .*

(i) *Then $\tilde{E} \cong E_0 \otimes L$ where E_0 is a (necessarily trivial) rank 2 complex $SU(2)$ vector bundle and L is a $U(1)$ line bundle over M ;*

(ii) *Every $SU(2) \times U(1)$ -connection on $\tilde{E} \cong E_0 \otimes L$ is of the form*

$$\nabla_A \otimes \text{Id}_L + \text{Id}_{E_0} \otimes \nabla_B$$

where ∇_A is an $SU(2)$ -connection on E_0 and ∇_B is an $U(1)$ -connection on L ;

(iii) *The curvature of this product connection looks like*

$$F_A + (F_B \oplus F_B)$$

hence an $SU(2) \times U(1)$ -connection on \tilde{E} is self-dual if and only if both its $SU(2)$ and $U(1)$ parts are self-dual;

(iv) *If an $SU(2) \times U(1)$ -connection on \tilde{E} has finite energy e then it admits a decomposition*

$$e = \frac{1}{8\pi^2} \|F_A\|_{L^2(M)}^2 + \frac{1}{8\pi^2} \|F_B \oplus F_B\|_{L^2(M)}^2 = \frac{1}{8\pi^2} \|F_A\|_{L^2(M)}^2 + \frac{1}{4\pi^2} \|F_B\|_{L^2(M)}^2.$$

Proof. (i) Standard obstruction theory says that over a non-compact oriented four-manifold M principal bundles with a connected, compact structure group G are classified by $H^2(M; \pi_1(G))$. Hence on the one hand all principal $SU(2)$ -bundles are trivial over M since $\pi_1(SU(2)) \cong 1$. Let us denote the associated complex rank 2 trivial bundle by E_0 . On the other hand principal $U(1)$ -bundles are classified by $H^2(M; \mathbb{Z})$ since $\pi_1(U(1)) \cong \mathbb{Z}$. Referring to the canonical isomorphism $\pi_1(U(1)) \cong \pi_1(SU(2) \times U(1))$ we obtain that a rank 2 complex $SU(2) \times U(1)$ vector bundle \tilde{E} can be uniquely written in the form $E_0 \otimes L$ where L is a $U(1)$ line bundle.

(ii) Let ∇_A be an $SU(2)$ -connection on E_0 and ∇_B be an $U(1)$ -connection on L . Taking the embeddings $\mathfrak{su}(2) \subset \mathfrak{su}(2) \times \mathfrak{u}(1) \cong \mathfrak{u}(2)$ as usual and $\mathfrak{u}(1) \subset \mathfrak{su}(2) \times \mathfrak{u}(1) \cong \mathfrak{u}(2)$ given by $B \mapsto \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix} = B \oplus B$ we can identify a product connection on $E_0 \otimes L$ with an $SU(2) \times U(1)$ -connection $\nabla_{A+(B \oplus B)}$ on the corresponding vector bundle \tilde{E} and vice versa.

(iii) Regarding the curvature we calculate

$$F_{A+(B\oplus B)} = F_A + (F_B \oplus F_B) + A \wedge (B \oplus B) + (B \oplus B) \wedge A = F_A + (F_B \oplus F_B)$$

since $A \wedge (B \oplus B) + (B \oplus B) \wedge A = 0$ because $B \oplus B$ is in the centre of $\mathfrak{u}(2)$. Consequently

$$*F_{A+(B\oplus B)} = *(F_A + (F_B \oplus F_B)) = *F_A + (*F_B \oplus *F_B) = F_A + (F_B \oplus F_B) = F_{A+(B\oplus B)}$$

demonstrating that our product connection continues to be self-dual if and only if its components are self-dual.

(iv) Finally the energy can be decomposed as claimed. \diamond

Consider an $SU(2) \times U(1)$ Yang–Mills instanton over some ALF space (M, g) . Then it follows from part (iv) of Lemma 4.1 that it is admissible² if and only if both its $SU(2)$ component ∇_A and $U(1)$ component ∇_B are admissible. We already know from Theorem 3.2 that if ∇_A is admissible then it has not only finite but even integer energy:

