DIMENSION OF SLICES OF SIERPIŃSKI-LIKE CARPETS

BALÁZS BÁRÁNY AND MICHAŁ RAMS

Abstract. We investigate the dimension of intersections of the Sierpiński-like carpets with lines. We show a sufficient condition that for a fixed rational slope the dimension of almost every intersection w.r.t the natural measure is strictly greater than $s-1$, and almost every intersection w.r.t the Lebesgue measure is strictly less than $s-1$, where s is the Hausdorff dimension of the carpet. Moreover, we give partial multifractal spectra for the Hausdorff and packing dimension of slices.

1. Introduction and Statements

Let $N \geq 2$ be an integer and let Ω be a subset of $\{0,\dots,N-1\}\times\{0,\dots,N-1\}$. Suppose that $N+1 \leq \sharp\Omega$. Let

$$F_{k,l}(x,y) := \frac{1}{N}(x,y) + \frac{1}{N}(k,l) \text{ for } (k,l) \in \Omega. \tag{1.1}$$

The attractor $\Lambda \subset \mathbb{R}^2$ of the iterated function system $\Psi = \{F_\omega\}_{\omega\in\Omega}$ is called a Sierpiński-like carpet. It is well known that Ψ satisfies the open set condition and $\dim_H \Lambda = \dim_P \Lambda = \dim_B \Lambda = \frac{\log \sharp\Omega}{\log N} > 1$, where $\dim_H \Lambda$ denotes the Hausdorff dimension, $\dim_P \Lambda$ denotes the packing dimension and $\dim_B \Lambda$ denotes the box (or Minkowski) dimension of the set Λ. For the definition and basic properties of the box, packing and Hausdorff dimensions we refer the reader to [2].

The main purpose of this paper is to investigate the dimension theory of the slices with fixed slope. For an angle θ denote proj_θ the θ-angle projection onto the y-axis. That is, $\mathrm{proj}_\theta(x,y) = y - x\tan\theta$. For a point $a \in \mathrm{proj}_\theta\Lambda$ let

$$L_{\theta,a} := \left\{(x,y) \in \mathbb{R}^2 : a = y - x\tan\theta\right\} \text{ and } E_{\theta,a} = L_{\theta,a} \cap \Lambda$$

be the corresponding slice of the attractor. Without loss of generality, by applying rotation and mirroring transformations on Λ, we may assume that $\theta \in [0,\pi/2)$.

The dimension theory of some special cases was examined before for example in [1, 9, 10, 15]. Liu, Xi and Zhao [9] proved for the usual Sierpński carpet (i.e. $N=3$ and $\Omega = \{0,1,2\}\times\{0,1,2\}\setminus\{(1,1)\}$) that the box and Hausdorff dimension of a slice $E_{\theta,a}$ for Lebesgue almost every point a are equal to a constant depending only on θ when the slope $\tan\theta$ is rational. Manning and Simon [10] showed that this constant is strictly less than $s-1$, where s is the dimension of the usual Sierpiński carpet. Later Bárány, Ferguson and Simon [1] proved analogous result for the usual Sierpiński gasket (i.e. $N=2$ and $\Omega = \{0,1\}\times\{0,1\}\setminus\{(1,1)\}$). Moreover, they showed that the box and Hausdorff dimension of a slice $E_{\theta,a}$ for almost every point a w.r.t the projection of the natural measure are equal to a constant depending only on θ strictly greater than $s-1$, when the slope $\tan\theta$ is rational , where s is

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the dimension of the gasket. Furthermore, Bárány, Ferguson and Simon [1] gave a non-complete multifractal spectra for the dimension of the slices. Our goal is to generalize the previous results.

Let ν be the unique self-similar measure satisfying

$$\nu = \sum_{\omega \in \Omega} \frac{1}{\sharp\Omega} \nu \circ F_{\omega}^{-1}.$$

We call the measure ν the natural measure supported on Λ. One may show that this measure is nothing else than the normalized s-dimensional Hausdorff measure restricted to Λ, i.e. $\nu = \frac{\mathcal{H}^s|_\Lambda}{\mathcal{H}^s(\Lambda)}$, where $s = \frac{\log \sharp\Omega}{\log N}$. We denote by $\nu_\theta = \nu \circ \mathrm{proj}_\theta^{-1}$ the projection of the natural measure.

First, we mention a weak dimension conservation phenomena for the Sierpiński-like carpets.

Proposition 1.1. *Let $N \geq 2$ be integer and $\Omega \subseteq \{0, \dots, N-1\} \times \{0, \dots, N-1\}$ then for every fixed $\theta \in [0, \pi/2)$*

$$\dim_H E_{\theta,a} = \dim_B E_{\theta,a} = \frac{\log \sharp\Omega}{\log N} - \dim_H \nu_\theta \textit{ for } \nu_\theta\textit{-a.e } a.$$

In particular,

$$\dim_H E_{\theta,a} = \dim_B E_{\theta,a} > \frac{\log \sharp\Omega}{\log N} - 1 \text{ for } \nu_\theta\text{-a.e } a. \Leftrightarrow \dim_H \nu_\theta < 1. \tag{1.2}$$

This inequality makes sense when $N + 1 \leq \sharp\Omega$. In the case of rational slopes we prove that the strict inequality is satisfied in (1.2) whenever $N \nmid \sharp\Omega$.

Theorem 1.2. *Let $N \geq 2$ be an integer and $\Omega \subseteq \{0, \dots, N-1\} \times \{0, \dots, N-1\}$ such that $N + 1 \leq \sharp\Omega$ and $N \nmid \sharp\Omega$. Then for every fixed $\theta \in [0, \pi/2)$ such that $\tan\theta \in \mathbb{Q}$ there exists a constant $\alpha(\theta)$ depending only on θ such that*

$$\alpha(\theta) = \dim_H E_{\theta,a} = \dim_B E_{\theta,a} > \frac{\log \sharp\Omega}{\log N} - 1 \textit{ for } \nu_\theta\textit{-a.e } a.$$

A similar theorem can be formalized for Lebesgue-typical points of the projection.

Theorem 1.3. *Let $N \geq 2$ be integer and $\Omega \subseteq \{0, \dots, N-1\} \times \{0, \dots, N-1\}$ such that $N + 1 \leq \sharp\Omega$ and $N \nmid \sharp\Omega$. For every fixed $\theta \in [0, \pi/2)$ such that $\tan\theta \in \mathbb{Q}$ and $\mathrm{proj}_\theta\Lambda = [-\tan\theta, 1]$ there exists a constant β depending only on θ such that*

$$\beta(\theta) = \dim_H E_{\theta,a} = \dim_B E_{\theta,a} < \frac{\log \sharp\Omega}{\log N} - 1 \textit{ Leb.-a.e. } a \in \mathrm{proj}_\theta\Lambda.$$

The proof of Theorem 1.2 and Theorem 1.3 uses a method different to one used in Manning, Simon [10] and Bárány, Ferguson, Simon [1]. In both of the papers the authors construct a finite set of matrices. They prove that this set of matrices satisfies a very strong irreducibility property (i.e. there exists a finite sequence of matrices such that the product has strictly positive elements) and using this fact they prove that the Lebesgue typical slice for a fixed rational slope has dimension strictly less than $s - 1$. The proof of this special irreducibility property is ad hoc, depends very much on the structure of the usual Sierpiński gasket and carpet and does not hold in general. We are going to modify this method as follows. We will construct the same type of matrices as in [1], [9]. Using the general properties of those matrices we will show that a ν_θ typical slice has dimension strictly greater than $s-1$ whenever $\tan\theta \in \mathbb{Q}$. Applying this fact and the results of Feng and Lau about nonnegative matrices [6] we will be able to prove the theorem about Lebesgue typical slices. For further details see Section 4.

Because of Theorem 1.2 and Theorem 1.3 one can claim that the dimension of the slices has a non-trivial multifractal spectra for rational slopes. Bárány, Ferguson and Simon [1] gave the incomplete spectrum of the dimension of the slices of the usual Sierpiński gasket. Precisely, they calculated the function

$$\delta \mapsto \dim_H \{a \in \mathrm{proj}_\theta\Lambda : \dim_H E_{\theta,a} = \delta\}$$

for any θ such that $\tan\theta \in \mathbb{Q}$ and the values $\delta \geq \beta(\theta)$, where $\beta(\theta)$ is the Lebesgue-typical dimension. Our aim is to generalize the previous result for the Hausdorff and packing dimension of the slices of the general Sierpiński-like carpets. Moreover, we will give the full spectra for the packing dimension of the slices of the usual Sierpiński gasket.

