

# ON THE LEDRAPPIER-YOUNG FORMULA FOR SELF-AFFINE MEASURES

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ABSTRACT. Ledrappier and Young [18] introduced a relation between entropy, Lyapunov exponents and dimension for invariant measures of diffeomorphisms on compact manifolds. In this paper, we show that self-affine measures on the plane satisfy the Ledrappier-Young formula if the corresponding iterated function systems (IFS) satisfies the strong separation condition and the linear parts satisfy the so called dominated splitting. We give a sufficient conditions, inspired by Ledrappier [16], that the dimensions of such self-affine measure is equal to the Lyapunov dimension. We show some applications, namely, we give another proof for Hueter-Lalley's classical theorem [14] and we consider self-affine measures and sets generated by lower triangular matrices.

## 1. INTRODUCTION

Let  $\mathcal{A} := \{A_1, A_2, \dots, A_N\}$  be a finite set of contracting, non-singular  $2 \times 2$  matrices, and let  $\Phi := \{f_i(\underline{x}) = A_i \underline{x} + \underline{t}_i\}_{i=1}^N$  be an iterated function system on the plane with affine mappings. Well known fact that there exists a unique non-empty compact subset  $\Lambda$  of  $\mathbb{R}^2$  such that

$$\Lambda = \bigcup_{i=1}^N f_i(\Lambda).$$

We call the set  $\Lambda$  as the attractor of  $\Phi$ .

The dimension theory of self-affine sets is far away from being well understood. One of the most natural approach of the Hausdorff and box dimension of self-affine sets is the so-called *subadditive pressure function*, introduced by Falconer [7]. Denote  $\alpha_i(A)$  the  $i$ th singular value of a  $2 \times 2$  non-singular matrix  $A$ . For  $s \geq 0$  define the singular value function  $\phi^s$  as follows

$$\phi^s(A) := \begin{cases} \alpha_1(A)^s & 0 \leq s \leq 1 \\ \alpha_1(A)\alpha_2(A)^{s-1} & 1 < s \leq 2 \\ (\alpha_1(A)\alpha_2(A))^{s/2} & s > 2. \end{cases}$$

We note that in this case,  $\alpha_1(A) = \|A\|$  and  $\alpha_2(A) = \|A^{-1}\|^{-1}$ , where  $\|\cdot\|$  is the usual matrix norm induced by the Euclidean norm on  $\mathbb{R}^2$ . Let us define the subadditive pressure function generated by  $\mathcal{A}$  for  $s \geq 0$  as

$$P(s) := \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{i_1, \dots, i_n=1}^N \phi^s(A_{i_1} \cdots A_{i_n}). \quad (1.1)$$

The function  $P(s)$  is continuous, strictly monotone decreasing on  $[0, \infty)$ , moreover  $P(0) = \log N$  and  $\lim_{s \rightarrow \infty} P(s) = -\infty$ . Falconer showed in [7] that the unique root  $s_0$  of the subadditive pressure function is always an upper bound for the box dimension of the attractor  $\Lambda$  and if  $\|A_i\| < 1/3$  for every  $i = 1, \dots, N$  then

$$\dim_H \Lambda = \dim_B \Lambda = \min \{2, s_0\} \text{ for Lebesgue almost every } \mathbf{t} = (t_1, \dots, t_N) \in \mathbb{R}^{2N}.$$

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The bound was later extended to  $1/2$  by Solomyak, see [25].

In the case of similarities (i.e.  $A_i = \rho_i U_i$ , where  $0 < \rho_i < 1$  and  $U_i$  are orthonormal matrices) the dimension theory of the attractors is well understood if a separation condition holds. In the case of strict affine mappings, it is very unclear. McMullen [21] showed a family of self-affine sets on the plane, where the Hausdorff and box dimension differs, however a separation condition holds. Later, such examples were constructed by Gatzouras and Lalley [12] and Barański [2]. In these cases the linear parts of the maps were diagonal matrices.

Falconer [8] proved that under some conditions and separation, the box dimension of a self-affine set is equal to the root of the subadditive pressure. However, the only known sufficient condition in general was given by Hueter and Lalley [14], which ensures that the Hausdorff and box dimension of a self-affine set coincide and equal to the root of the subadditive pressure.

One way to understand the Hausdorff dimension of self-affine sets depends on the understand of Hausdorff dimension of self-affine measures. We call a measure  $\mu$  self-affine if its compactly supported with support  $\Lambda$  and there exists a  $\underline{p} = (p_1, \dots, p_N)$  probability vector such that

$$\mu = \sum_{i=1}^N p_i \mu \circ f_i^{-1}. \quad (1.2)$$

Ledrappier and Young [17, 18] introduced a formula for the Hausdorff dimension of invariant measures of diffeomorphisms on compact manifolds. It is a widespread claim that self-affine measures satisfy this formula but it was proven just in a very few cases. Basically, the first result on a class of self-affine measures and sets, for which the formula hold, was proven by Przytycki and Urbański [23]. Later, Feng and Hu [11] proved that if the linear parts of the mappings are diagonal matrices then the Ledrappier-Young formula holds for the Hausdorff dimension of the self-affine measures, without assuming any separation condition or condition on the norm of the matrices. Moreover, Ledrappier [16] proved that the formula is valid for a special family of self-affine measures, namely when the support is the graphs of a Weierstrass functions.

Our main goal is to generalize Ledrappier's result [16] for a more general family of self-affine measures.

An other important dimension theoretical property of a self-affine measure is its exactness. Denote  $B_r(\underline{x})$  the two dimensional ball centered at  $\underline{x} \in \mathbb{R}^2$  with radius  $r$ . Then we call

$$\underline{d}_\mu(\underline{x}) = \liminf_{r \rightarrow 0^+} \frac{\log \mu(B_r(\underline{x}))}{\log r} \quad \text{and} \quad \bar{d}_\mu(\underline{x}) = \limsup_{r \rightarrow 0^+} \frac{\log \mu(B_r(\underline{x}))}{\log r}$$

the lower and upper local dimension of  $\mu$  at the point  $\underline{x}$ , if the limit exists then we say that the measure has local dimension  $d_\mu(\underline{x})$  at the point  $\underline{x}$ . It is well known fact that

$$\dim_H \mu = \mu - \text{esssup}_\mu \underline{d}_\mu(\underline{x}) \quad (1.3)$$

for any  $\mu$  Radon measure. Moreover, we call the measure  $\mu$  exact dimensional if the local dimension exists for  $\mu$  almost every points and equals to  $\dim_H \mu$ . Feng and Hu [11] proved that self-similar measures, and self-affine measures if the linear parts are diagonal matrices, are exact dimensional. Ledrappier [16] proved that for the graphs of Weierstrass functions. We extend this phenomena, as well.

To analyse self-affine measures, it is convenient to handle it as a natural projection of Bernoulli measures. That is, let  $\Sigma^+ = \{1, \dots, N\}^{\mathbb{N}}$  be the symbolic space of one side infinite length words and let  $\nu = \{p_1, \dots, p_N\}^{\mathbb{N}}$  be a Bernoulli measure, where  $\underline{p} = (p_1, \dots, p_N)$  is a probability vector. If  $\pi_+ : \Sigma^+ \mapsto \Lambda$  denotes the natural projection, i.e.  $\pi_+(i_0, i_1, \dots) = \lim_{n \rightarrow \infty} f_{i_0} \circ \dots \circ f_{i_n}(\underline{0})$ , then  $\mu = (\pi_+)_* \nu = \nu \circ \pi_+^{-1}$ .

According to the classical result of Oseledec Multiplicative Ergodic Theorem [22] for  $\nu$ -almost every  $\mathbf{i} \in \Sigma^+$  there exist constants  $0 < \chi_\mu^s \leq \chi_\mu^{ss}$  such that

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \log \alpha_1(A_{i_0} \cdots A_{i_{n-1}}) &= -\chi_\mu^s \text{ and} \\ \lim_{n \rightarrow \infty} \frac{1}{n} \log \alpha_2(A_{i_0} \cdots A_{i_{n-1}}) &= -\chi_\mu^{ss} \text{ for } \nu\text{-a.e. } \mathbf{i} = (i_0, i_1, \dots) \in \Sigma^+. \end{aligned}$$

Denote the entropy of  $\nu$  by  $h_\nu = -\sum_{i=1}^N p_i \log p_i$  then we can define the Lyapunov-dimension of the measure  $\mu$  by

$$\dim_{\text{Lyap}} \mu = \min \left\{ 2, \frac{h_\nu}{\chi_\mu^s}, 1 + \frac{h_\nu - \chi_\mu^s}{\chi_\mu^{ss}} \right\}.$$

It is known that the Lyapunov dimension of a self-affine measure is always an upper bound for the Hausdorff dimension, see [15]. We show also a sufficient condition (based on the idea of Ledrappier [16]) which implies that the Lyapunov- and Hausdorff dimension of a self-affine measure coincide.

Throughout the paper we will follow the method of Ledrappier [16] and Ledrappier and Young [17, 18]. At the end of the paper we give an alternative proof for the Hueter-Lalley Theorem and we show some applications for triangular matrices.

## 2. PRELIMINARIES AND RESULTS

Let  $\mathcal{A} := \{A_1, A_2, \dots, A_N\}$  be a finite set of contracting, non-singular  $2 \times 2$  matrices, and let  $\Phi := \{f_i(\underline{x}) = A_i \underline{x} + \underline{t}_i\}_{i=1}^N$  be an iterated function system on the plane with affine mappings.