$$\frac{1}{8\pi^2} \|F_A\|_{L^2(M)}^2 = k.$$

Regarding the Abelian part we can take the embedding $U(1) \subset SU(2)$ and identify ∇_B on the line bundle L with the reducible $SU(2)$ Yang–Mills instanton $\nabla_B \mapsto \nabla_B \oplus \nabla_{-B}$ on the split bundle $L \oplus L^{-1}$. This is an admissible solution hence we can apply either Theorem 3.1 to conclude that its energy must be congruent to an $U(1)$ Chern–Simons invariant of the boundary (1) hence must be integer or we can simply refer to Theorem 3.2 again to obtain that it has integer energy $\frac{1}{8\pi^2} \|F_B \oplus F_{-B}\|_{L^2(M)}^2 = l$. Hence

$$\frac{1}{8\pi^2} \|F_B \oplus F_B\|_{L^2(M)}^2 = \frac{1}{8\pi^2} \|F_B \oplus F_{-B}\|_{L^2(M)}^2 = l.$$

Therefore an admissible $SU(2) \times U(1)$ instanton $\nabla_{A+(B\oplus B)}$ has integer energy by part (iv) of Lemma 4.1.

Finally note that if E is an $U(2)$ vector bundle with $w_2(E^{\mathbb{R}}) = 0 \in H^2(M; \mathbb{Z}_2)$ then it uniquely lifts to an $SU(2) \times U(1)$ vector bundle \tilde{E} . Consequently any connection on E can be uniquely lifted to a connection on \tilde{E} . Taking into account the canonical isomorphism $\mathfrak{u}(2) \cong \mathfrak{su}(2) \times \mathfrak{u}(1)$ we find that the energies of the original and the lifted connections are equal. Consequently our considerations continue to hold for admissible $U(2)$ Yang–Mills instantons on bundles with vanishing second Stiefel–Whitney class.

Summing up we obtain

Theorem 4.1. *Let (M, g) be an ALF space with an end $W \cong N \times \mathbb{R}^+$. Let E be a $U(2)$ vector bundle with $w_2(E^{\mathbb{R}}) = 0 \in H^2(M; \mathbb{Z}_2)$ carrying an admissible $U(2)$ Yang–Mills instanton ∇_A i.e., a smooth self-dual connection satisfying Definition 2.1. Then*

$$\frac{1}{8\pi^2} \|F_A\|_{L^2(M)}^2 \in \mathbb{N}$$

that is, its energy is always integer.

Regarding the asymptotical flat connection $\nabla_\Gamma|_W$ in part (i) of Definition 2.1 either ∇_A is flat and $\nabla_\Gamma|_W$ extends smoothly over the whole M such that $\nabla_A = \nabla_\Gamma$ (in this case if $\nabla_\Gamma \neq \nabla_\Theta$ then $\pi_1(M) \neq 1$) or ∇_A is non-flat and $\nabla_\Gamma|_W$ is a flat connection with trivial local holonomy $m = 0$ in (5) (in this case if $\nabla_\Gamma|_W \neq \nabla_\Theta|_W$ then $\pi_1(B_{+\infty}) \neq 1$ in (1) and (4)). \diamond

²To be precise for this we generalize Definition 2.1 i.e., the definition of $SU(2)$ -admissibility to arbitrary G -admissibility in a straightforward way.

Remark. We would like to point out the relevance of admissibility in the case of Abelian instantons. In this case it is easy to see in the framework of L^2 -cohomology that without imposing admissibility a continuous energy spectrum would destroy everything.

So let ∇_B be an Abelian instanton over an ALF space (M, g) . By definition its energy is finite but in general $[\frac{1}{2\pi\mathbf{i}}F_B] \in \overline{H}_{L^2}^2(M, g)$ i.e., the curvature lives only in the second (reduced) L^2 -cohomology group of the non-compact but complete Riemannian 4-manifold (M, g) with ALF asymptotics. It turns out [23] that for such geometries this subtle cohomology reduces to ordinary cohomology of the Hausel–Hunsicker–Mazzeo compactification (4). Therefore $\overline{H}_{L^2}^2(M, g) \cong H^2(X; \mathbb{R})$ and using (1) and (4) the Mayer–Vietoris sequence gives

$$\dots \rightarrow H^1(N; \mathbb{R}) \rightarrow \overline{H}_{L^2}^2(M, g) \rightarrow H_c^2(M; \mathbb{R}) \oplus H^2(B_{+\infty}; \mathbb{R}) \rightarrow H^2(N; \mathbb{R}) \rightarrow \dots$$

Consequently these cohomology classes can be divided into two parts as follows: we say that an L^2 -cohomology class on a complete Riemannian manifold (M, g) is *topological* if its image lies in the ordinary compactly supported de Rham cohomology $H_c^2(M; \mathbb{R})$ under the homomorphism above. Otherwise it is called *non-topological* i.e., if its image is in $H^2(B_{+\infty}; \mathbb{R})$. Note that representatives of non-topological L^2 -cohomology classes are necessarily exact 2-forms on M . Roughly speaking non-topological L^2 -cohomology classes are not predicatable by topological means [32].