Consider the projected IFS $\psi = \{f_\omega\}$ of $\Psi = \{F_\omega\}_{\omega\in\Omega}$, i.e.

$$f_{k,l}(x) = \frac{x}{N} + \frac{-k\tan\theta + l}{N}, \text{ for every } (k,l) \in \Omega. \tag{1.3}$$

By straightforward calculations and [11, Theorem 2.7] we see that ψ satisfies the finite type condition for $\tan\theta \in \mathbb{Q}$ and therefore, the weak separation property.

Let us divide the interval $I = [-\tan\theta, 1] = \mathrm{proj}_\theta\Lambda$ into $p+q$ equal intervals, i.e. $I_k = [\frac{k-1-p}{q}, \frac{k-p}{q}]$ for $k = 1,\dots,p+q$. Moreover, let us divide I_k for every k into N equal parts. That is, $I_k^\xi = [\frac{k-1-p}{q} + \frac{\xi}{Nq}, \frac{k-1-p}{q} + \frac{\xi+1}{Nq}]$ for $\xi = 0,\dots,N-1$. For every $\xi = 0,\dots,N-1$ let us define a $(p+q)\times(p+q)$ real matrix A_ξ in the following way

$$(A_\xi)_{i,j} := \sharp\left\{\omega \in \Omega : f_\omega(I_j) = I_i^\xi\right\}. \tag{1.4}$$

By some simple calculations the matrices A_n, $n = 0,\dots,N-1$ can be written in the form

$$(A_n)_{i,j} = \sharp\{(k,l) \in \Omega : iN + n = kp + (N-1-l)q + j + N - 1\}.$$

Denote by $P(t)$ the pressure function which is defined as

$$P(t) = \lim_{n\to\infty} \frac{1}{n\log N} \log \sum_{\xi_1,\dots,\xi_n=0}^{N-1} \left(\underline{e}^T A_{\xi_1}\cdots A_{\xi_n}\underline{e}\right)^t, \tag{1.5}$$

where $\underline{e} = (1,\dots,1)^T \in \mathbb{R}^{p+q}$, and let us define

$$b_{\min} = \lim_{t\to-\infty} \frac{P(t)}{t} \text{ and } b_{\max} = \lim_{t\to\infty} \frac{P(t)}{t}. \tag{1.6}$$

Theorem 1.4. *Let $N \geq 2$ be integer and $\Omega \subseteq \{0,\dots,N-1\}\times\{0,\dots,N-1\}$. Then for every fixed $\theta \in [0,\pi/2)$ such that $\tan\theta \in \mathbb{Q}$ and $[-\tan\theta, 1] = \mathrm{proj}_\theta\Lambda$ we have*

$$\dim_H \{a \in \mathrm{proj}_\theta\Lambda : \dim_H E_{\theta,a} = \delta\} = \dim_H \{a \in \mathrm{proj}_\theta\Lambda : \dim_P E_{\theta,a} = \delta\} = P^*(\delta) \text{ for every } \delta \in [\beta(\theta), b_{\max}],$$

where $P^*(\delta) := \inf_t\{-\delta t + P(t)\}$. Moreover, the function $P^*(\delta)$ is continuous, concave and monotone decreasing on $[\beta(\theta), b_{\max}]$.

Because of the special structure of the usual Sierpiński gasket (see Lemma 4.10), it is possible to give complete spectrum for the packing dimension of the slices.

Proposition 1.5. *Let Λ be the usual Sierpiński gasket, i.e. $N = 2$ and $\Omega = \{0,1\}^2 \setminus \{(1,1)\}$. Then for every fixed $\theta \in [0, \pi/2)$ such that $\tan\theta \in \mathbb{Q}$*

$$\dim_H \{a \in \mathrm{proj}_\theta \Lambda : \dim_P E_{\theta,a} = \delta\} = P^*(\delta) \text{ for every } \delta \in [b_{\min}, b_{\max}].$$

The organization of the paper is as follows, in Section 2 we prove Proposition 1.1. In Section 3 we will construct our matrices according to the rational projection and using their general properties we prove Theorem 1.2. In Section 4 we define the so-called pressure function corresponding to our nonnegative matrices and using previous results of Feng and Lau [3],[4],[6] we prove Theorem 1.3 and Theorem 1.4.

2. Proof of Proposition 1.1

Before we prove Proposition 1.1, we state a general dimension conservation phenomena for self-similar measures of Sierpiński-like carpets. Let $N \geq 2$ be integer and $\Omega \subseteq \{0, \ldots, N-1\} \times \{0, \ldots, N-1\}$. Then it is well known that for every positive probability vector $(p_\omega)_{\omega\in\Omega}$ there exists a unique probability measure μ satisfying

$$\mu = \sum_{\omega\in\Omega} p_\omega \mu \circ F_\omega^{-1},$$

where the IFS $\Psi = \{F_\omega\}_{\omega\in\Omega}$ are defined in (1.1). Denote by Λ the attractor of $\{F_\omega\}_{\omega\in\Omega}$.

Proposition 2.1. *For any $\theta \in [0, \pi/2)$*

$$\dim_H \mu_\theta + \dim_H \mu_a^\theta = \dim_H \mu \text{ for } \mu_\theta\text{-a.e. } a,$$

where $\mu_\theta = \mu \circ \mathrm{proj}_\theta^{-1}$ and $\{\mu_a^\theta\}_{a\in\mathrm{proj}_\theta\Lambda}$ denote the canonical system of conditional measures with respect to the partition $\{\mathrm{proj}_\theta^{-1}(a) : a \in \mathrm{proj}_\theta\Lambda\}$. In particular, for the natural measure $\nu = \frac{\mathcal{H}^s|_\Lambda}{\mathcal{H}^s(\Lambda)}$, where $s = \frac{\log \sharp\Omega}{\log N}$ (the measure corresponding to the probabilistic vector $p_\omega = (1/\sharp\Omega, \ldots, 1/\sharp\Omega)$), we have

$$\frac{\log \sharp\Omega}{\log N} - \dim_H \nu_\theta \leq \dim_H E_{\theta,a} \text{ for } \nu_\theta\text{-a.e. } x.$$

Proof. To prove the proposition we apply the results of Furstenberg [7] about ergodic CP-chains.

We define a measurable map $T : \mathcal{P}([0,1]^2) \times [0,1]^2 \mapsto \mathcal{P}([0,1]^2) \times [0,1]^2$, where $\mathcal{P}(\Lambda)$ denotes the probability measures of $[0,1]^2$, as follows

$$T(\vartheta, x) := \left(\frac{\vartheta|_{[\frac{k}{N},\frac{k+1}{N})\times[\frac{l}{N},\frac{l+1}{N})} \circ F_{k,l}}{\vartheta([\frac{k}{N},\frac{k+1}{N}) \times [\frac{l}{N},\frac{l+1}{N}))}, Nx \mod 1 \right),$$

where $x \in [\frac{k}{N},\frac{k+1}{N}) \times [\frac{l}{N},\frac{l+1}{N})$. Moreover, let us define a probability measure Θ on $\mathcal{P}([0,1]^2) \times [0,1]^2$ that $d\Theta(\vartheta, x) = d\vartheta(x) d\delta_\mu(\vartheta)$, where μ is a given self-similar measure of Λ. Then it is easy to see that the measure Θ is T-invariant and ergodic. The statement of proposition follows from [7, Theorem 3.1]. $\square$

For an alternative proof we refer the reader to [5, Proposition 4.14, Remark 4.15].