**Definition 2.1.** *We say that  $\Phi$  satisfies the strong separation condition (SSC) if there exists an open and bounded set  $\mathcal{O} \subset \mathbb{R}^2$  such that*

- (1) *for every  $i = 1, \dots, N$ ,  $f_i(\overline{\mathcal{O}}) \subseteq \mathcal{O}$  and*
- (2) *for every  $i \neq j$ ,  $f_i(\overline{\mathcal{O}}) \cap f_j(\overline{\mathcal{O}}) = \emptyset$ ,*

where  $\overline{\mathcal{O}}$  denotes the closure of  $\mathcal{O}$ .

If the IFS satisfies the SSC then

$$f_i(\Lambda) \cap f_j(\Lambda) = \emptyset \text{ for every } i \neq j, \quad (2.1)$$

where  $\Lambda$  denotes the attractor of  $\Phi$ . One can show that (2.1) is actually equivalent to SSC. Moreover,

$$\Lambda = \bigcap_{n=1}^{\infty} \bigcup_{i_1, \dots, i_n=1}^N f_{i_1} \circ \cdots \circ f_{i_n}(\overline{\mathcal{O}}).$$

Let us denote by  $\mathcal{S} = \{1, \dots, N\}$  the set of symbols and by  $\Sigma = \mathcal{S}^{\mathbb{Z}}$  the symbolic space of two side infinite words. Moreover, let  $\Sigma^+ = \mathcal{S}^{\mathbb{N}}$  be the set of right- and  $\Sigma^- = \mathcal{S}^{\mathbb{Z}^-}$  be the set of left side infinite length words. For a two sided infinite length word  $\mathbf{i} = (\dots, i_{-2}, i_{-1}; i_0, i_1, i_2, \dots)$  let us denote the left hand side by  $\mathbf{i}_-$  and the right-hand side by  $\mathbf{i}_+$ , i.e.  $\mathbf{i}_- = (\dots, i_{-2}, i_{-1})$  and  $\mathbf{i}_+ = (i_0, i_1, i_2, \dots)$ . Denote  $\Sigma^* = \bigcup_{n=0}^{\infty} \mathcal{S}^n$  the set of finite length words. The length of a finite length word  $\underline{i}$  is denoted by  $|\underline{i}|$  and for an infinite length word  $\mathbf{i} \in \Sigma$  we denote by  $\mathbf{i}|_n^k$  the elements of  $\mathbf{i}$  between  $n$  and  $k$ , i.e.  $\mathbf{i}|_n^k = (i_n, \dots, i_k)$ . Let us define also the cylinder sets on  $\Sigma$  (and on  $\Sigma^+$  respectively) by

$$[\mathbf{i}|_n^k] = \left\{ \mathbf{j} \in \Sigma : \mathbf{j}|_n^k = \mathbf{i}|_n^k \right\}.$$

We denote the composition of functions of  $\Phi$  for a finite length word  $\underline{i} = (i_1, \dots, i_n) \in \Sigma^*$  by  $f_{\underline{i}} = f_{i_1} \circ \cdots \circ f_{i_n}$ .

Now let us introduce a dynamical system  $F$  acting on  $\overline{\mathcal{O}} \times \Sigma^+$  by

$$F(\underline{x}, \mathbf{i}) := (f_{i_0}(\underline{x}), \sigma \mathbf{i}),$$

where  $\mathcal{O}$  is the open and bounded set from Definition 2.1. Since  $F$  is hyperbolic map acting  $\overline{\mathcal{O}} \times \Sigma^+$ , the unique non-empty and compact set, which is  $F$ -invariant, is  $\bigcap_{n=0}^{\infty} F^n(\overline{\mathcal{O}} \times \Sigma^+) = \Lambda \times \Sigma^+$ .

Define  $\pi_- : \Sigma^- \mapsto \Lambda$  (similarly to  $\pi_+$ ) by

$$\pi_-(\dots, i_{-2}, i_{-1}) = \lim_{n \rightarrow \infty} f_{i_{-1}} \circ \dots \circ f_{i_{-n}}(0) = \sum_{n=1}^{\infty} A_{i_{-1}} \cdots A_{i_{-n+1}} t_{i_{-n}}.$$

If  $\sigma$  is the left-shift operator on  $\Sigma$  then it is easy to see that  $F$  is conjugate to  $\sigma$  by the projection  $\pi : \Sigma \mapsto \Lambda \times \Sigma^+$ , where  $\pi(\mathbf{i}) := (\pi_-(\mathbf{i}_-), \mathbf{i}_+)$ . That is,

$$\pi \circ \sigma = F \circ \pi.$$

Let  $\underline{p} = (p_1, \dots, p_N)$  be a probability vector and let  $\nu = \{p_1, \dots, p_N\}^{\mathbb{N}}$  be the corresponding left-shift invariant and ergodic Bernoulli-probability measure on  $\Sigma^+$ . Denote  $\widehat{\nu} = \{p_1, \dots, p_N\}^{\mathbb{Z}}$  the natural extension of  $\nu$  to  $\Sigma$ . Let us define its projection to  $\Lambda \times \Sigma^+$  by  $\widehat{\mu} := \pi_* \widehat{\nu} = \widehat{\nu} \circ \pi^{-1}$ . Then  $\widehat{\mu}$  is an  $F$ -invariant and ergodic probability measure on  $\Lambda \times \Sigma^+$ , moreover  $\widehat{\mu} = \mu \times \nu$ , where  $\mu$  is self-affine measure defined in (1.2).

For the analysis of the dimension theoretical point of view, we need an assumption for the matrices  $\mathcal{A}$ , which ensures for us that there is a dynamically invariant foliation on  $\overline{\mathcal{O}} \times \Sigma^+$ .

**Definition 2.2.** *We say that the set  $\mathcal{A}$  of matrices satisfies the dominated splitting if there are constants  $C, \delta > 0$  such that*

$$\frac{\alpha_1(A_{\underline{i}})}{\alpha_2(A_{\underline{i}})} \geq C e^{\delta n} \text{ for all } \underline{i} \in \Sigma^* \text{ with } |\underline{i}| = n.$$

For example, family of matrices with strictly positive entries satisfies dominated splitting, see [1].

Let us define a map from  $\Sigma$  to  $\mathcal{A}$  in a natural way, i.e.  $A(\mathbf{i}) := A_{i_0}$ . Denote the product by  $A^{(n)}(\mathbf{i}) := A(\sigma^{n-1}\mathbf{i}) \cdots A(\mathbf{i})$  for  $\mathbf{i} \in \Sigma$  and  $n \geq 1$ . Now we are going to state some useful properties for set  $\mathcal{A}$  of matrices, satisfying dominated splitting.

**Lemma 2.3** ([4],[26]). *The set  $\mathcal{A}$  of matrices satisfies the dominated splitting if and only if for every  $\mathbf{i} \in \Sigma$  there are two one-dimensional subspaces  $e_{ss}(\mathbf{i}), e_s(\mathbf{i})$  of  $\mathbb{R}^2$  such that*

- (1)  $A(\mathbf{i})e_i(\mathbf{i}) = e_i(\sigma\mathbf{i})$  for every  $\mathbf{i} \in \Sigma$  and  $i = s, ss$ ,
- (2) there are constants  $C, \delta > 0$  such that

$$\frac{\|A^{(n)}(\mathbf{i})|e_s(\mathbf{i})\|}{\|A^{(n)}(\mathbf{i})|e_{ss}(\mathbf{i})\|} \geq C e^{\delta n} \text{ for all } \mathbf{i} \in \Sigma \text{ and } n \geq 1.$$

We call the family of subspaces  $e_{ss}(\mathbf{i})$  as strong stable directions.

We note that the dependence of the subspaces  $e_i$  on  $\mathbf{i} \in \Sigma$  is continuous, that is  $e_i : \Sigma \mapsto \mathbf{P}^1$  is continuous with the standard metrics, where  $\mathbf{P}^1$  denotes the projective space, see [6, Section B.1].

**Lemma 2.4** ([5]). *Let  $\mathcal{A}$  be a set of matrices satisfying the dominated splitting and let  $e_{ss}(\mathbf{i}), e_s(\mathbf{i})$  be the two one-dimensional subspaces of  $\mathbb{R}^2$  defined in Lemma 2.3. Then there exists a constant  $C > 0$  such that*

$$C^{-1} \|A^{(n)}(\mathbf{i})|e_s(\mathbf{i})\| \leq \alpha_1(A^{(n)}(\mathbf{i})) \leq C \|A^{(n)}(\mathbf{i})|e_s(\mathbf{i})\| \text{ and} \\ C^{-1} \|A^{(n)}(\mathbf{i})|e_{ss}(\mathbf{i})\| \leq \alpha_2(A^{(n)}(\mathbf{i})) \leq C \|A^{(n)}(\mathbf{i})|e_{ss}(\mathbf{i})\|.$$

In particular,

$$\chi_{\mu}^i = - \lim_{n \rightarrow \infty} \frac{1}{n} \log \|A^{(n)}(\mathbf{i})|e_i(\mathbf{i})\| = - \int \log \|A(\mathbf{i})|e_i(\mathbf{i})\| d\widehat{\nu}(\mathbf{i}) \text{ for } \widehat{\nu}\text{-a.e. } \mathbf{i} \text{ and } i = s, ss. \quad (2.2)$$

The dominated splitting property implies that the Lyapunov exponents are always distinct, actually  $\chi_\mu^s + \delta \leq \chi_\mu^{ss}$  for any  $\mu$  self-affine measure.

Let  $C_+ := \{(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\} : xy \geq 0\}$  be the standard positive cone. A cone is an image of  $C_+$  by a linear isomorphism and a multicone is a disjoint union of finitely many cones.