We make two assumptions. The first is that $H^{1,2}(N; \mathbb{R}) = 0$. In this case unambiguously

$$\frac{1}{2\pi\mathbf{i}}F_B = \sum_{i=1}^{b_c^2(M)} k_i \omega_i + \omega_0$$

where $\{\omega_i\}$ are harmonic representatives of the basis of the compactly supported integer cohomology $H_c^2(M; \mathbb{R}) \cap H^2(M; \mathbb{Z})$ and $k_i \in \mathbb{Z}$ and $\omega_0 = d\beta$ represents the exact non-topological part. Obviously ∇_B lives on a line bundle L with Chern class $c_1(L) = [\sum k_i \omega_i] \in H_c^2(M; \mathbb{R}) \cap H^2(M; \mathbb{Z})$. The finite energy of this Abelian instanton looks like

$$\begin{aligned} \frac{1}{8\pi^2} \|F_B\|_{L^2(M)}^2 &= \frac{1}{2} \int_M \left(\frac{1}{2\pi\mathbf{i}}F_B \right) \wedge * \left(\frac{1}{2\pi\mathbf{i}}F_B \right) \\ &= \frac{1}{2} \sum_{i,j} k_i k_j \int_M \omega_i \wedge * \omega_j + \sum_i k_i \int_M \omega_0 \wedge * \omega_i + \frac{1}{2} \int_M \omega_0 \wedge * \omega_0. \end{aligned}$$

The second assumption is that all $k_i = 0$. In this case the curvature is an exact self-dual non-compactly supported 2-form on (M, g) still having finite energy. The corresponding Abelian instanton ∇_{B_0} with $F_{B_0} = \omega_0$ lives on the trivial bundle $L_0 \cong M \times \mathbb{C}$. Taking into account the triviality of L_0 as well as the Abelian nature of ∇_{B_0} if it is self-dual on L_0 with $\frac{1}{8\pi^2} \|F_{B_0}\|_{L^2(M)}^2 = 1$ then for any $c \in \mathbb{R}$ the rescaled connection ∇_{cB_0} remains smooth and self-dual on L_0 with energy $\frac{1}{8\pi^2} \|F_{cB_0}\|_{L^2(M)}^2 = c^2$. This unexpected continuous energy phenomenon occurs for example in the important case of the multi-Taub–NUT spaces [20, Sections 2 and 4]: the connection 1-form B_0 arises as the metric dual of the Killing field associated to the $U(1)$ isometry of the metric.

Consequently in this case admissibility is indeed to be imposed which gives $c = k$ hence the energy of the $U(1)$ Yang–Mills instanton ∇_{kB_0} is a half-integer. Hence the energy of the reducible $SU(2)$ Yang–Mills instanton $\nabla_{kB_0 \oplus (-kB_0)}$ or the $U(2)$ one $\nabla_{kB_0 \oplus kB_0}$ will be integer in agreement with Theorems 3.2 or 4.1.

5 Conclusion and outlook toward quantum theory

In this paper we proved that the energy spectrum of a natural class of $SU(2)$ or $U(2)$ Yang–Mills instantons (called admissible instantons) over a generic (i.e., not necessarily hyper-Kähler) ALF space consists of non-negative integers only. This sharpens the previously known result that the energy must be congruent to a Chern–Simons invariant of the infinitely distant boundary N hence to a rational number. In this context the stronger result is surprising because N as defined in (1) has many non-integer Chern–Simons invariants, cf. [24, Theorem 4.3]. We have seen that the reason behind this integrality phenomenon is the existence of a smooth compactification of the original space and a powerful codimension 2 singularity removal result.

In this closing section we would like to push one step further the role played by this Hausel–Hunsicker–Mazzeo compactification and this Sibner–Sibner–Råde singularity removal theorem in the ALF scenario. Namely, we ask ourselves whether the emergence of the infinitely distant surface $B_{+\infty}$ in (1) and (4) is a topological hint that the thing lurks behind the concept of a four dimensional Yang–Mills theory over an ALF space is in fact a 2 dimensional conformal field theory. If this is the case then it would help one to construct the underlying (twisted $N = 2$ supersymmetric) quantum gauge theory.

