Averages along the squares and related topics

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Abstract

After discussing some earlier results we turn to the answer to a question raised by J-P. Conze. We show that for any $x, \alpha \in \mathbb{T}$, $\alpha \notin \mathbb{Q}$ there exist $f \in L^1(\mathbb{T})$, $f \geq 0$ such that the averages

$$(\star) \qquad \frac{1}{N} \sum_{n=1}^{N} f(y + nx + n^2\alpha)$$

diverge for a.e. y. By Birkhoff's Ergodic Theorem applied on $\mathbb{T}^2$ for the transformation $(x, y) \mapsto (x + \alpha, y + 2x + \alpha)$ for almost every $x \in \mathbb{T}$ the averages $(\star)$ converge for a.e. y. We show that given $\alpha \notin \mathbb{Q}$ one can find $f \in L^1(\mathbb{T})$ for which the set $D_{\alpha,f} \stackrel{\text{def}}{=} \{x \in \mathbb{T} : (\star)$ diverges for a.e. y as $N \to \infty\}$ is of Hausdorff dimension one. We also show that for a polynomial $p(n)$ of degree two with integer coefficients the sequence $p(n)$ is universally L^1-bad.