**Lemma 2.5** ([1], [4]). *A set  $\mathcal{A}$  of matrices satisfies the dominated splitting if and only if  $\mathcal{A}$  has a forward invariant multicone, i.e. there is a multicone  $M$  such that  $\bigcup_{i=1}^N A_i(M) \subset M^\circ$ , where  $M^\circ$  denotes the interior of  $M$ .*

Note that if  $M$  is a forward-invariant multicone w.r.t  $\mathcal{A} = (A_1, \dots, A_N)$  then the closure of its complement is backward-invariant, i.e. forward-invariant for  $\mathcal{A}^{-1} = (A_1^{-1}, \dots, A_N^{-1})$ .

**Lemma 2.6** ([1], [4]). *Let  $\mathcal{A}$  be a set of matrices satisfying the dominated splitting and let  $M$  be a forward-invariant multicone. Then for every  $\mathbf{i} \in \Sigma$*

$$e_s(\mathbf{i}) = \bigcap_{n=1}^{\infty} A_{i_{-1}} \cdots A_{i_{-n}}(M) \text{ and } e_{ss}(\mathbf{i}) = \bigcap_{n=1}^{\infty} A_{i_0}^{-1} \cdots A_{i_{n-1}}^{-1}(\overline{M^c}),$$

where  $M^c$  denotes the complement of  $M$ . In particular,  $e_s(\mathbf{i})$  depends only on  $\mathbf{i}_-$  and  $e_{ss}(\mathbf{i})$  depends only on  $\mathbf{i}_+$ .

An easy consequence of Lemma 2.5 and Lemma 2.6 is that there exists a constant  $\alpha > 0$  such that

$$\sphericalangle_{e_s(\mathbf{i}), e_{ss}(\mathbf{j})} \geq \alpha > 0, \text{ for every } \mathbf{i}, \mathbf{j} \in \Sigma, \quad (2.3)$$

where  $\sphericalangle$  denotes the included angle.

Let us denote the orthogonal projection from  $\mathbb{R}^2$  to the subspace perpendicular to  $e_{ss}(\mathbf{i})$  by  $\text{proj}_{\mathbf{i}}^{ss}$ . We call the family of projections of  $\mu$  along the strong stable directions as *transversal measures* and we denote by

$$\mu_{\mathbf{i}}^T := (\text{proj}_{\mathbf{i}}^{ss})_* \mu = \mu \circ (\text{proj}_{\mathbf{i}}^{ss})^{-1}. \quad (2.4)$$

Now we are ready to state our main theorem.

**Theorem 2.7.** *Let  $\mathcal{A} = \{A_1, A_2, \dots, A_N\}$  be a finite set of contracting, non-singular  $2 \times 2$  matrices, and let  $\Phi = \{f_i(\underline{x}) = A_i \underline{x} + \underline{t}_i\}_{i=1}^N$  be an iterated function system on the plane with affine mappings. Let  $\nu$  be a left-shift invariant and ergodic Bernoulli-probability measure on  $\Sigma^+$ , and  $\mu$  is the corresponding self-affine measure. If*

- (1)  $\mathcal{A}$  satisfies the dominated splitting,
- (2)  $\Phi$  satisfies the strong separation condition

then  $\mu$  is exact dimensional and

$$\dim_H \mu = \frac{h_\nu}{\chi_\mu^{ss}} + \left(1 - \frac{\chi_\mu^s}{\chi_\mu^{ss}}\right) \dim_H \mu_{\mathbf{i}}^T \text{ for } \nu\text{-almost every } \mathbf{i} \in \Sigma^+, \quad (2.5)$$

where  $h_\nu$  denotes the entropy of  $\nu$  and  $\chi_\mu^s, \chi_\mu^{ss}$  are the Lyapunov exponents, defined in (2.2).

We note that, unfortunately, the strong separation condition cannot be relaxed to the open set condition. Let  $A_1$  and  $A_2$  be two matrices with strictly positive entries such that the IFS  $\{f_i(\underline{x}) = A_i \underline{x}\}_{i=1,2}$  maps the closed unit square into itself and  $f_1((0, 1)^2) \cap f_2((0, 1)^2) = \emptyset$ . Then the IFS satisfies the open set condition, however its attractor is only a single point. Hence, (2.5) cannot hold for any self-affine measure, which are just the Dirac measure.

Since the transversal measures  $\mu_{\mathbf{i}}^T$  are the orthogonal projections of  $\mu$ ,  $\dim_H \mu_{\mathbf{i}}^T \leq \min\{1, \dim_H \mu\}$ . By (2.5), simple algebraic manipulations show that

$$\dim_H \mu = \dim_{\text{Lyap}} \mu \Leftrightarrow \dim_H \mu_{\mathbf{i}}^T = \min\{1, \dim_H \mu\} \text{ for } \nu\text{-a.e. } \mathbf{i} \in \Sigma^+. \quad (2.6)$$

If the distribution of the strong stable directions  $e_{ss}$  has large dimension then one can claim that the right-hand side of (2.6) holds. Let us consider the map  $e_{ss} : \Sigma^+ \mapsto \mathbf{P}^1$  which maps an  $\mathbf{i} \in \Sigma^+$  to the element of the projective space associated to  $e_{ss}(\mathbf{i})$ . Let us define the push-down measure of  $\nu$  by  $e_{ss}$  on  $\mathbf{P}^1$  as

$$\nu_{ss} := (e_{ss})_*\nu = \nu \circ (e_{ss})^{-1}. \quad (2.7)$$

**Theorem 2.8.** *Let  $\mathcal{A} = \{A_1, A_2, \dots, A_N\}$  be a finite set of contracting, non-singular  $2 \times 2$  matrices, and let  $\Phi = \{f_i(\underline{x}) = A_i \underline{x} + \underline{t}_i\}_{i=1}^N$  be an iterated function system on the plane with affine mappings. Let  $\nu$  be a left-shift invariant and ergodic Bernoulli-probability measure on  $\Sigma^+$ , and  $\mu$  is the corresponding self-affine measure. If*

- (1)  $\mathcal{A}$  satisfies the dominated splitting,
- (2)  $\Phi$  satisfies the strong separation condition,
- (3)  $\dim_H \nu_{ss} \geq \min \{1, \dim_{\text{Lyap}} \mu\}$

then

$$\dim_H \mu = \dim_{\text{Lyap}} \mu = \min \left\{ \frac{h_\nu}{\chi_\mu^s}, 1 + \frac{h_\nu - \chi_\mu^s}{\chi_\mu^{ss}} \right\}.$$

The proof of Theorem 2.8 is based on the idea of Ledrappier [16, Lemma 1], and uses a modification of the classical result of Marstrand [20].

*Proof of Theorem 2.8.* By Theorem 2.7, we know that  $\dim_H \mu_{\mathbf{i}}^T$  is a constant for  $\nu$ -almost every  $\mathbf{i} \in \Sigma^+$ . Using Lemma 5.1 we have that for every  $\varepsilon > 0$  there exists a set  $A \subseteq \Sigma^+$  such that  $\nu(A) > 0$  and for every  $\mathbf{i} \in A$   $\dim_H \mu_{\mathbf{i}}^T \geq \min \{1, \dim_H \mu\} - \varepsilon$ . This implies that  $\dim_H \mu_{\mathbf{i}}^T \geq \min \{1, \dim_H \mu\} - \varepsilon$  for  $\nu$ -almost every  $\mathbf{i} \in \Sigma^+$ . Since  $\varepsilon > 0$  was arbitrary we get

$$\dim_H \mu_{\mathbf{i}}^T = \min \{1, \dim_H \mu\} \text{ for } \nu\text{-almost every } \mathbf{i} \in \Sigma^+.$$

The statement of the theorem follows by (2.6).  $\square$

Lemma 5.1 is stated and proved in the Appendix, Section 5.

### 3. PROOF OF LEDRAPPIER-YOUNG FORMULA

The proof of Theorem 2.7 is decomposed into four propositions, Proposition 3.1, 3.3, 3.7, and 3.8. The proofs of Proposition 3.1 and 3.3 follow the proof of [16, Proposition 2]. Proposition 3.7 is a modified versions of [18, Lemma 11.3.1] and Proposition 3.8 is a modification of [18, Section (10.2)].

Let  $\nu$  be the left-shift invariant and ergodic Bernoulli-probability measure on  $\Sigma^+$  and  $\mu$  is self-affine measure defined in (1.2). Let  $\hat{\mu} = \mu \times \nu$  be the  $F$ -invariant and ergodic probability measure on  $\Lambda \times \Sigma^+$ , defined in the previous section. Denote  $\mathcal{B}$  the usual Borel  $\sigma$ -algebra on  $\Lambda \times \Sigma^+$ .