We want to calculate the partition function of our quantum gauge theory over the ALF space (M, g) . Put

$$\mathcal{A}(E_0) := \{\nabla_A \mid \nabla_A \text{ is admissible on } E_0 \text{ in the sense of Definition 2.1}\}$$

and let $\mathcal{B}(E_0) := \mathcal{A}(E_0)/\mathcal{G}(E_0)$ be the quotient space of $L^2_{2,\Gamma}$ gauge equivalence classes. Taking the complex coupling constant $\tau := \frac{\theta}{2\pi} + \frac{4\pi}{e^2}\mathbf{i} \in \mathbb{C}^+$ and suppressing the supersymmetric terms the partition function is a complex number given by the formal integral

$$Z(M, g, \tau, SU(2)) = \int_{\mathcal{B}(E_0)} e^{\frac{1}{2e^2}\|F_A\|_{L^2(M)}^2 - \frac{i\theta}{16\pi^2}(F_A, *F_A)_{L^2(M)}} \mathcal{D}[\nabla_A].$$

For simplicity we suppose that (i) the integral above localizes to classical solutions; (ii) all finite energy classical solutions are self-dual; (iii) all self-dual solutions are admissible in the sense of Definition 2.1; (iv) M is simply connected and the surface $B_{+\infty}$ is orientable. Let ∇_{A_0} be a finite energy classical solution to the Yang–Mills equations. Then by Theorem 3.2 there exists an integer $k \in \mathbb{N}$ and a flat connection $\nabla_\Gamma|_W$ with $m = 0$ i.e., which is a pullback of a flat connection $\nabla_{\Gamma_{+\infty}}$ on $B_{+\infty}$, such that $([\nabla_{A_0}], \Gamma) \in \mathcal{M}(k, \Gamma)$. Then the previous integral cuts down to

$$Z(M, g, \tau, SU(2)) = Z_{\text{Quant}} \sum_{k \in \mathbb{N}} \int_{\mathcal{M}(B_{+\infty})} \int_{\widehat{\mathcal{M}}(k, \Gamma)} e^{\frac{1}{2e^2}\|F_{A_0}\|_{L^2(M)}^2 - \frac{i\theta}{16\pi^2}(F_{A_0}, *F_{A_0})_{L^2(M)}} \mathcal{D}[\nabla_{A_0}] \mathcal{D}[\nabla_{\Gamma_{+\infty}}]$$

where $\mathcal{M}(B_{+\infty})$ is the modulis space of flat connections over $B_{+\infty}$. We may try to calculate this integral within the framework of (topological) quantum field theory. Since elements of $\mathcal{B}(E_0)$ extend as finite energy objects over X and in particular elements of $\widehat{\mathcal{M}}(k, \Gamma)$ remain smooth on it it is plausible to replace M by its compactification X as in (4) and suppose that $Z(M, g, \tau, SU(2)) = Z(X, \tau, SU(2))$ where $Z(X, \tau, SU(2)) : \mathcal{H}_{-\infty}(\emptyset) \rightarrow \mathcal{H}_{+\infty}(\emptyset)$ is a linear map and $\mathcal{H}_{\pm\infty}(\emptyset) \cong \mathbb{C}$ are the Hilbert spaces attached to the past and future boundaries of the closed space X now considered as a cobordism between two emptysets. Assume that for a fixed $0 < R < +\infty$ the space X is cut up along $\partial\overline{M}_R = N \times \{R\}$ as follows:

$$X = (M \setminus V_R) \cup_{N \times \{R\}} V_R$$

and let $\mathcal{H}_R(N)$ denote the Hilbert space associated to $\partial\overline{M}_R \cong N$. By the standard axioms we expect that there exist vectors $v_R \in \mathcal{H}_R(N)$ and $w_R \in \mathcal{H}_R(N)^*$ such that $Z(X, \tau, \text{SU}(2)) = (v_R, w_R)$ and the left hand side is independent of the particular value of R . Therefore taking the limit $R \rightarrow +\infty$ we formally obtain

$$Z(M, g, \tau, \text{SU}(2)) = Z(X, \tau, \text{SU}(2)) = (v_{+\infty}, w_{+\infty}) \quad (10)$$

where $v_{+\infty} \in \mathcal{H}_{+\infty}(N) \cong \mathcal{H}(B_{+\infty})$ and similarly $w_{+\infty} \in \mathcal{H}(B_{+\infty})^*$ since in the limit $R \rightarrow +\infty$ the fibration (1) cuts down to $B_{+\infty} \subset X$ as in (4).