For a finite length word $\underline{\omega} \in \Omega^n$ let $F_{\underline{\omega}} = F_{\omega_0} \circ \cdots \circ F_{\omega_{n-1}}$ and denote by $G_n(\theta, a)$ the set of nth level cylinders intersecting the line $L_{\theta,a}$. That is,

$$G_n(\theta, a) := \left\{\underline{\omega} \in \Omega^n : F_{\underline{\omega}}(\Lambda) \cap L_{\theta,a} \neq \emptyset\right\}. \tag{2.1}$$

Standard calculation gives us

Lemma 2.2. *For any $\theta \in [0, \pi/2)$*

$$\underline{\dim}_B E_{\theta,a} = \liminf_{n\to\infty} \frac{\log \sharp G_n(\theta, a)}{n \log N} \text{ and } \overline{\dim}_B E_{\theta,a} = \limsup_{n\to\infty} \frac{\log \sharp G_n(\theta, a)}{n \log N}.$$

Lemma 2.3. *For any $\theta \in [0, \pi/2)$*

$$\underline{d}_{\nu_\theta}(a) + \overline{\dim}_B E_{\theta,a} \leq \frac{\log \sharp\Omega}{\log N} \text{ for every } a \in \mathrm{proj}_\theta\Lambda.$$

Proof. First, let us observe that

$$\nu_\theta(B_{N^{-n}}(a)) \geq \frac{\sharp G_n(\theta, a)}{\sharp\Omega^n}.$$

Hence,

$$\underline{d}_{\nu_\theta}(a) = \liminf_{n\to\infty} \frac{\log \nu_\theta(B_{N^{-n}}(a))}{-n \log N} \leq \liminf_{n\to\infty} \frac{\log \frac{\sharp G_n(\theta,a)}{\sharp\Omega^n}}{-n \log N} =$$
$$\frac{\log \sharp\Omega}{\log N} - \limsup_{n\to\infty} \frac{\log \sharp G_n(\theta, a)}{n \log N} = \frac{\log \sharp\Omega}{\log N} - \overline{\dim}_B E_{\theta,a},$$

where the last inequality follows form the previous lemma. □

Proof of Proposition 1.1. Since $d_{\nu_\theta}(a) = \dim_H \nu_\theta$ for ν_θ-almost every $a \in \mathrm{proj}_\theta\Lambda$, the combination of Proposition 2.1 and Lemma 2.3 proves the statement. □

3. Proof of Theorem 1.2

Through this section we always assume that $N \nmid \sharp\Omega$ and $N + 1 \leq \sharp\Omega$. Moreover, let $\theta \in [0, \pi/2)$ and $\tan\theta = \frac{p}{q}$ be arbitrary but fixed. Let us recall the definition of projected IFS (1.3) and the definition of matrices (1.4). The projected IFS $\psi = \{f_\omega\}$ of $\Psi = \{F_\omega\}_{\omega\in\Omega}$ according to proj_θ is

$$f_{k,l}(x) = \frac{x}{N} + \frac{-kp + lq}{Nq}, \text{ for every } (k, l) \in \Omega.$$

Divide the interval $I = [-\frac{p}{q}, 1]$ into $p + q$ equal intervals, i.e. $I_k = [\frac{k-1-p}{q}, \frac{k-p}{q}]$ for $k = 1, \ldots, p+q$. Furthermore, divide I_k for every $k = 1, \ldots, p+q$ into N equal parts. That is, $I_k^\xi = [\frac{k-1-p}{q} + \frac{\xi}{Nq}, \frac{k-1-p}{q} + \frac{\xi+1}{Nq}]$ for $\xi = 0, \ldots, N-1$. For every $\xi = 0, \ldots, N-1$ let us define a $(p+q) \times (p+q)$ real matrix A_ξ in the following way

$$(A_\xi)_{i,j} := \sharp\left\{\omega \in \Omega : f_\omega(I_j) = I_i^\xi\right\}.$$

From the definition of the matrices (1.4) it is easy to see that

$$\sum_{i=1}^{p+q} \sum_{\xi=0}^{N-1} (A_\xi)_{i,j} = \sharp\Omega \text{ for every } j = 1, \ldots, p+q. \tag{3.1}$$

In general, for $\xi_1,\dots,\xi_n\in\{0,\dots,N-1\}$ let $I_j^{\xi_1,\dots,\xi_n}$ be the interval

$$I_j^{\xi_1,\dots,\xi_n}=\left[\frac{j-1-p}{q}+\frac{1}{q}\sum_{k=1}^{n}\frac{\xi_k}{N^k},\frac{j-1-p}{q}+\frac{1}{q}\sum_{k=1}^{n}\frac{\xi_k}{N^k}+\frac{1}{qN^n}\right].$$

By the definition, for the products of the matrices hold

$$\left(A_{\xi_1}\cdots A_{\xi_n}\right)_{i,j}=\sharp\left\{\underline{\omega}\in\Omega^n:f_{\underline{\omega}}(I_j)=I_i^{\xi_1,\dots,\xi_n}\right\}. \tag{3.2}$$

Because of (3.1) the matrix

$$P=\frac{1}{\sharp\Omega}\sum_{n=0}^{N-1}A_n^T$$

defines a Markov-chain on $\Xi:=\{1,\dots,p+q\}$. Let us divide the set of states into two parts. Let

$$\Xi_r=\{i\in\Xi:\nu_\theta(I_i)>0\}$$
$$\Xi_t=\{i\in\Xi:\nu_\theta(I_i)=0\}.$$

Lemma 3.1. *The set Ξ_r is a recurrent class and Ξ_t is a transient class of the Markov-chain defined by P. Moreover, Ξ_r is aperiodic.*

Proof. First, we show that if $i\in\Xi_r$ and $P_{i,j}>0$ then $j\in\Xi_r$. Since $P_{i,j}>0$ there exist $\omega\in\Omega$ and $n\in\{0,\dots,N-1\}$ such that $f_\omega(I_i)=I_j^n$. Therefore $0<\nu_\theta(f_\omega(I_i))=\nu_\theta(I_j^n)\leq\nu_\theta(I_j)$.

On the other hand, for every $K>0$ sufficiently large and for every $j\in\Xi_r$ there exists a $\underline{\omega}\in\Omega^K$ such that $f_{\underline{\omega}}(I)\subseteq I_j$. This implies that for every $j\in\Xi_r$ and every $i\in\Xi$, $(P^K)_{i,j}>0$, which proves the statement. □

We note that if $\mathrm{proj}_\theta\Lambda=[-\tan\theta,1]$ then $\Xi_r=\Xi$ and $\Xi_t=\emptyset$. It is well known from the theory of Markov-chains that there exists a unique probability vector $\underline{p}$ such that $\underline{p}$ is the stationary distribution of P, i.e. $\underline{p}^TP=\underline{p}^T$. In particular,

$$\left(\sum_{\xi=0}^{N-1}A_\xi\right)\underline{p}=\sharp\Omega\cdot\underline{p}.$$

Lemma 3.2. *For every $i\in\{1,\dots,p+q\}$ and $(\xi_1,\dots,\xi_n)\in\{0,\dots,N-1\}^n$*

$$\nu_\theta(I_i^{\xi_1,\dots,\xi_n})=\frac{\underline{e}_iA_{\xi_1}\cdots A_{\xi_n}\underline{p}}{\sharp\Omega^n},$$

where $\underline{e}_i$ denotes the ith element of the natural basis of $\mathbb{R}^{p+q}$.

Proof. First, let us observe that $\underline{p}_i=\nu_\theta(I_i)$. That is,

$$\nu_\theta(I_i)\;=\;\sum_{\xi=0}^{N-1}\nu_\theta(I_i^\xi)\;=\;\sum_{\xi=0}^{N-1}\sum_{j=1}^{p+q}\sum_{\omega\in\Omega:f_\omega(I_j)=I_i^\xi}\frac{\nu_\theta(I_j)}{\sharp\Omega}\;=\;\sum_{j=1}^{p+q}\frac{\nu_\theta(I_j)}{\sharp\Omega}\sum_{\xi=0}^{N-1}\left(A_\xi\right)_{i,j}.$$

At the second equality we have used that ν_θ is a self-similar measure. Therefore the vector $(\nu_\theta(I_i))_{i=1}^{p+q}$ is a probability right-eigenvector of $\sum_{\xi=0}^{N-1}A_\xi$. Thus, in general,

$$\nu_\theta(I_i^{\xi_1,\dots,\xi_n})=\sum_{j=1}^{p+q}\sum_{\underline{\omega}\in\Omega^n:f_{\underline{\omega}}(I_j)=I_i^{\xi_1,\dots,\xi_n}}\frac{\nu_\theta(I_j)}{\sharp\Omega^n}=\sum_{j=1}^{p+q}\frac{\nu_\theta(I_j)}{\sharp\Omega^n}\left(A_{\xi_1}\cdots A_{\xi_n}\right)_{i,j}.$$