If  $\zeta$  is a measurable partition of  $\Lambda \times \Sigma^+$  then by the classical result of Rokhlin [24], there exists a canonical system of conditional measures, i.e. for  $\hat{\mu}$ -a.e.  $\underline{\mathbf{y}} \in \Lambda \times \Sigma$  there exists a measure  $\mu_{\underline{\mathbf{y}}}^\zeta$  supported on  $\zeta(\underline{\mathbf{y}})$ , the element of  $\zeta$  containing  $\underline{\mathbf{y}}$ . These measures are uniquely defined up to a 0 measure set and for every measurable set  $A$  the function  $\underline{\mathbf{y}} \mapsto \mu_{\underline{\mathbf{y}}}^\zeta(A)$  is  $\mathcal{B}_\zeta$ -measurable, where  $\mathcal{B}_\zeta$  is the sub- $\sigma$ -algebra of  $\mathcal{B}$  whose elements are union of elements of  $\zeta$ . Moreover,

$$\hat{\mu}(A) = \int \mu_{\underline{\mathbf{y}}}^\zeta(A) d\hat{\mu}(\underline{\mathbf{y}}). \quad (3.1)$$

For two measurable partitions  $\zeta_1$  and  $\zeta_2$  we define the common refinement  $\zeta_1 \vee \zeta_2$  such that for every  $\underline{\mathbf{y}}$ ,  $(\zeta_1 \vee \zeta_2)(\underline{\mathbf{y}}) = \zeta_1(\underline{\mathbf{y}}) \cap \zeta_2(\underline{\mathbf{y}})$ . Moreover, let us define the image of the partition  $\zeta$  in the natural way, i.e. for every  $\underline{\mathbf{y}}$ ,  $(F\zeta)(\underline{\mathbf{y}}) = F(\zeta(F^{-1}(\underline{\mathbf{y}})))$ .

We define a dynamically invariant foliation on  $\Lambda \times \Sigma^+$  with respect to the strong stable directions. Denote  $e_{ss}$  the family of one-dimensional strong stable directions defined in Lemma 2.3. Since  $e_{ss}$

depends only on  $\mathbf{i}_+$  by Lemma 2.5, it defines a foliation on  $\bar{\mathcal{O}}$  for every  $\mathbf{i}_+ \in \Sigma^+$ . Hence, it defines a foliation  $\xi^{ss}$  on  $\Lambda \times \Sigma^+$ . Namely, for an  $\underline{\mathbf{y}} = (\underline{x}, \mathbf{i}) \in \Lambda \times \Sigma^+$  let  $l_{ss}(\underline{\mathbf{y}})$  be the line trough  $\underline{x}$  parallel to  $e_{ss}(\mathbf{i})$  on  $\mathbb{R}^2 \times \{\mathbf{i}\}$ . Let the partition element  $\xi^{ss}(\underline{\mathbf{y}})$  be the intersection of the line  $l_{ss}(\underline{\mathbf{y}})$  with  $\Lambda \times \{\mathbf{i}\}$ . It is easy to see that  $F\xi^{ss}$  is a refinement of  $\xi^{ss}$ , that is, for every  $\underline{\mathbf{y}}$ ,  $(F\xi^{ss})(\underline{\mathbf{y}}) \subset \xi^{ss}(\underline{\mathbf{y}})$ .

Let us define the conditional entropy of  $F\xi^{ss}$  with respect to  $\xi^{ss}$  in the usual way,

$$H(F\xi^{ss}|\xi^{ss}) := - \int \log \widehat{\mu}_{\underline{\mathbf{y}}}^{\xi^{ss}}((F\xi^{ss})(\underline{\mathbf{y}})) d\widehat{\mu}(\underline{\mathbf{y}}).$$

**Proposition 3.1.** *For  $\widehat{\mu}$ -a.e.  $\underline{\mathbf{y}} \in \Lambda \times \Sigma^+$  the measure  $\widehat{\mu}_{\underline{\mathbf{y}}}^{\xi^{ss}}$  is exact dimensional and*

$$\dim_H \widehat{\mu}_{\underline{\mathbf{y}}}^{\xi^{ss}} = \frac{H(F\xi^{ss}|\xi^{ss})}{\chi_{\mu}^{ss}}.$$

Before we prove the proposition, we define another partition  $\mathcal{P} = \{f_i(\Lambda) \times \Sigma^+\}_{i=1}^N$ . It is easy to see that

$$\mathcal{P} \vee \xi^{ss} = F\xi^{ss}. \quad (3.2)$$

Let us denote the ball with radius  $r$  centered at  $\underline{\mathbf{y}}$  by  $B_r(\underline{\mathbf{y}})$ . Let  $B_r^{ss}(\underline{\mathbf{y}})$  be the restriction of the ball to  $\xi^{ss}(\underline{\mathbf{y}})$ . That is,

$$B_r^{ss}(\underline{\mathbf{y}}) = \{\underline{\mathbf{z}} \in \xi^{ss}(\underline{\mathbf{y}}) : |\underline{\mathbf{y}} - \underline{\mathbf{z}}| \leq r\},$$

where  $|\cdot|$  denotes the usual Euclidean norm on  $\mathbb{R}^2$ .

**Lemma 3.2.** *There is a constant  $c_1 > 0$  that for every  $n \geq 1$  and  $\underline{\mathbf{y}} = (\underline{x}, \mathbf{i}) \in \Lambda \times \Sigma^+$  with  $\underline{x} = \pi_-(\dots, i_{-2}, i_{-1})$*

$$B_{c_1^{-1}\alpha_2(A_{i_{-1}} \dots A_{i_{-n}})}^{ss}(\underline{\mathbf{y}}) \subseteq \left( \bigvee_{k=0}^{n-1} F^k \mathcal{P} \vee \xi^{ss} \right) (\underline{\mathbf{y}}) \subseteq B_{c_1 \alpha_2(A_{i_{-1}} \dots A_{i_{-n}})}^{ss}(\underline{\mathbf{y}}),$$

where  $\alpha_2(\cdot)$  is the second singular value of a matrix.

*Proof.* Let us fix an  $n \geq 1$  and  $\underline{\mathbf{y}} \in \Lambda \times \Sigma^+$  and let  $F^{-n}(\underline{\mathbf{y}}) = (\underline{x}', \mathbf{i}')$  then  $\mathbf{i}' = (i_{-n}, \dots, i_{-1}, i_0, \dots)$ . Denote  $D = \text{diam}(\bar{\mathcal{O}})$  the diameter of  $\bar{\mathcal{O}}$ . By the definition of strong stable directions, see Lemma 2.3, we have

$$\text{diam} \left( \left( \bigvee_{k=0}^{n-1} F^k \mathcal{P} \vee \xi^{ss} \right) (\underline{\mathbf{y}}) \right) \leq D \|A^{(n)}(\mathbf{i}')|e_{ss}(\mathbf{i}')\|.$$

On the other hand, let  $\kappa = \min_{i \neq j} \text{dist}(f_i(\Lambda), f_j(\Lambda))$ . Since the IFS  $\Phi$  satisfies the strong separation condition, see Definition 2.1,  $\kappa > 0$ . Then for every  $F^{-n}(\underline{\mathbf{y}}) = (\underline{x}', \mathbf{i}') \in \Lambda \times \Sigma^+$  if  $\underline{x}' \in f_i(\Lambda)$  then  $\text{dist}(\underline{x}', f_j(\Lambda)) > \kappa/2$  for every  $j \neq i$ . So

$$F^n(B_{\frac{\kappa}{2}}^{ss}(F^{-n}(\underline{\mathbf{y}}))) \subseteq \left( \bigvee_{k=0}^{n-1} F^k \mathcal{P} \vee \xi^{ss} \right) (\underline{\mathbf{y}}).$$

Applying again Lemma 2.3, we get  $F^n(B_{\frac{\kappa}{2}}^{ss}(F^{-n}(\underline{\mathbf{y}}))) = B_{\frac{\kappa}{2} \|A^{(n)}(\mathbf{i}')|e_{ss}(\mathbf{i}')\|}^{ss}(\underline{\mathbf{y}})$ .

Let  $C > 0$  be the constant defined in Lemma 2.4, then by choosing  $c_1 := C \max\{D, (\frac{\kappa}{2})^{-1}\}$ , the statement of the lemma follows.  $\square$

*Proof of Proposition 3.1.* To prove the statement of the proposition it is enough to show that

$$\lim_{r \rightarrow 0^+} \frac{\log \widehat{\mu}_{\underline{\mathbf{y}}}^{\xi^{ss}}(B_r^{ss}(\underline{\mathbf{y}}))}{\log r} = \frac{H(F\xi^{ss}|\xi^{ss})}{\chi_{\mu}^{ss}} \text{ for } \widehat{\mu}\text{-a.e. } \underline{\mathbf{y}}.$$

By Lemma 3.2, it is equivalent to show that

$$\lim_{n \rightarrow \infty} \frac{\log \widehat{\mu}_{\underline{\mathbf{y}}}^{\xi^{ss}} \left( \left( \bigvee_{k=0}^{n-1} F^k \mathcal{P} \vee \xi^{ss} \right) (\underline{\mathbf{y}}) \right)}{\log \alpha_2(A_{i_{-1}} \cdots A_{i_{-n}})} = \frac{H(F\xi^{ss}|\xi^{ss})}{\chi_{\mu}^{ss}} \text{ for } \widehat{\mu}\text{-a.e } \underline{\mathbf{y}}. \quad (3.3)$$

Then by using the definition of the partitions and conditional measures,

$$\begin{aligned} \log \widehat{\mu}_{\underline{\mathbf{y}}}^{\xi^{ss}} \left( \left( \bigvee_{k=0}^{n-1} F^k \mathcal{P} \vee \xi^{ss} \right) (\underline{\mathbf{y}}) \right) &= \log \widehat{\mu}_{\underline{\mathbf{y}}}^{\xi^{ss}} (\mathcal{P}(\underline{\mathbf{y}}) \cap \cdots \cap F^{n-1}(\mathcal{P}(F^{-n+1}(\underline{\mathbf{y}})))) = \\ &\log \widehat{\mu}_{\underline{\mathbf{y}}}^{\xi^{ss}} (\mathcal{P}(\underline{\mathbf{y}})) + \log \widehat{\mu}_{\underline{\mathbf{y}}}^{\xi^{ss} \vee \mathcal{P}} (F(\mathcal{P}(F^{-1}(\underline{\mathbf{y}})))) + \cdots + \log \widehat{\mu}_{\underline{\mathbf{y}}}^{\xi^{ss} \vee \mathcal{P} \vee \cdots \vee F^{n-2} \mathcal{P}} (F^{n-1}(\mathcal{P}(F^{-n+1}(\underline{\mathbf{y}})))) . \end{aligned}$$