What sort of space is $\mathcal{H}(B_{+\infty})$ here? We can follow the original ideas of Witten [36]. Theorem 3.2 says that if M is simply connected then all admissible $\text{SU}(2)$ Yang–Mills instantons approach a flat connection $\nabla_\Gamma|_{V_R}$ which in the limit $R \rightarrow +\infty$ smoothly reduces to a flat $\text{SU}(2)$ connection $\nabla_{\Gamma_{+\infty}}$ on $B_{+\infty}$. Therefore by the principles of geometric quantization we would expect that

$$\mathcal{H}(B_{+\infty}) = \bigoplus_{k \in \mathbb{N}} H^0(\mathcal{M}(B_{+\infty}); L^k)$$

where L is the usual quantizing line bundle over $\mathcal{M}(B_{+\infty})$. To regard H^0 as the space of holomorphic sections we need a complex structure on $\mathcal{M}(B_{+\infty})$ which is inherited from one on $B_{+\infty}$. However the whole procedure and hence the space $\mathcal{H}(B_{+\infty})$ is expected to be independent of any particular complex structure which leads to the usual conclusion that $H^0(\mathcal{M}(B_{+\infty}); L^k)$ should be the space of conformal blocks of some conformal field theory (probably the $\text{SU}(2)$ Wess–Zumino–Witten model over $B_{+\infty}$ at level k).

Therefore $\mathcal{H}(B_{+\infty})$ would carry a representation of the symmetry algebra of some conformal field theory and in particular of the mapping class group of $B_{+\infty}$; these might lead to the understanding of the modular properties of the original partition function $Z(M, g, \tau, \text{SU}(2))$ if written in the form (10). In this way apparently a very geometric link between $4d$ YM theory over an ALF space and $2d$ CFT emerges which supports some recent investigations [1, 2, 4, 33].

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References

- [1] Alday, L.F., Gaiotto, D., Tachikawa, Y.: *Liouville correlation functions from four-dimensional gauge theories*, Lett. Math. Phys. **91**, 167-197 (2010);
- [2] Alfimov, M.N., Tarnopolosky, G.M.: *Parafermionic Liouville field theory and instanton moduli spaces on ALE spaces*, preprint, 21pp., arXiv: 1110.5628 [hep-th] (2011);
- [3] Atiyah, M.F., Hitchin, N.J.: *The geometry and dynamics of magnetic monopoles*, Princeton University Press, Princeton (1988);

- [4] Belavin, A.A., Bershtein, M.A., Feigin, B.L., Litvinov, A.V., Tarnopolosky, G.M.: *Instanton moduli spaces and bases in coset conformal field theory*, preprint, 31pp., arXiv: 1111.2803 [hep-th] (2011);
- [5] Bianchi, M., Fucito, F., Rossi, G., Martenilli, M.: *Explicit construction of Yang–Mills instantons on ALE spaces*, Nucl. Phys. **B473**, 367-404 (1996);
- [6] Bruckmann, F., N ogr adi, D., van Baal, P.: *Constituent monopoles through the eyes of fermion zero modes*, Nucl. Phys. **B666**, 197-229 (2003);
- [7] Bruckmann, F., N ogr adi, D., van Baal, P.: *Higher charge calorons with non-trivial holonomy*, Nucl. Phys. **B698**, 233-254 (2004);
- [8] Charbonneau, B., Hurtubise, J.: *The Nahm transform for calorons*, in: *The many facets of geometry: a tribute to Nigel Hitchin* (ed.: O.G. Prada, J.-P. Bourguignon, S. Salamon), 34-70, Oxford Univ. Press (2010);
- [9] Charbonneau, B., Hurtubise, J.: *Calorons, Nahm’s equations and bundles over $\mathbb{P}^1 \times \mathbb{P}^1$* , Comm. Math. Phys. **280**, 315-349 (2008);
- [10] Cherkis, S.A.: *Moduli spaces of instantons on the Taub–NUT space*, Comm. Math. Phys. **290**, 719-736 (2009);
- [11] Cherkis, S.A.: *Instantons on the Taub–NUT space*, Adv. Theor. Math. Phys. **14** 609-642 (2010);
- [12] Cherkis, S.A.: *Instantons on gravitons*, Comm. Math. Phys. **306**, 449-483 (2011);
- [13] Cherkis, S.A., Hitchin, N.J.: *Gravitational instantons of type D_k* , Comm. Math. Phys. **260**, 299-317 (2005);
- [14] Derek, H.: *Large scale and large period limits of symmetric calorons*, Journ. Math. Phys. **48**, 082905 (2007);
- [15] Eguchi, T., Gilkey, P.B., Hanson, A.J.: *Gravity, gauge theories and differential geometry*, Phys. Rep. **66**, 213-393 (1980);
- [16] Etesi, G.: *The topology of asymptotically locally flat gravitational instantons*, Phys. Lett. **B641**, 461-465 (2006);
- [17] Etesi, G., Hausel, T.: *Geometric interpretation of Schwarzschild instantons*, Journ. Geom. Phys. **37**, 126-136 (2001);
- [18] Etesi, G., Hausel, T.: *Geometric construction of new Yang–Mills instantons over Taub–NUT space*, Phys. Lett. **B514**, 189-199 (2001);
- [19] Etesi, G., Hausel, T.: *On Yang–Mills instantons over multi-centered gravitational instantons*, Comm. Math. Phys. **235**, 275-288 (2003);
- [20] Etesi, G., Jardim, M.: *Moduli spaces of self-dual connections over asymptotically locally flat gravitational instantons*, Comm. Math. Phys. **280**, 285-313 (2008), Erratum: *ibid.* **288**, 799-800 (2009);