□

Denote A^r_ξ the submatrix of A_ξ by deleting the rows and columns of Ξ_t. If $j \in \Xi_r$ and $i \in \Xi_t$ then $(A_\xi)_{i,j} = 0$ for every $\xi = 0, \dots, N-1$. Hence,

$$\sum_{i\in\Xi_r}\sum_{\xi=0}^{N-1}\left(A^r_\xi\right)_{i,j} = \sharp\Omega \text{ for every } j \in \Xi_r. \tag{3.3}$$

Lemma 3.3. *For any $i, j \in \Xi_r$ and $\xi_1, \dots, \xi_n \in \{0, \dots, N-1\}$*

$$\left(A_{\xi_1}\cdots A_{\xi_n}\right)_{i,j} = \left(A^r_{\xi_1}\cdots A^r_{\xi_n}\right)_{i,j}.$$

Proof. Let us prove by induction. For $n = 2$

$$\left(A_{\xi_1}A_{\xi_2}\right)_{i,j} = \sum_{k=1}^{p+q}\left(A_{\xi_1}\right)_{i,k}\left(A_{\xi_2}\right)_{k,j} = \sum_{k\in\Xi_r}\left(A_{\xi_1}\right)_{i,k}\left(A_{\xi_2}\right)_{k,j} = \left(A^r_{\xi_1}A^r_{\xi_2}\right)_{i,j}.$$

We used in the second equation that $\left(A_{\xi_2}\right)_{k,j} = 0$ whenever $k \in \Xi_t$. Then

$$\left(A_{\xi_1}\cdots A_{\xi_n}A_{\xi_{n+1}}\right)_{i,j} = \sum_{k=1}^{p+q}\left(A_{\xi_1}\cdots A_{\xi_n}\right)_{i,k}\left(A_{\xi_{n+1}}\right)_{k,j}.$$

Again, $\left(A_{\xi_{n+1}}\right)_{k,j} = 0$ whenever $k \in \Xi_t$, so

$$\sum_{k\in\Xi_r}\left(A_{\xi_1}\cdots A_{\xi_n}\right)_{i,k}\left(A_{\xi_{n+1}}\right)_{k,j} = \sum_{k\in\Xi_r}\left(A^r_{\xi_1}\cdots A^r_{\xi_n}\right)_{i,k}\left(A_{\xi_{n+1}}\right)_{k,j} = \left(A^r_{\xi_1}\cdots A^r_{\xi_{n+1}}\right)_{i,j}.$$

□

In particular, an important consequence of Lemma 3.3 is that for every $\xi_1, \dots, \xi_n \in \{0, \dots, N-1\}$ and $i \in \Xi_r$

$$\nu_\theta(I_i^{\xi_1,\dots,\xi_n}) = \frac{\underline{\widehat{e}_i}^T A^r_{\xi_1}\cdots A^r_{\xi_n}\underline{\widehat{p}}}{\sharp\Omega^n}, \tag{3.4}$$

where $\underline{\widehat{p}} = (\nu_\theta(I_j))_{j\in\Xi_r}$ and $\underline{\widehat{e}_i}$ is the ith element of the natural basis of $\mathbb{R}^{\sharp\Xi_r}$. Now, we define a left-shift invariant measure η on the symbolic space $\Sigma = \{0, \dots, N-1\}^{\mathbb{N}}$. Endow Σ with the metric $d(\underline{\xi}, \underline{\zeta}) = N^{-n}$ for $\underline{\xi} = (\xi_1, \xi_2, \dots)$ and $\underline{\zeta} = (\zeta_1, \zeta_2, \dots)$, where n is the largest integer such that $\xi_i = \zeta_i (1 \le i \le n)$. For a cylinder set $[\xi_1, \dots, \xi_n] = \{(\zeta_1, \zeta_2, \dots) \in \Sigma : \zeta_k = \xi_k,\ k = 1, \dots, n\}$ let

$$\eta([\xi_1, \dots, \xi_n]) := \frac{\underline{\widehat{e}}^T A^r_{\xi_1}\cdots A^r_{\xi_n}\underline{\widehat{p}}}{\sharp\Omega^n}, \tag{3.5}$$

where $\underline{\widehat{e}} = \sum_{i\in\Xi_r}\underline{\widehat{e}_i}$. By (3.3), η is a probability measure.

Lemma 3.4. *The probability measure η is σ-invariant and mixing and hence ergodic, where σ denotes the left-shift operator on Σ.*

Proof. First, we prove the invariance. It is enough to prove for the cylinder sets. Since the vector $\underline{\widehat{e}}$ is a left-eigenvector of $\sum_{\xi=0}^{N-1} A^r_\xi$ (3.3), then for a cylinder set

$[\xi_1,\dots,\xi_n]$

$$\eta(\sigma^{-1}[\xi_1,\dots,\xi_n])=\sum_{\xi=0}^{N-1}\eta([\xi,\xi_1,\dots,\xi_n])=\sum_{\xi=0}^{N-1}\frac{\underline{\widehat{e}}^TA_\xi A^r_{\xi_1}\cdots A^r_{\xi_n}\underline{\widehat{p}}}{\sharp\Omega^{n+1}}=\frac{\underline{\widehat{e}}^TA^r_{\xi_1}\cdots A^r_{\xi_n}\underline{\widehat{p}}}{\sharp\Omega^{n}}=\eta([\xi_1,\dots,\xi_n]).$$

To prove the mixing property it is enough to show that for any cylinder sets $[\xi_1,\dots,\xi_k]$ and $[\zeta_1,\dots,\zeta_l]$

$$\lim_{n\to\infty}\eta([\xi_1,\dots,\xi_k]\cap\sigma^{-n}[\zeta_1,\dots,\zeta_l])=\eta([\xi_1,\dots,\xi_k])\eta([\zeta_1,\dots,\zeta_l]).$$

By the definition of η (3.5), for sufficiently large n

$$\eta([\xi_1,\dots,\xi_k]\cap\sigma^{-n}[\zeta_1,\dots,\zeta_l])=\sum_{i_1,\dots,i_{n-k}=0}^{N-1}\frac{\underline{\widehat{e}}^TA^r_{\xi_1}\cdots A^r_{\xi_k}A^r_{i_1}\cdots A^r_{i_{n-k}}A^r_{\zeta_1}\cdots A^r_{\zeta_l}\underline{\widehat{p}}}{\sharp\Omega^{n+l}}=\frac{\underline{\widehat{e}}^TA^r_{\xi_1}\cdots A^r_{\xi_k}\left(\sum_{i=0}^{N-1}A^r_i\right)^{n-k}A^r_{\zeta_1}\cdots A^r_{\zeta_l}\underline{\widehat{p}}}{\sharp\Omega^{n+l}}.$$

Applying Lemma 3.1 and the basic properties of aperiodic, irreducible Markov chains, we have

$$\lim_{n\to\infty}\frac{\left(\sum_{i=0}^{N-1}A^r_i\right)^{n-k}}{\sharp\Omega^{n-k}}=\underline{\widehat{p}}\,\underline{\widehat{e}}^T,$$

which implies the mixing property. □

Lemma 3.5. *Denote by h_η the entropy of measure η. If $N\nmid\sharp\Omega$ and $N+1\le\sharp\Omega$ then $h_\eta<\log N$.*

Proof. We argue by contradiction. Suppose that $h_\eta=\log N$. By [14, Theorem 4.10] and [14, Theorem 4.18] we have that

$$h_\eta=\lim_{n\to\infty}-\frac{1}{n}\sum_{\xi_1,\dots,\xi_n=0}^{N-1}\frac{\underline{\widehat{e}}^TA^r_{\xi_1}\cdots A^r_{\xi_n}\underline{\widehat{p}}}{\sharp\Omega^n}\log\frac{\underline{\widehat{e}}^TA^r_{\xi_1}\cdots A^r_{\xi_n}\underline{\widehat{p}}}{\sharp\Omega^n},$$

and the right hand side decreases as $n\to\infty$. That is, $h_\eta=\log N$ if and only if

$$\frac{\underline{\widehat{e}}^TA^r_{\xi_1}\cdots A^r_{\xi_n}\underline{\widehat{p}}}{\sharp\Omega^n}=\frac{1}{N^n},\ \text{for every } n\ge1 \text{ and } \xi_1,\dots,\xi_n\in\{0,\dots,N-1\}.\qquad(3.6)$$