Applying (3.2)

$$\begin{aligned} \log \widehat{\mu}_{\underline{\mathbf{y}}}^{\xi^{ss} \vee \mathcal{P} \vee \cdots \vee F^{k-1} \mathcal{P}} \left( F^k(\mathcal{P}(F^{-k}(\underline{\mathbf{y}}))) \right) &= \log \widehat{\mu}_{\underline{\mathbf{y}}}^{F^k \xi^{ss}} \left( F^k(\mathcal{P}(F^{-k}(\underline{\mathbf{y}}))) \right) = \\ &\log \widehat{\mu}_{F^{-k}(\underline{\mathbf{y}})}^{\xi^{ss}} \left( \mathcal{P}(F^{-k}(\underline{\mathbf{y}})) \right) , \end{aligned}$$

where in the last equality we have used the invariance of the measure  $\widehat{\mu}$ . Hence

$$\frac{1}{n} \log \widehat{\mu}_{\underline{\mathbf{y}}}^{\xi^{ss}} \left( \left( \bigvee_{k=0}^{n-1} F^k \mathcal{P} \vee \xi^{ss} \right) (\underline{\mathbf{y}}) \right) = \frac{1}{n} \sum_{k=0}^{n-1} \log \widehat{\mu}_{F^{-k}(\underline{\mathbf{y}})}^{\xi^{ss}} \left( \mathcal{P}(F^{-k}(\underline{\mathbf{y}})) \right)$$

Since  $\widehat{\mu}$  is ergodic

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \widehat{\mu}_{\underline{\mathbf{y}}}^{\xi^{ss}} \left( \left( \bigvee_{k=0}^{n-1} F^k \mathcal{P} \vee \xi^{ss} \right) (\underline{\mathbf{y}}) \right) = \int \log \widehat{\mu}_{\underline{\mathbf{y}}}^{\xi^{ss}} (\mathcal{P}(\underline{\mathbf{y}})) d\widehat{\mu}(\underline{\mathbf{y}}) = -H(\mathcal{P}|\xi^{ss}). \quad (3.4)$$

Using the property of conditional entropy and (3.2),  $H(\mathcal{P}|\xi^{ss}) = H(\mathcal{P} \vee \xi^{ss}|\xi^{ss}) = H(F\xi^{ss}|\xi^{ss})$ . Applying Oseledec's Theorem, we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \alpha_2(A_{i_{-1}} \cdots A_{i_{-n}}) = -\chi_{\mu}^{ss} \text{ for } \nu\text{-a.e } \mathbf{i},$$

which together with (3.4) implies (3.3).  $\square$

The next proposition is devoted to prove that the transversal measures  $\mu_{\mathbf{i}}^T = \mu \circ (\text{proj}_{\mathbf{i}}^{ss})^{-1}$  are exact dimensional measures for  $\nu$ -a.e  $\mathbf{i} \in \Sigma^+$  and calculate the typical Hausdorff dimension, where  $\text{proj}_{\mathbf{i}}^{ss}$  is the orthogonal projection from  $\mathbb{R}^2$  to the subspace perpendicular to  $e_{ss}(\mathbf{i})$ .

**Proposition 3.3.** *For  $\nu$ -a.e.  $\mathbf{i} \in \Sigma^+$  the measure  $\mu_{\mathbf{i}}^T$  is exact dimensional and*

$$\dim_H \mu_{\mathbf{i}}^T = \frac{h_{\nu} - H(F\xi^{ss}|\xi^{ss})}{\chi_{\mu}^s}.$$

We define an other invariant foliation  $\xi^s$  with respect to the stable plane. That is, for every  $\underline{\mathbf{y}} = (\underline{x}, \mathbf{i}) \in \Lambda \times \Sigma^+$ ,  $\xi^s(\underline{\mathbf{y}}) = \Lambda \times \{\mathbf{i}\}$ . Then the foliation  $\xi^s$  has similar properties to  $\xi^{ss}$ , i.e.  $F\xi^s$  is a refinement of  $\xi^s$  and  $\mathcal{P} \vee \xi^s = F\xi^s$ . Moreover, it is easy to see that for every  $\underline{\mathbf{y}}$

$$\widehat{\mu}_{\underline{\mathbf{y}}}^{\xi^s} = \mu \quad (3.5)$$

The proof of the proposition uses a slight modification of the classical result of Maker [19]. We state and prove a simpler version of the theorem in the Appendix, Section 5.



For the examination of the local dimension of the projected measure, instead of looking at the balls on the projection we introduce the transversal stable balls associated to the projection. Let  $B_r^t(\underline{x}, \mathbf{i})$  be transversal stable ball with radius  $r$ , i.e

$$B_r^t(\underline{x}, \mathbf{i}) = \{(\underline{y}, \mathbf{j}) : \mathbf{i} = \mathbf{j} \ \& \ \text{dist}(l_{ss}(\underline{x}, \mathbf{i}), l_{ss}(\underline{x}, \mathbf{j})) \leq 2r\},$$

where  $l_{ss}(\underline{x}, \mathbf{i})$  denotes the line trough  $\underline{x}$  parallel to  $e_{ss}(\mathbf{i})$ .

By technical reasons, for the examination we have to introduce the modified transversal stable ball. Since the IFS  $\Phi$  satisfies the SSC, by Lemma 2.6, for an  $\underline{\mathbf{y}} = (\underline{x}, \mathbf{i}) \in \Lambda \times \Sigma^+$  we can define the stable direction  $e_s(\underline{\mathbf{y}})$  of  $\underline{\mathbf{y}}$  by  $e_s(\underline{\mathbf{y}}) := e_s(\underline{x}) := e_s(\mathbf{i}_-)$ , where  $\pi_-(\mathbf{i}_-) = \underline{x}$ . Denote  $\text{dist}_{e_s(\underline{\mathbf{y}})}$  the natural Euclidean distance on the subspace  $e_s(\underline{\mathbf{y}})$ .

Then for an  $(\underline{x}, \mathbf{i}) \in \Lambda \times \Sigma^+$ , we define the modified transversal stable ball with radius  $\delta$  by

$$B_\delta^T(\underline{x}, \mathbf{i}) = \{(\underline{y}, \mathbf{j}) \in \Lambda \times \Sigma^+ : \mathbf{i} = \mathbf{j} \ \& \ \text{dist}_{e_s(\underline{x}, \mathbf{i})}(l_{ss}(\underline{x}, \mathbf{i}), l_{ss}(\underline{y}, \mathbf{j})) \leq \delta\}.$$

For a visualisation, see Figure 1.

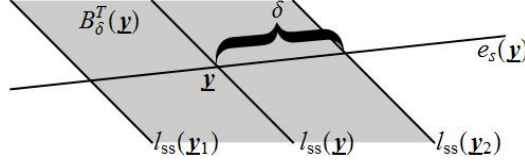


FIGURE 1. A visualisation of the modified transversal ball  $B_\delta^T(\underline{\mathbf{y}})$ .

By (2.3) there exists a constant  $c > 0$  that for every  $\underline{\mathbf{y}} \in \Lambda \times \Sigma^+$  and  $r > 0$

$$B_{c^{-1}r}^T(\underline{x}, \mathbf{i}) \subseteq B_r^t(\underline{x}, \mathbf{i}) \subseteq B_{cr}^T(\underline{x}, \mathbf{i}). \quad (3.6)$$

**Lemma 3.4.** For any  $\underline{\mathbf{y}} = (\underline{x}, \mathbf{i}) \in \Lambda \times \Sigma^+$  with  $\underline{x} = \pi_-(\dots, i_{-1})$

$$\mu(B_\delta^T(\underline{\mathbf{y}}) \cap \mathcal{P}(\underline{\mathbf{y}})) = \mu\left(B_{\|A_{i_{-1}}|e_s(F^{-1}(\underline{\mathbf{y}}))\|^{-1}\delta}^T(F^{-1}(\underline{\mathbf{y}}))\right) p_{i_{-1}}.$$

*Proof.* Since the directions  $e_s$  are  $F$ -invariant, we get for any  $\underline{\mathbf{y}}' = (\underline{x}', \mathbf{i}')$  and  $\infty > \delta' > 0$

$$F(B_{\delta'}^T(\underline{\mathbf{y}}') \times [i'_0]) = B_{\|A_{i'_0}|e_s(\underline{\mathbf{y}}')\|\delta'}^T(F(\underline{\mathbf{y}}')) \cap \mathcal{P}(F(\underline{\mathbf{y}}')) \times \Sigma^+.$$

The map  $F$  is invertible, hence

$$B_{\delta'}^T(\underline{\mathbf{y}}') \times [i'_0] = F^{-1}\left(B_{\|A_{i'_0}|e_s(\underline{\mathbf{y}}')\|\delta'}^T(F(\underline{\mathbf{y}}')) \cap \mathcal{P}(F(\underline{\mathbf{y}}')) \times \Sigma^+\right).$$