- [21] Etesi, G., Szabó, Sz.: *Harmonic functions and instanton moduli spaces on the multi-Taub–NUT space*, Comm. Math. Phys. **301**, 175-214 (2011);
- [22] Gibbons, G.W., Hawking, S.W.: *Gravitational multi-instantons*, Phys. Lett. **B78**, 430-432 (1978);
- [23] Hausel, T., Hunsicker, E., Mazzeo, R.: *Hodge cohomology of gravitational instantons*, Duke Math. Journ. **122**, 485-548 (2004);
- [24] Kirk, P., Klassen, E.: *Chern–Simons invariants of 3-manifolds decomposed along tori and the circle bundle over the representation space of T^2* , Comm. Math. Phys. **153**, 521-557 (1993);
- [25] Lee, Kimyeong, Yi, Sang-Heon: *1/4 BPS dyonic calorons*, Phys. Rev. **D67**, 025012 (2003);
- [26] Mosna, R.A., Tavares, G.M.: *New self-dual solutions of SU(2) Yang–Mills theory in Euclidean Schwarzschild space*, Phys. Rev. **D80**, 105006 (2009);
- [27] Murray, M., Vozzo, R.F.: *The caloron correspondence and higher string classes for loop groups*, Journ. Geom. Phys. **90**, 1235-1250 (2010);
- [28] Nakamura, A., Sakaguchi, J.: *Multicalorons revisited*, Journ. Math. Phys. **51**, 043503 (2010);
- [29] Nye, T.M.W.: *The geometry of calorons*, PhD Thesis, University of Edinburgh, 147 pp, arXiv: hep-th/0311215 (2001);
- [30] Råde, J.: *Singular Yang–Mills fields. Local theory II.*, Journ. Reine Angew. Math. **456**, 197-219 (1994);
- [31] Sibner, I.M., Sibner, R.J.: *Classification of singular Sobolev connections by their holonomy*, Comm. Math. Phys. **144**, 337-350 (1992);
- [32] Segal, G., Selby, A.: *The cohomology of the space of magnetic monopoles*, Comm. Math. Phys. **177**, 775–787 (1996);
- [33] Tachikawa, Y.: *A strange relationship between 2d CFT and 4d gauge theory* (in Japanese), preprint, 38pp., arXiv: 1108.5632 [hep-th] (2011);
- [34] Ward, R.S.: *Symmetric calorons*, Phys. Lett. **B582**, 203-210 (2004);
- [35] Wehrheim, K.: *Energy identity for anti-self-dual instantons on $\mathbb{C} \times \Sigma$* , Math. Res. Lett. **13**, 161-166 (2006);
- [36] Witten, E.: *Quantum field theory and the Jones polynomial*, Comm. Math. Phys. **121**, 351-399 (1989);
- [37] Witten, E.: *Branes, instantons and Taub–NUT spaces*, Journ. High Energy Phys. 0906:067 (2009).