By Lemma 3.1 there exists a $K>0$ such that $\left(\sum_{\xi=0}^{N-1}A^r_\xi\right)^K>0$, i.e. each element of the matrix is strictly positive. Without loss of generality, we may assume that $K>(p+q)^2+1$. Then there exists a word $(\zeta_1,\dots,\zeta_K)$ of length K such that $\left(\sum_{\xi=0}^{N-1}A^r_\xi\right)^K-A^r_{\zeta_1}\cdots A^r_{\zeta_K}>0$. Let $\mathcal{A}:=\{0,\dots,N-1\}^K\setminus\{(\zeta_1,\dots,\zeta_K)\}$. By Perron-Frobenius theorem there exists a $\rho>0$ and $\underline{u},\underline{v}$ vectors such that ρ is the largest eigenvalue of the matrix $\sum_{\underline{\xi}\in\mathcal{A}}A^r_{\underline{\xi}}$ and $\underline{u},\underline{v}$ are the corresponding left and right eigenvectors. Moreover,

$$\lim_{n\to\infty}\frac{1}{\rho^n}\left(\sum_{\underline{\xi}\in\mathcal{A}}A^r_{\underline{\xi}}\right)^n=\underline{v}\underline{u}^T. \tag{3.7}$$

By our assumption (3.6)

$$\frac{1}{n}\log\underline{\widehat{e}}^T\left(\sum_{\underline{\xi}\in\mathcal{A}}A^r_{\underline{\xi}}\right)^n\underline{\widehat{p}}=\log\frac{\sharp\Omega^K\sharp\mathcal{A}}{N^K}=\log\frac{\sharp\Omega^K(N^K-1)}{N^K}.$$

On the other hand, by (3.7)

$$\lim_{n\to\infty}\frac{1}{n}\log\underline{\widehat{e}}^T\left(\sum_{\underline{\xi}\in\mathcal{A}}A^r_{\underline{\xi}}\right)^n\underline{\widehat{p}}=\log\rho.$$

So $\rho=\sharp\Omega^K-\frac{\sharp\Omega^K}{N^K}$ but this is a contradiction since $\sharp\Omega^K-\frac{\sharp\Omega^K}{N^K}\in\mathbb{Q}\backslash\mathbb{Z}$ cannot be a root of characteristic polynomial of $\sum_{\underline{\xi}\in\mathcal{A}}A^r_{\underline{\xi}}$, which is a matrix of integer coefficients. □

Proof of Theorem 1.2. Let Γ be the natural projection from Σ to interval $[0,1]$, that is,

$$\Gamma(\xi_1,\xi_2,\dots)=\sum_{n=1}^{\infty}\frac{\xi_n}{N^n}. \tag{3.8}$$

Denote h_k the linear function, mapping I_k to $[0,1]$, that is, $h_k(x)=qx-(k-1-p)$. The measure

$$\widetilde{\nu}_\theta:=\sum_{k\in\Xi_r}\nu_\theta|_{I_k}\circ h_k^{-1}=\eta\circ\Gamma^{-1}$$

is $Nx \mod 1$ invariant and ergodic by (3.4) and Lemma 3.4. Moreover,

$$\dim_H\widetilde{\nu}_\theta=\min_{1\le k\le p+q}\dim_H\nu_\theta|_{I_k}\circ h_k^{-1}=\dim_H\nu_\theta. \tag{3.9}$$

By the Volume Lemma [13, Theorem 10.4.1,Theorem 10.4.2] and Lemma 3.5, we have

$$\dim_H\widetilde{\nu}_\theta=\frac{h_\eta}{\log N}<1. \tag{3.10}$$

The statement of the theorem follows from (3.9), (3.10) and Proposition 1.1. □

4. Proof of Theorem 1.3 and Theorem 1.4

In the rest of the paper we assume that $\mathrm{proj}_\theta\Lambda=[-\tan\theta,1]$. In the previous section we have shown that the matrices, constructed in (1.4) can be used for determine the dimension of the projected natural measure. In this section we show that the matrices can be used for determine the box dimension of the slices, with the additional assumption that the projection is an interval.

We note that if $\mathrm{proj}_\theta\Lambda=[-\tan\theta,1]$ then $\Xi_r=\Xi$ and $\Xi_t=\emptyset$. In particular, $A^r_\xi=A_\xi$ for every $\xi\in\{0,\dots,N-1\}$.

Lemma 4.1. *Let* $\Omega \subseteq \{0,\dots,N-1\}^2$ *and* $\theta \in [0,\pi/2)$ *such that* $\tan\theta = \frac{p}{q}$ *and* $\mathrm{proj}_\theta\Lambda = [-\tan\theta, 1]$*. Then for* $a = \frac{k-1-p}{q} + \frac{1}{q}\sum_{n=1}^{\infty}\frac{\xi_n}{N^n}$

$$\underline{\dim}_B E_{\theta,a} = \liminf_{n\to\infty}\frac{\log \underline{e}_k A_{\xi_1}\cdots A_{\xi_n}\underline{e}}{n\log N},\ \text{and}\ \overline{\dim}_B E_{\theta,a} = \limsup_{n\to\infty}\frac{\log \underline{e}_k A_{\xi_1}\cdots A_{\xi_n}\underline{e}}{n\log N},$$

where $\underline{e}_k$ *is the* k*th element of the natural basis of* $\mathbb{R}^{p+q}$*.*

Proof. Let $a = \frac{k-1-p}{q} + \frac{1}{q}\sum_{n=1}^{\infty}\frac{\xi_n}{N^n}$. Let us recall the definition (2.1) of $G_n(\theta,a)$, which is the number of cylinder sets intersecting the line $L_{\theta,a}$. Since $\mathrm{proj}_\theta\Lambda = [-\tan\theta, 1]$ let us observe that for every $n \geq 1$ and every $\underline{\omega} \in \Omega^n$

$$F_{\underline{\omega}}([0,1]^2) \cap L_{\theta,a} \neq \emptyset \iff F_{\underline{\omega}}(\Lambda) \cap L_{\theta,a} \neq \emptyset.$$

Hence

$$\sharp G_n(\theta,a) = \sharp\left\{\underline{\omega} \in \Omega^n : F_{\underline{\omega}}([0,1]^2) \cap L_{\theta,a} \neq \emptyset\right\}.$$

Since $\tan\theta$ is rational,

$$F_{\underline{\omega}}([0,1]^2) \cap L_{\theta,a} \neq \emptyset \Leftrightarrow \text{there exists a } 1 \leq j \leq p+q \text{ such that } f_{\underline{\omega}}(I_j) = I_k^{\xi_1,\dots,\xi_n}$$

Using (3.2) we have $\underline{e}_k A_{\xi_1}\cdots A_{\xi_n}\underline{e} = \sharp G_n(\theta,a)$. The statement follows from Lemma 2.2. □

Proposition 4.2. *Let* $\Omega \subseteq \{0,\dots,N-1\}^2$ *and* $\theta \in [0,\pi/2)$ *such that* $\tan\theta = \frac{p}{q}$ *and* $\mathrm{proj}_\theta\Lambda = [-\tan\theta, 1]$*. Then there exists a constant* $\beta = \beta(\theta)$ *depending only on* θ *such that*

$$\dim_H E_{\theta,a} = \dim_B E_{\theta,a} = \beta(\theta) \text{ for Leb.-a.e. } a \in \mathrm{proj}_\theta\Lambda.$$

For the proof of Proposition 4.2 we refer to [9, Section 7].