By taking  $\underline{\mathbf{y}}' = F(\underline{\mathbf{y}})$  we have  $\|A_{i'_0}|e_s(\underline{\mathbf{y}}')\| = \|A_{i_{-1}}|e_s(F^{-1}(\underline{\mathbf{y}}))\|$  and by taking  $\delta = \|A_{i_{-1}}|e_s(F^{-1}(\underline{\mathbf{y}}))\|\delta'$

$$B_{\|A_{i_{-1}}|e_s(F^{-1}(\underline{\mathbf{y}}))\|^{-1}\delta}^T(F^{-1}(\underline{\mathbf{y}})) \times [i_{-1}] = F^{-1}(B_\delta^T(\underline{\mathbf{y}}) \cap \mathcal{P}(\underline{\mathbf{y}}) \times \Sigma^+).$$

The measure  $\hat{\mu}$  is  $F$ -invariant, therefore

$$\begin{aligned} \mu(B_\delta^T(\underline{\mathbf{y}}) \cap \mathcal{P}(\underline{\mathbf{y}})) &= \hat{\mu}(B_\delta^T(\underline{\mathbf{y}}) \cap \mathcal{P}(\underline{\mathbf{y}}) \times \Sigma^+) = \hat{\mu}(F^{-1}(B_\delta^T(\underline{\mathbf{y}}) \cap \mathcal{P}(\underline{\mathbf{y}}) \times \Sigma^+)) = \\ &= \hat{\mu}(B_{\|A_{i_{-1}}|e_s(F^{-1}(\underline{\mathbf{y}}))\|^{-1}\delta}^T(F^{-1}(\underline{\mathbf{y}})) \times [i_{-1}]) = \mu(B_{\|A_{i_{-1}}|e_s(F^{-1}(\underline{\mathbf{y}}))\|^{-1}\delta}^T(F^{-1}(\underline{\mathbf{y}}))) p_{i_{-1}}. \end{aligned}$$

□

**Lemma 3.5.** For every  $\underline{\mathbf{y}} = (\underline{\mathbf{x}}, \mathbf{i}) \in \Lambda \times \Sigma^+$

$$\|A^{(2)}(\mathbf{i})|e_s(\underline{\mathbf{y}})\| = \|A(\mathbf{i})|e_s(\underline{\mathbf{y}})\| \|A(\sigma\mathbf{i})|e_s(F(\underline{\mathbf{y}}))\|.$$

In particular,

$$\|A^{(n)}(\mathbf{i})|e_s(\underline{\mathbf{y}})\| = \prod_{k=0}^{n-1} \|A(\sigma^k\mathbf{i})|e_s(F^k(\underline{\mathbf{y}}))\|.$$

*Proof.* By definition  $\|A(\mathbf{i})|e_s(\underline{\mathbf{y}})\| = \sup_{v \in e_s(\underline{\mathbf{y}})} \frac{\|A(\mathbf{i})v\|}{\|v\|}$ . On the other hand for every  $v_1, v_2 \in e_s(\underline{\mathbf{y}})$  there exists a constant  $c \in \mathbb{R}$  such that  $cv_1 = v_2$ . Therefore  $\|A(\mathbf{i})|e_s(\underline{\mathbf{y}})\| = \|A(\mathbf{i})v\|$  with any vector  $v \in e_s(\underline{\mathbf{y}})$  with  $\|v\| = 1$ . Hence,

$$\|A^{(2)}(\mathbf{i})|e_s(\underline{\mathbf{y}})\| = \|A^{(2)}(\mathbf{i})v\| = \|A(\sigma\mathbf{i}) \frac{A(\mathbf{i})v}{\|A(\mathbf{i})v\|}\| \|A(\mathbf{i})v\| = \|A(\sigma\mathbf{i})|e_s(F(\underline{\mathbf{y}}))\| \|A(\mathbf{i})|e_s(\underline{\mathbf{y}})\|,$$

where we used in the last equation that  $A(\mathbf{i})v \in e_s(F(\underline{\mathbf{y}}))$ .  $\square$

Let us define functions  $g(\underline{\mathbf{y}}) := \widehat{\mu}_{\underline{\mathbf{y}}}^{\xi_{ss}}(\mathcal{P}(\underline{\mathbf{y}}))$  and  $g_\delta(\underline{\mathbf{y}}) := \frac{\mu(B_\delta^T(\underline{\mathbf{y}}) \cap \mathcal{P}(\underline{\mathbf{y}}))}{\mu(B_\delta^T(\underline{\mathbf{y}}))}$ . By definition and (3.5),  $g_\delta \rightarrow g$  as  $\delta \rightarrow 0+$  for  $\mu$  almost everywhere and, since  $g_\delta$  is uniformly bounded, (3.1) implies  $g_\delta \rightarrow g$  in  $L^1(\widehat{\mu})$  as  $\delta \rightarrow 0+$ .

**Lemma 3.6.** The function  $\sup_{\delta > 0} \{-\log g_\delta\}$  is in  $L^1(\widehat{\mu})$ .

*Proof.* To verify the statement of the lemma, it is enough to show that

$$\sum_{k=1}^{\infty} \widehat{\mu} \left\{ \underline{\mathbf{y}} : \inf_{\delta > 0} g_\delta(\underline{\mathbf{y}}) < e^{-k} \right\} < \infty.$$

By (3.5)

$$\widehat{\mu} \left\{ \underline{\mathbf{y}} : \inf_{\delta > 0} g_\delta(\underline{\mathbf{y}}) < e^{-k} \right\} = \sum_{i=1}^N \int \mu \left\{ \underline{\mathbf{x}} \in f_i(\Lambda) : \inf_{\delta > 0} \frac{\mu(B_\delta^T(\underline{\mathbf{x}}, \mathbf{i}) \cap f_i(\Lambda))}{\mu(B_\delta^T(\underline{\mathbf{x}}, \mathbf{i}))} < e^{-k} \right\} d\nu(\mathbf{i}). \quad (3.7)$$

For a fixed  $\mathbf{i} \in \Sigma^+$  denote  $E_{k,\mathbf{i}}^i$  the set  $\left\{ \underline{\mathbf{x}} \in f_i(\Lambda) : \inf_{\delta > 0} \frac{\mu(B_\delta^T(\underline{\mathbf{x}}, \mathbf{i}) \cap f_i(\Lambda))}{\mu(B_\delta^T(\underline{\mathbf{x}}, \mathbf{i}))} < e^{-k} \right\}$ . Let

$$\mathcal{E}_{k,\mathbf{i}}^i := \left\{ B_\delta^T(\underline{\mathbf{x}}, \mathbf{i}) : \frac{\mu(B_\delta^T(\underline{\mathbf{x}}, \mathbf{i}) \cap f_i(\Lambda))}{\mu(B_\delta^T(\underline{\mathbf{x}}, \mathbf{i}))} < e^{-k} \right\}$$

be the collection of closed transversal balls. It is needless to say that  $\mathcal{E}_{k,\mathbf{i}}^i$  is a cover of  $E_{k,\mathbf{i}}^i$ . Then by Besicovitch covering Theorem there exists a constant  $c > 0$  independent of  $\mathbf{i}$ ,  $i$  and  $k$ , and a countable family of balls  $\bigcup_{n=1}^c \mathcal{F}_n \subseteq \mathcal{E}_{k,\mathbf{i}}^i$  such that

$$E_{k,\mathbf{i}}^i \subseteq \bigcup_{n=1}^c \bigcup_{B \in \mathcal{F}_n} B \text{ and } B' \cap B'' = \emptyset \text{ if } B', B'' \in \mathcal{F}_j \text{ for } j = 1, \dots, n.$$

Hence,

$$\mu \left\{ \underline{\mathbf{x}} \in f_i(\Lambda) : \inf_{\delta > 0} \frac{\mu(B_\delta^T(\underline{\mathbf{x}}, \mathbf{i}) \cap f_i(\Lambda))}{\mu(B_\delta^T(\underline{\mathbf{x}}, \mathbf{i}))} < e^{-k} \right\} \leq \sum_{c=1}^N \sum_{B \in \mathcal{F}_n} \mu(B \cap f_i(\Lambda)) \leq e^{-k} \sum_{n=1}^c \sum_{B \in \mathcal{F}_n} \mu(B) \leq ce^{-k}.$$

Therefore, by (3.7)

$$\sum_{k=1}^{\infty} \widehat{\mu} \left\{ \underline{\mathbf{y}} : \inf_{\delta > 0} g_\delta(\underline{\mathbf{y}}) < e^{-k} \right\} \leq \sum_{k=1}^{\infty} cNe^{-k} < \infty.$$

$\square$

*Proof of Proposition 3.3.* By the definition of the transversal measure, the statement of the proposition is equivalent to

$$\lim_{\delta \rightarrow 0^+} \frac{\log \mu(B_\delta^t(\underline{\mathbf{y}}))}{\log \delta} = \frac{h_\nu - H(F\xi^{ss}|\xi^{ss})}{\chi_\mu^s} \text{ for } \hat{\mu}\text{-a.e } \underline{\mathbf{y}}.$$

Hence, by (3.6) it is enough to show that

$$\lim_{\delta \rightarrow 0^+} \frac{\log \mu(B_\delta^T(\underline{\mathbf{y}}))}{\log \delta} = \frac{h_\nu - H(F\xi^{ss}|\xi^{ss})}{\chi_\mu^s} \text{ for } \hat{\mu}\text{-a.e } \underline{\mathbf{y}}.$$

By Lemma 2.4, if  $\underline{\mathbf{y}} = (\underline{\mathbf{x}}, \mathbf{i}) \in \Lambda \times \Sigma^+$  with  $\underline{\mathbf{x}} = \pi_-(\dots, i_{-2}, i_{-1})$ , it is sufficient to show

$$\lim_{n \rightarrow \infty} \frac{\log \mu \left( B_{\|A_{i_{-1}} \dots A_{i_{-n}}|e_s(F^{-n}(\underline{\mathbf{y}}))\|}^T(\underline{\mathbf{y}}) \right)}{\log \alpha_1(A_{i_{-1}} \dots A_{i_{-n}})} = \frac{h_\nu - H(F\xi^{ss}|\xi^{ss})}{\chi_\mu^s} \text{ for } \hat{\mu}\text{-a.e } \underline{\mathbf{y}}. \quad (3.8)$$