Now, let us recall the definition of the pressure function $P(t)$, and $b_{\max}$ defined in (1.5), (1.6), i.e.

$$P(t) = \lim_{n\to\infty}\frac{1}{n\log N}\log\sum_{\xi_1,\dots,\xi_n=0}^{N-1}\left(\underline{e}^T A_{\xi_1}\cdots A_{\xi_n}\underline{e}\right)^t,\ \text{and}\ b_{\max} = \lim_{t\to\infty}\frac{P(t)}{t}.$$

Lemma 4.3. *The pressure function* $P(t)$ *exists for every* $t \in \mathbb{R}$*, and monotone increasing, convex and continuous. Moreover,* $P(t)$ *is continuously differentiable for every* $t > 0$*.*

Proof. By Lemma 3.1, there exists a $K > 0$ such that $\left(\sum_{\xi=0}^{N-1} A_\xi\right)^K > 0$. Then the existence follows from [3, Lemma 2.2]. The differentiability follows from [6, Theorem 3.3], and the monotonicity, convexity, and continuity property can be proven by standard argument. The continuity of the derivative is not explicitely mentioned in [6, Theorem 3.3], but it follows from convexity. □

Theorem 4.4. [3, Theorem 1.1] *Let* A_ξ *be non-negative matrices for* $\xi = 0,\dots,N-1$*. If there exists a* $K > 0$ *such that* $\sum_{n=0}^{K}\left(\sum_{\xi=0}^{N-1} A_\xi\right)^n > 0$ *then*

$$\dim_H\left\{\underline{\xi} \in \Sigma : \lim_{n\to\infty}\frac{\log \underline{e}^T A_{\xi_1}\cdots A_{\xi_n}\underline{e}}{n\log N} = \alpha\right\} = \inf_t\{-\alpha t + P(t)\} =: P^*(\alpha),$$

where $\dim_H$ *is defined according to the metric* $d(\underline{\xi},\underline{\zeta}) = N^{-n}$ *for* $\underline{\xi} = (\xi_1,\xi_2,\dots)$ *and* $\underline{\zeta} = (\zeta_1,\zeta_2,\dots)$*, where* n *is the largest integer such that* $\xi_i = \zeta_i (1 \leq i \leq n)$*.*

Lemma 4.5. *For every $t > 0$ there is a unique ergodic, left-shift invariant Gibbs measure μ_t on Σ such that there exists a $C > 0$ that for any $(\xi_1, \dots, \xi_n) \in \{0, \dots, N-1\}^*$*

$$C^{-1} \le \frac{\mu_t([\xi_1, \dots, \xi_n])}{\left(\underline{e}^T A_{\xi_1} \cdots A_{\xi_n} \underline{e}\right)^t N^{-nP(t)}} \le C.$$

Moreover,

$$\dim_H \mu_t = -tP'(t) + P(t)$$

and

$$\lim_{n\to\infty} \frac{\log \underline{e}^T A_{\xi_1} \cdots A_{\xi_n} \underline{e}}{n \log N} = P'(t) \text{ for } \mu_t\text{-a.a. } (\xi_1, \xi_2, \dots) \in \Sigma.$$

The proof of the lemma follows from [6, Theorem 3.2] and [6, Proof of Theorem 1.3].

Lemma 4.6. *For every $t > 0$*

$$P'(t) = \lim_{n\to\infty} \frac{1}{n \log N} \sum_{\xi_1, \dots, \xi_n = 0}^{N-1} \mu_t([\xi_1, \dots, \xi_n]) \log \underline{e}^T A_{\xi_1} \cdots A_{\xi_n} \underline{e} =$$

$$\inf_{n \ge 1} \frac{1}{n \log N} \sum_{\xi_1, \dots, \xi_n = 0}^{N-1} \mu_t([\xi_1, \dots, \xi_n]) \log \underline{e}^T A_{\xi_1} \cdots A_{\xi_n} \underline{e},$$

where μ_t is the Gibbs measure defined in Lemma 4.5.

The proof of the lemma follows from [4, Theorem 1.2] and [4, Lemma 2.2(ii)].

Lemma 4.7. *For any $\delta > 0$*

$$\dim_H \left\{ \underline{\xi} \in \Sigma : \limsup_{n\to\infty} \frac{\log \underline{e}^T A_{\xi_1} \cdots A_{\xi_n} \underline{e}}{n \log N} \ge \delta \right\} \le \inf_{t>0} \{-\delta t + P(t)\}$$

Proof. We will prove the upper bound with the method of Olsen and Winter [12].

Let $\varepsilon > 0$ be arbitrary but fixed. Let us define the following set of cylinders:

$$\mathbf{A}_n(\varepsilon) := \left\{ [\xi_1, \dots, \xi_k] : k \ge n,\ \delta - \varepsilon \le \frac{\log \underline{e}^T A_{\xi_1} \cdots A_{\xi_k} \underline{e}}{k \log N} \right\}.$$

It is easy to see that the set

$$\bigcup_{[\xi_1, \cdots, \xi_k] \in \mathbf{A}_n(\varepsilon)} [\xi_1, \dots, \xi_k]$$

covers the set $G_\delta := \left\{ \underline{\xi} \in \Sigma : \limsup_{n\to\infty} \frac{\log \underline{e}^T A_{\xi_1} \cdots A_{\xi_n} \underline{e}}{n \log N} \ge \delta \right\}$. Let $\mathbf{B}_n(\varepsilon)$ be the set of disjoint cylinders in $\mathbf{A}_n(\varepsilon)$ such that

$$\bigcup_{[\xi_1, \cdots, \xi_k] \in \mathbf{B}_n(\varepsilon)} [\xi_1, \dots, \xi_k] = \bigcup_{[\xi_1, \cdots, \xi_k] \in \mathbf{A}_n(\varepsilon)} [\xi_1, \dots, \xi_k].$$

Then for every $t > 0$

$$\mathcal{H}_{N^{-n}}^{-\delta t + P(t) + 2\varepsilon}(G_\delta) \le \sum_{[\xi_1, \dots, \xi_k] \in \mathbf{B}_n(\varepsilon)} N^{-k(-\delta t + P(t) + 2\varepsilon)} \le$$

$$N^{-n\varepsilon} \sum_{[\xi_1, \dots, \xi_k] \in \mathbf{B}_n(\varepsilon)} N^{-kP(t)} \left(\underline{e}^T A_{\xi_1} \cdots A_{\xi_k} \underline{e}\right)^t.$$

By Lemma 4.5

$$\mathcal{H}_{N^{-n}}^{-\delta t+P(t)+2\varepsilon}(G_\delta) \leq N^{-n\varepsilon} \sum_{[\xi_1,\dots,\xi_k]\in\mathbf{B}_n(\varepsilon)} \mu_t([\xi_1,\dots,\xi_k]) \leq N^{-n\varepsilon}.$$

Since $\varepsilon > 0$ and $t > 0$ were arbitrary,

$$\dim_H G_\delta \leq \inf_{t>0}\left\{-\delta t + P(t)\right\}.$$

□

Before we prove our main theorems let us introduce $p+q$ projecting maps from Σ to I_k. That is,

$$\Gamma_k(\underline{\xi}) := \frac{k-1-p}{q} + \frac{1}{q}\sum_{k=1}^{\infty}\frac{\xi_k}{N^k}.$$

Denote $E_{\theta,\Gamma(\underline{\xi})}$ the union of slices corresponding to $\Gamma_k(\underline{\xi})$, i.e.

$$E_{\theta,\Gamma(\underline{\xi})} := \bigcup_{k=1}^{p+q} E_{\theta,\Gamma_k(\underline{\xi})}.$$

Proof of Theorem 1.3. By Proposition 4.2, it is enough to show that

$$\dim_H\left\{a\in\mathrm{proj}_\theta\Lambda : \dim_H E_{\theta,a} = \dim_B E_{\theta,a} = s-1\right\} < 1 \tag{4.1}$$

(we remind that $s = \log\sharp\Omega/\log N$ is the Hausdorff dimension of the carpet). However,

$$\dim_H\left\{a\in\mathrm{proj}_\theta\Lambda : \dim_H E_{\theta,a} = \dim_B E_{\theta,a} = s-1\right\} \leq$$

$$\dim_H\left\{a\in\mathrm{proj}_\theta\Lambda : \dim_B E_{\theta,a} = s-1\right\} = \dim_H \bigcup_{k=1}^{p+q}\left\{a\in I_k : \dim_B E_{\theta,a} = s-1\right\} =$$

$$\dim_H \bigcup_{k=1}^{p+q}\left\{\underline{\xi}\in\Sigma : \dim_B E_{\theta,\Gamma_k(\underline{\xi})} = s-1\right\} \leq \dim_H\left\{\underline{\xi}\in\Sigma : \dim_B E_{\theta,\Gamma(\underline{\xi})} \geq s-1\right\}.$$

By Lemma 4.1 and Lemma 4.7

$$\dim_H\left\{\underline{\xi}\in\Sigma : \dim_B E_{\theta,\Gamma(\underline{\xi})} \geq s-1\right\} \leq \inf_{t>0}\left\{-(s-1)t + P(t)\right\}.$$

By the definition of pressure function $P(t)$ we have $P(0) = 1$, $P(1) = s$. Moreover, by Lemma 4.3 and Lemma 4.5, we have $P'(1) = s - \dim_H\eta > s-1$, where η is the probability measure defined in (3.5). Then there exists a $t' \in [0,1]$, such that $P(t') < 1 + (s-1)t'$. Hence

$$\inf_{t>0}\left\{-(s-1)t + P(t)\right\} \leq -(s-1)t' + P(t') < 1,$$

which implies (4.1) and completes the proof. □

Before we prove Theorem 1.4, we need two technical lemmas.