We write the measure of the ball as

$$\begin{aligned} & \mu \left( B_{\|A_{i_{-1}} \dots A_{i_{-n}}|e_s(F^{-n}(\underline{\mathbf{y}}))\|}^T(\underline{\mathbf{y}}) \right) = \\ & \mu(B_1^T(F^{-n}(\underline{\mathbf{y}}))) \frac{\mu(B_{\|A_{i_{-n}}|e_s(F^{-n}(\underline{\mathbf{y}}))\|}^T(F^{-n+1}(\underline{\mathbf{y}})))}{\mu(B_1^T(F^{-n}(\underline{\mathbf{y}})))} \prod_{k=1}^{n-1} \frac{\mu \left( B_{\|A_{i_{-k}} \dots A_{i_{-n}}|e_s(F^{-n}(\underline{\mathbf{y}}))\|}^T(F^{-k+1}(\underline{\mathbf{y}})) \right)}{\mu \left( B_{\|A_{i_{-k-1}} \dots A_{i_{-n}}|e_s(F^{-n}(\underline{\mathbf{y}}))\|}^T(F^{-k}(\underline{\mathbf{y}})) \right)}. \end{aligned}$$

Applying Lemma 3.4 and Lemma 3.5 we get for every  $k = 1, \dots, n$

$$\begin{aligned} & \mu \left( B_{\|A_{i_{-k-1}} \dots A_{i_{-n}}|e_s(F^{-n}(\underline{\mathbf{y}}))\|}^T(F^{-k}(\underline{\mathbf{y}})) \right) = \\ & \mu \left( B_{\|A_{i_{-k}}|e_s(F^{-k}(\underline{\mathbf{y}}))\|^{-1}\|A_{i_{-k}} \dots A_{i_{-n}}|e_s(F^{-n}(\underline{\mathbf{y}}))\|}^T(F^{-k}(\underline{\mathbf{y}})) \right) = \\ & \mu \left( B_{\|A_{i_{-k}} \dots A_{i_{-n}}|e_s(F^{-n}(\underline{\mathbf{y}}))\|}^T(F^{-k+1}(\underline{\mathbf{y}})) \cap \mathcal{P}(F^{-k+1}(\underline{\mathbf{y}})) \right) p_{i_{-k}}^{-1}. \end{aligned}$$

Hence,

$$\begin{aligned} & \frac{1}{n} \log \mu \left( B_{\|A_{i_{-1}} \dots A_{i_{-n}}|e_s(F^{-n}(\underline{\mathbf{y}}))\|}^T(\underline{\mathbf{y}}) \right) = \\ & \frac{1}{n} \log \mu(B_1^T(F^{-n}(\underline{\mathbf{y}}))) - \frac{1}{n} \sum_{k=1}^n \log g_{\|A_{i_{-k}} \dots A_{i_{-n}}|e_s(F^{-n}(\underline{\mathbf{y}}))\|}(F^{-k+1}(\underline{\mathbf{y}})) + \frac{1}{n} \sum_{k=1}^n \log p_{i_{-k}}. \end{aligned}$$

Let us define a function  $h_{n,k}(\underline{\mathbf{y}}) := \log g_{\|A_{i_{-1}} \dots A_{i_{-n+k-1}}|e_s(F^{-n+k-1}(\underline{\mathbf{y}}))\|}(\underline{\mathbf{y}})$ . Then

$$\begin{aligned} & \frac{1}{n} \log \mu \left( B_{\|A_{i_{-1}} \dots A_{i_{-n}}|e_s(F^{-n}(\underline{\mathbf{y}}))\|}^T(\underline{\mathbf{y}}) \right) = \\ & \frac{1}{n} \log \mu(B_1^T(F^{-n}(\underline{\mathbf{y}}))) - \frac{1}{n} \sum_{k=1}^n h_{n,k}(F^{-k+1}(\underline{\mathbf{y}})) + \frac{1}{n} \sum_{k=1}^n \log p_{i_{-k}}. \end{aligned}$$

Since  $\|A_{i_{-1}} \dots A_{i_{-n+k-1}}|e_s(F^{-n+k-1}(\underline{\mathbf{y}}))\| \rightarrow 0$  uniformly on  $\Lambda \times \Sigma^+$  as  $n-k \rightarrow \infty$ ,  $\lim_{n-k \rightarrow \infty} h_{n,k} = \log g$  in  $L^1(\hat{\mu})$  and for  $\hat{\mu}$ -almost everywhere. Moreover, by Lemma 3.6 we can apply Maker's ergodic theorem Lemma 5.2, thus by (3.4)

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n h_{n,k}(F^{-k+1}(\underline{\mathbf{y}})) = \int \log g(\underline{\mathbf{y}}) d\hat{\mu}(\underline{\mathbf{y}}) = -H(F\xi^{ss}|\xi^{ss}) \text{ for } \hat{\mu}\text{-a.e } \underline{\mathbf{y}}.$$

On the other hand

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \log \mu(B_1^T(F^{-n}(\underline{\mathbf{y}}))) &= 0 \text{ for every } \underline{\mathbf{y}} \in \Lambda \times \Sigma^+ \text{ and} \\ \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \log p_{i_{-k}} &= \sum_{i=1}^N p_i \log p_i = -h_\nu \text{ for } \hat{\mu}\text{-a.e } \underline{\mathbf{y}} \in \Lambda \times \Sigma^+, \end{aligned}$$

which implies

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mu \left( B_{\|A_{i_{-1}} \cdots A_{i_{-n}}\| e_s(F^{-n}(\underline{\mathbf{y}}))}^T(\underline{\mathbf{y}}) \right) = -h_\nu + H(F\xi^{ss}|\xi^{ss}) \quad (3.9)$$

Applying Oseledec's Theorem, we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \alpha_1(A_{i_{-1}} \cdots A_{i_{-n}}) = -\chi_\mu^s \text{ for } \nu\text{-a.e } \mathbf{i},$$

which together with the equation (3.9) implies (3.8).  $\square$

Denote  $B_r^s(\underline{\mathbf{x}}, \mathbf{i})$  the square on  $\Lambda \times \{\mathbf{i}\}$  with a side parallel to  $e_{ss}(\mathbf{i})$  and length  $2r$  centered at  $(\underline{\mathbf{x}}, \mathbf{i}) \in \Lambda \times \Sigma^+$ . It is easy to see that there exists a constant  $c > 0$  that for every  $(\underline{\mathbf{x}}, \mathbf{i}) \in \Lambda \times \Sigma^+$ ,

$$B_{c^{-1}r}(\underline{\mathbf{x}}) \subseteq B_r^s(\underline{\mathbf{x}}, \mathbf{i}) \subseteq B_{cr}(\underline{\mathbf{x}}), \quad (3.10)$$

where  $B_r(\underline{\mathbf{x}})$  is the usual Euclidean ball on  $\mathbb{R}^2$ .

**Proposition 3.7.** *For  $\mu$ -a.e  $\underline{\mathbf{x}} \in \Lambda$*

$$\liminf_{r \rightarrow 0^+} \frac{\log \mu(B_r(\underline{\mathbf{x}}))}{\log r} \geq \frac{H(F\xi^{ss}|\xi^{ss})}{\chi_\mu^{ss}} + \frac{h_\nu - H(F\xi^{ss}|\xi^{ss})}{\chi_\mu^s}.$$

*Proof.* By (3.10), it is enough to show that

$$\liminf_{r \rightarrow 0^+} \frac{\log \mu(B_r^s(\underline{\mathbf{y}}))}{\log r} \geq \frac{H(F\xi^{ss}|\xi^{ss})}{\chi_\mu^{ss}} + \frac{h_\nu - H(F\xi^{ss}|\xi^{ss})}{\chi_\mu^s} \text{ for } \hat{\mu}\text{-a.e } \underline{\mathbf{y}}. \quad (3.11)$$

For simplicity, let  $d_s := \frac{H(F\xi^{ss}|\xi^{ss})}{\chi_\mu^{ss}}$ . By Proposition 3.1, the measure  $\hat{\mu}_{\underline{\mathbf{y}}}^{\xi^{ss}}$  is exact dimensional and by Egorov's Theorem for every  $\varepsilon > 0$  there exists a set  $J_1 \subseteq \Lambda \times \Sigma^+$  with  $\hat{\mu}(J_1) > 1 - \varepsilon$  such that there exists an  $M_1 > 0$  that for every  $m \geq M_1$  and  $\underline{\mathbf{y}} \in J_1$

$$\hat{\mu}_{\underline{\mathbf{y}}}^{\xi^{ss}}(B_{2e^{-m}}^{ss}(\underline{\mathbf{y}})) \leq e^{m(-d_s+\varepsilon)}.$$

By the definition of  $B_r^s(\underline{\mathbf{y}})$  it is easy to see that  $B_r^s(\underline{\mathbf{y}}) \cap \xi^{ss}(\underline{\mathbf{y}}) = B_r^s(\underline{\mathbf{y}})$  and by the definition of conditional measures

$$\hat{\mu}_{\underline{\mathbf{z}}}^{\xi^{ss}}(B_{2e^{-m}}^s(\underline{\mathbf{y}})) \leq e^{m(-d_s+\varepsilon)} \text{ for every } \underline{\mathbf{y}} \in J_1 \text{ and } \xi^{ss}(\underline{\mathbf{z}}) = \xi^{ss}(\underline{\mathbf{y}}). \quad (3.12)$$