Lemma 4.8. *Let μ_t be the measure defined in Lemma 4.5. Then for μ_t-a.e. $\underline{\xi}\in\Sigma$*

$$\dim_H E_{\theta,\Gamma(\underline{\xi})} = \dim_P E_{\theta,\Gamma(\underline{\xi})} = \dim_B E_{\theta,\Gamma(\underline{\xi})}.$$

Proof. Let $H:(x,y)\mapsto(x,px-qy \mod 1)$ be a map of $S^1\times S^1$ into itself. Then $H(\Lambda)\subseteq S^1\times S^1$ compact, $T_N\times T_N$-invariant. Since μ_t is left-shift invariant then $\mu_t\circ\Gamma^{-1}$ is T_N-invariant. Using [8, Proposition 2.6] we have for $\mu_t\circ\Gamma^{-1}$-a.e. x

$$\dim_H\pi^{-1}(x)=\dim_P\pi^{-1}(x)=\dim_B\pi^{-1}(x),$$

where $\pi:H(\Lambda)\mapsto S^1$ is the projection to the first coordinate.

Let $J:x\mapsto -qx \mod 1$ be the mapping $\mathrm{proj}_\theta\Lambda$ into S^1. Then for every $k,l\in\Xi$ and $\underline{\xi}\in\Sigma$, $J(\Gamma_k(\underline{\xi}))=J(\Gamma_l(\underline{\xi}))=\Gamma(\underline{\xi})$, where $\Gamma:\Sigma\mapsto[0,1]$ is defined in (3.8). Observe that $\pi^{-1}(\Gamma(\underline{\xi}))=H(E_{\theta,\Gamma(\underline{\xi})})$. The proof is completed by the fact that $\dim H(E_{\theta,\Gamma(\xi)})=\dim E_{\theta,\Gamma(\xi)}$, where dim denotes packing, Hausdorff and box dimension simultaneously. □

Lemma 4.9. *For every $\delta\in(\beta(\theta),b_{\max})$ there exists a $0<t=t_\delta$ such that*

$$P'(t_\delta)=\delta.$$

Proof. By Lemma 4.3, the function $P'(t)$ is monotone increasing and continuous for $t>0$, hence it is enough to show that $\beta(\theta)=\lim_{t\to0+}P'(t)$.

First, we prove that

$$1=\dim_H\left\{\underline{\xi}\in\Sigma:\lim_{n\to\infty}\frac{\log\underline{e}^TA_{\xi_1}\cdots A_{\xi_n}\underline{e}}{n\log N}=\beta(\theta)\right\}=\inf_t\left\{-\beta(\theta)t+P(t)\right\}. \quad (4.2)$$

The second equality follows from Theorem 4.4. Using Theorem 1.3, we have that for every $k\in\Xi$

$$\mathcal{L}(\{a\in I_k:\dim_B E_{\theta,a}=\beta(\theta)\})=\mathcal{L}(I_k)=\frac{1}{q},$$

where $\mathcal{L}$ denotes the Lebesgue measure on the real line. Let λ be the uniform Bernoulli measure on Σ. Using that $q*\mathcal{L}|_{I_k}=\lambda\circ\Gamma_k^{-1}$, we have

$$1=q*\mathcal{L}(\{a\in I_k:\dim_B E_{\theta,a}=\beta(\theta)\})=\lambda(\left\{\underline{\xi}\in\Sigma:\dim_B E_{\theta,\Gamma_k(\underline{\xi})}=\beta(\theta)\right\})$$

Hence,

$$1=\lambda(\bigcap_{k=1}^{p+q}\left\{\underline{\xi}\in\Sigma:\dim_B E_{\theta,\Gamma_k(\underline{\xi})}=\beta(\theta)\right\})\leq$$
$$\lambda(\left\{\underline{\xi}\in\Sigma:\lim_{n\to\infty}\frac{\log\underline{e}^TA_{\xi_1}\cdots A_{\xi_n}\underline{e}}{n\log N}=\beta(\theta)\right\}). \quad (4.3)$$

Since $\dim_H\lambda=1$, we get the first equation in (4.2).

The other consequence of (4.3) combined with the sub-additive ergodic theorem [14, p. 231] is that

$$\beta(\theta)=\lim_{n\to\infty}\frac{1}{n\log N}\sum_{\xi_1,\dots,\xi_n=0}^{N-1}\frac{1}{N^n}\log\underline{e}^TA_{\xi_1}\cdots A_{\xi_n}\underline{e}.$$

Moreover, it follows from the definition of Gibbs measures $\{\mu_t\}_{t>0}$, defined in Lemma 4.5, that $\mu_t \to \lambda$ weakly as $t \to 0+$. Therefore, by Lemma 4.6,

$$\lim_{t\to 0+} P'(t) \leq \lim_{t\to 0+} \frac{1}{n\log N} \sum_{\xi_1,\dots,\xi_n=0}^{N-1} \mu_t([\xi_1,\dots,\xi_n]) \log \underline{e}^T A_{\xi_1}\cdots A_{\xi_n}\underline{e} =$$

$$\frac{1}{n\log N} \sum_{\xi_1,\dots,\xi_n=0}^{N-1} \frac{1}{N^n} \log \underline{e}^T A_{\xi_1}\cdots A_{\xi_n}\underline{e}.$$

Since it holds for every $n \geq 1$, we have $\lim_{t\to 0+} P'(t) \leq \beta(\theta)$.

On the other hand, it follows from Theorem 4.4 that for every $t > 0$, $1 \leq -\beta(\theta)t + P(t)$. Since $P(0) = 0$ and $P(t)$ is continuously differentiable for $t > 0$, we have $\beta(\theta) \leq \lim_{t\to 0+} P'(t)$. □

Proof of Theorem 1.4. Denote by dim either the Hausdorff or packing dimension and let $\delta \in [\beta(\theta), b_{\max})$ then

$$\dim_H \{a \in \mathrm{proj}_\theta\Lambda : \dim E_{\theta,a} = \delta\} = \dim_H \bigcup_{k=1}^{p+q} \{a \in I_k : \dim E_{\theta,a} = \delta\}.$$

Then using the properties of $\Gamma_k : \Sigma \mapsto I_k$, we get

$$\dim_H \bigcup_{k=1}^{p+q} \{a \in I_k : \dim E_{\theta,a} = \delta\} = \dim_H \left\{\underline{\xi} \in \Sigma : \exists k \in \Xi,\ \dim E_{\theta,\Gamma_k(\underline{\xi})} = \delta\right\}.$$

By simple property of dimension, we get

$$\dim_H \left\{\underline{\xi} \in \Sigma : \exists k \in \Xi,\ \dim E_{\theta,\Gamma_k(\underline{\xi})} = \delta\right\} \geq$$

$$\dim_H \left\{\underline{\xi} \in \Sigma : \dim_H E_{\theta,\Gamma(\underline{\xi})} = \dim_P E_{\theta,\Gamma(\underline{\xi})} = \dim_B E_{\theta,\Gamma(\underline{\xi})} = \delta\right\}.$$

There are two possibilities, if $\delta = \beta(\theta)$ than we consider the uniform measure λ and $\dim_H \lambda = P^*(\beta(\theta)) = 1$. Otherwise, by Lemma 4.9, there exists a $t_\delta \geq 0$ such that $P'(t_\delta) = \delta$. Lemma 4.5 implies that $\dim_H \mu_{t_\delta} = -t_\delta P'(t_\delta) + P(t_\delta) = P^*(\delta)$. Using Lemma 4.8

$$\dim_H \{a \in \mathrm{proj}_\theta\Lambda : \dim E_{\theta,a} = \delta\} \geq$$

$$\dim_H \left\{\underline{\xi} \in \Sigma : \dim_H E_{\theta,\Gamma(\underline{\xi})} = \dim_P E_{\theta,\Gamma(\underline{\xi})} = \dim_B E_{\theta,\Gamma(\underline{\xi})} = \delta\right\} \geq \dim_H \mu_{t_\delta} = P^*(\delta),$$

which proves the lower bound. For the upper bound, using Lemma 4.7

$$\dim_H \left\{\underline{\xi} \in \Sigma : \exists k \in \Xi,\ \dim E_{\theta,\Gamma_k(\underline{\xi})} = \delta\right\} \leq \dim_H \left\{\underline{\xi} \in \Sigma : \overline{\dim}_B E_{\theta,\Gamma(\underline{\xi})} \geq \delta\right\} \leq \inf_{t>0}\{-\delta t + P(t)\}.$$

The function $P(t)$ is convex (Lemma 4.3), hence $t \mapsto -\delta t + P(t)$ is convex as well. So either $\delta = \beta(\theta)$ then $\lim_{t\to 0+} P'(t) = \delta = \beta(\theta)$ or $\delta > \beta(\theta)$ then the convexity of the function implies that

$$\inf_t\{-\delta t + P(t)\} = \inf_{t>0}\{-\delta t + P(t)\} \Leftrightarrow \text{ there exists a } t > 0 \text{ that } P'(t) = \delta.$$

Therefore,

$$\dim_H \{a \in \mathrm{proj}_\theta\Lambda : \dim E_{\theta,a} = \delta\} \leq \inf_{t>0}\{-\delta t + P(t)\} = P^*(\delta),$$

which completes the proof. $\square$

Now we will turn to the special case of Sierpiński gasket.

Lemma 4.10. *Suppose that Λ is the Sierpiński gasket (i.e. $N=2$ and $\Omega=\{0,1\}^2\setminus\{(1,1)\}$) then for any $\theta\in[0,\pi/2)$ such that $\tan\theta\in\mathbb{Q}$ the set*

$$M:=\bigcup_{k=-\infty}^{\infty}\sigma^k\left\{(\xi_1,\xi_2,\dots)\in\Sigma:\forall n\geq 1\exists i,j\in\Xi,\left(A_{\xi_1}\cdots A_{\xi_n}\right)_{i,j}=0\right\}$$

has Hausdorff dimension 0*. Moreover, for every $\underline{\xi}\in\Sigma\backslash N$*

$$\overline{\dim}_B E_{\theta,\Gamma_k(\underline{\xi})}=\overline{\dim}_B E_{\theta,\Gamma(\underline{\xi})}=\overline{\dim}_B E_{\theta,\Gamma(\sigma\underline{\xi})}\text{ for every }k=1,\dots,p+q.$$

Proof. The first part of the lemma follows from [1, Proposition 3.2].

To prove the rest of the statement, let us observe $\overline{\dim}_B E_{\theta,\Gamma_k(\underline{\xi})}\leq\overline{\dim}_B E_{\theta,\Gamma(\underline{\xi})}$ for every $\underline{\xi}\in\Sigma$ and $k\in\Xi$. Moreover, since $\underline{e}^T A_{\xi}\leq\sharp\Omega\cdot\underline{e}^T$ and

$$\overline{\dim}_B E_{\theta,\Gamma(\underline{\xi})}=\limsup_{n\to\infty}\frac{\log\underline{e}^T A_{\xi_1}\cdots A_{\xi_n}\underline{e}}{n\log N},$$

we have $\overline{\dim}_B E_{\theta,\Gamma(\underline{\xi})}\leq\overline{\dim}_B E_{\theta,\Gamma(\sigma\underline{\xi})}$.

If $\underline{\xi}\notin M$ then there exists a $K=K(\underline{\xi})$ such that

$$A_{\xi_1}\cdots A_{\xi_K}>0.$$

Therefore, for every $n\geq K+1$, $\underline{e}_k^T A_{\xi_1}\cdots A_{\xi_n}\underline{e}\geq\underline{e}^T A_{\xi_{K+1}}\cdots A_{\xi_n}\underline{e}$ for any $k=1,\dots,p+q$. This implies that $\underline{\dim}_B E_{\theta,\Gamma_k(\underline{\xi})}\geq\underline{\dim}_B E_{\theta,\Gamma(\sigma^K\underline{\xi})}$. Hence,

$$\overline{\dim}_B E_{\theta,\Gamma(\sigma^K\underline{\xi})}\geq\overline{\dim}_B E_{\theta,\Gamma(\sigma\underline{\xi})}\geq\overline{\dim}_B E_{\theta,\Gamma(\underline{\xi})}\geq\overline{\dim}_B E_{\theta,\Gamma_k(\underline{\xi})}\geq\overline{\dim}_B E_{\theta,\Gamma(\sigma^K\underline{\xi})}.$$

$\square$

Proposition 4.11. *If Λ is the Sierpiński gasket then for every $\underline{\xi}\notin M$ and $k\in\Xi$*

$$\overline{\dim}_B E_{\theta,\Gamma_k(\underline{\xi})}=\dim_P E_{\theta,\Gamma_k(\underline{\xi})}.$$

Proof. Let $\underline{\xi}\notin M$ and $k\in\Xi$. Moreover, let $\{A_i\}$ be an arbitrary countable decomposition of $E_{\theta,\Gamma_k(\underline{\xi})}$. Since the set $E_{\theta,\Gamma_k(\underline{\xi})}$ is compact, there exists a j such that A_j contains a non-empty interior in $E_{\theta,\Gamma_k(\underline{\xi})}$. That is, there exists an $\varepsilon>0$ and $x\in E_{\theta,\Gamma_k(\underline{\xi})}$ such that $B_\varepsilon(x)\cap E_{\theta,\Gamma_k(\underline{\xi})}\subseteq A_j$. In particular, there exists an $n\geq 1$ and $(\omega_0,\dots,\omega_{n-1})\in\Omega^n$ such that $F_{\omega_0,\dots,\omega_{n-1}}(\Lambda)\cap E_{\theta,\Gamma_k(\underline{\xi})}\subseteq A_j$. It is easy to see that $F_{\omega_0,\dots,\omega_{n-1}}(\Lambda)\cap E_{\theta,\Gamma_k(\underline{\xi})}=F_{\omega_0,\dots,\omega_{n-1}}(E_{\theta,\Gamma_i(\sigma^n\underline{\xi})})$ for an $i\in\{1,\dots,p+q\}$.

Using Lemma 4.10 and the fact that M is σ invariant, we get

$$\overline{\dim}_B A_j\geq\overline{\dim}_B F_{\omega_0,\dots,\omega_{n-1}}(E_{\theta,\Gamma_i(\sigma^n\underline{\xi})})=\overline{\dim}_B E_{\theta,\Gamma_i(\sigma^n\underline{\xi})}=\overline{\dim}_B E_{\theta,\Gamma_k(\underline{\xi})}.$$

The statement follows from the definition of packing dimension. $\square$

Proof of Proposition 1.5. Let $\delta\in[b_{\min},b_{\max}]$ arbitrary, then

$$\dim_H\{a\in\mathrm{proj}_\theta\Lambda:\dim_P E_{\theta,a}=\delta\}=\dim_H\left\{\underline{\xi}\in\Sigma:\exists k\in\Xi\dim_P E_{\theta,\Gamma_k(\underline{\xi})}=\delta\right\}.$$

Using Proposition 4.11 and Lemma 4.10

$$\dim_H\left\{\underline{\xi}\in\Sigma:\exists k\in\Xi,\ \dim_P E_{\theta,\Gamma_k(\underline{\xi})}=\delta\right\}=$$
$$\dim_H\left\{\underline{\xi}\in\Sigma:\exists k\in\Xi,\ \overline{\dim}_B E_{\theta,\Gamma_k(\underline{\xi})}=\delta\right\}=\dim_H\left\{\underline{\xi}\in\Sigma:\overline{\dim}_B E_{\theta,\Gamma(\underline{\xi})}=\delta\right\}.$$

The statement follows from Lemma 4.1 and Theorem 4.4. □

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Balázs Bárány, Institute of Mathematics, Polish Academy of Sciences, ul. Sniadeckich 8, 00-956 Warszawa, Poland
E-mail address: balubsheep@gmail.com

Michał Rams, Institute of Mathematics, Polish Academy of Sciences, ul. Sniadeckich 8, 00-956 Warszawa, Poland
E-mail address: M.Rams@impan.gov.pl