The combination of Lebesgue density Theorem and Egorov's Theorem implies that there exists a set  $J_2 \subseteq J_1$  with  $\hat{\mu}(J_2) > 1 - 2\varepsilon$  and  $M_2 > 0$  such that for every  $m \geq M_2$  and  $\underline{\mathbf{y}} \in J_2$

$$\mu(J_1 \cap B_{e^{-m}}^s(\underline{\mathbf{y}})) \geq \frac{1}{2} \mu(B_{e^{-m}}^s(\underline{\mathbf{y}})).$$

By (3.12), for every  $\underline{\mathbf{z}} \in B_{e^{-m}}^t(\underline{\mathbf{y}})$  such that there exists a  $\underline{\mathbf{z}}' \in \xi^{ss}(\underline{\mathbf{z}}) \cap B_{e^{-m}}^s(\underline{\mathbf{y}}) \cap J_1$

$$\hat{\mu}_{\underline{\mathbf{z}}}^{\xi^{ss}}(B_{e^{-m}}^s(\underline{\mathbf{y}}) \cap J_1) \leq \hat{\mu}_{\underline{\mathbf{z}}}^{\xi^{ss}}(B_{2e^{-m}}^s(\underline{\mathbf{z}}') \cap J_1) \leq e^{m(-d_s+\varepsilon)}.$$

If  $\xi^{ss}(\underline{\mathbf{z}}) \cap B_{e^{-m}}^s(\underline{\mathbf{y}}) \cap J_1 = \emptyset$  then the bound above is trivial. Hence, for every  $\underline{\mathbf{y}} \in J_2$

$$\mu(B_{e^{-m}}^s(\underline{\mathbf{y}})) \leq 2 \int_{B_{e^{-m}}^t(\underline{\mathbf{y}})} \hat{\mu}_{\underline{\mathbf{z}}}^{\xi^{ss}}(B_{e^{-m}}^s(\underline{\mathbf{y}}) \cap J_1) d\hat{\mu}(\underline{\mathbf{z}}) \leq 2e^{m(-d_s+\varepsilon)} \mu(B_{e^{-m}}^t(\underline{\mathbf{y}})).$$

Since  $\varepsilon > 0$  was arbitrary, inequality (3.11) follows by Proposition 3.3.  $\square$

**Proposition 3.8.** For  $\mu$ -a.e  $\underline{x} \in \Lambda$

$$\limsup_{r \rightarrow 0^+} \frac{\log \mu(B_r(\underline{x}))}{\log r} \leq \frac{H(F\xi^{ss}|\xi^{ss})}{\chi_\mu^{ss}} + \frac{h_\nu - H(F\xi^{ss}|\xi^{ss})}{\chi_\mu^s}.$$

*Proof.* For simplicity, let  $h_s := H(F\xi^{ss}|\xi^{ss})$ . By applying Egorov's Theorem for Proposition 3.1 and for Shannon-McMillan-Breiman Theorem, we get that for every  $\varepsilon > 0$  there exists a set  $J_1$  with  $\widehat{\mu}(J_1) > 1 - \varepsilon$  and  $M_1 > 0$  such that for every  $\underline{y} = (\underline{x}, \mathbf{i}) \in J_1$  and every  $m \geq M_1$

$$\widehat{\mu}_{\underline{y}}^{\xi^{ss}}(B_{\frac{\kappa}{2}} e^{-m(\chi_\mu^s - 2\varepsilon)}(\underline{y})) \geq e^{-m(\chi_\mu^s - 2\varepsilon)(d_s + \varepsilon)}, \quad (3.13)$$

$$\widehat{\mu}_{\underline{y}}^{\xi^{ss}}\left(\left(\bigvee_{k=0}^{m-1} F^k \mathcal{P}\right)(\underline{y})\right) \leq e^{-m(h_s - \varepsilon)}, \quad (3.14)$$

$$\left(\bigvee_{k=0}^{m-1} F^k \mathcal{P}\right)(\underline{y}) \subseteq B_{e^{-m(\chi_\mu^s - 2\varepsilon)}}(\underline{x}) \times \Sigma^+, \quad (3.15)$$

$$\widehat{\mu}\left(\left(\bigvee_{k=0}^{m-1} F^k \mathcal{P}\right)(\underline{y})\right) \geq e^{-m(h_\nu + \varepsilon)}. \quad (3.16)$$

Applying Lebesgue's density Theorem and Egorov's Theorem, there exists a set  $J_2 \subseteq J_1$  with  $\widehat{\mu}(J_1) > 1 - 2\varepsilon$  and  $M_2 \geq M_1$  such that for every  $\underline{y} = (\underline{x}, \mathbf{i}) \in J_2$  and every  $m \geq M_2$

$$\frac{\widehat{\mu}_{\underline{y}}^{\xi^{ss}}(B_{\frac{\kappa}{2}} e^{-m(\chi_\mu^s - 2\varepsilon)}(\underline{y}) \cap J_1)}{\widehat{\mu}_{\underline{y}}^{\xi^{ss}}(B_{\frac{\kappa}{2}} e^{-m(\chi_\mu^s - 2\varepsilon)}(\underline{y}))} \geq \frac{1}{2}. \quad (3.17)$$

For every  $m \geq M_2$  we can define a finite sequence  $\{\underline{y}_i\}_{i=1}^{N^m}$  such that for any  $i \neq j$

$$\left(\bigvee_{k=0}^{m-1} F^k \mathcal{P}\right)(\underline{y}_i) \cap \left(\bigvee_{k=0}^{m-1} F^k \mathcal{P}\right)(\underline{y}_j) = \emptyset,$$

and  $\underline{y}_i \in J_1$  whenever  $J_1 \cap \left(\bigvee_{k=0}^{m-1} F^k \mathcal{P}\right)(\underline{y}_i) \neq \emptyset$ . For simplicity, we introduce the notation  $L_m(\underline{y}) := B_{\frac{\kappa}{2}} e^{-m(\chi_\mu^s - 2\varepsilon)}(\underline{y}) \cap J_1$ . By (3.13) and (3.17), for any  $\underline{y} \in J_2$

$$\widehat{\mu}_{\underline{y}}^{\xi^{ss}}(L_m(\underline{y})) \geq \frac{1}{2} \widehat{\mu}_{\underline{y}}^{\xi^{ss}}(B_{\frac{\kappa}{2}} e^{-m(\chi_\mu^s - 2\varepsilon)}(\underline{y})) \geq \frac{1}{2} e^{-m(\chi_\mu^s - 2\varepsilon)(d_s + \varepsilon)}. \quad (3.18)$$

Then by (3.14)

$$\begin{aligned} \widehat{\mu}_{\underline{y}}^{\xi^{ss}}(L_m(\underline{y})) &\leq \sum_{i: \underline{y}_i \in J_1} \widehat{\mu}_{\underline{y}_i}^{\xi^{ss}}\left(\left(\bigvee_{k=0}^{m-1} F^k \mathcal{P}\right)(\underline{y}_i) \cap L_m(\underline{y})\right) \\ &\leq \#\left\{\underline{y}_i \in J_1 : L_m(\underline{y}) \cap \left(\bigvee_{k=0}^{m-1} F^k \mathcal{P}\right)(\underline{y}_i) \neq \emptyset\right\} \cdot \max_{i: \underline{y}_i \in J_1} \left\{\widehat{\mu}_{\underline{y}_i}^{\xi^{ss}}\left(\left(\bigvee_{k=0}^{m-1} F^k \mathcal{P}\right)(\underline{y}_i)\right)\right\} \\ &\leq \#\left\{\underline{y}_i \in J_1 : L_m(\underline{y}) \cap \left(\bigvee_{k=0}^{m-1} F^k \mathcal{P}\right)(\underline{y}_i) \neq \emptyset\right\} e^{-m(h_s - \varepsilon)}. \end{aligned}$$

Hence,

$$\widehat{\mu}_{\underline{\mathbf{y}}}^{\xi^{ss}}(L_m(\underline{\mathbf{y}}))e^{m(h_s-\varepsilon)} \leq \#\left\{\underline{\mathbf{y}}_i \in J_1 : L_m(\underline{\mathbf{y}}) \cap \left(\bigvee_{k=0}^{m-1} F^k\mathcal{P}\right)(\underline{\mathbf{y}}_i) \neq \emptyset\right\}. \quad (3.19)$$

On the other hand, if  $L_m(\underline{\mathbf{y}}) \cap \left(\bigvee_{k=0}^{m-1} F^k\mathcal{P}\right)(\underline{\mathbf{y}}_i) \neq \emptyset$  then by (3.15)

$$\left(\bigvee_{k=0}^{m-1} F^k\mathcal{P}\right)(\underline{\mathbf{y}}_i) \subseteq B_{2e^{-m(\chi_\mu^s-2\varepsilon)}}(\underline{\mathbf{x}}) \times \Sigma^+.$$

Therefore,

$$\begin{aligned} \mu(B_{2e^{-m(\chi_\mu^s-2\varepsilon)}}(\underline{\mathbf{x}})) &= \widehat{\mu}(B_{2e^{-m(\chi_\mu^s-2\varepsilon)}}(\underline{\mathbf{x}}) \times \Sigma^+) \\ &\geq \#\left\{\underline{\mathbf{y}}_i \in J_1 : L_m(\underline{\mathbf{y}}) \cap \left(\bigvee_{k=0}^{m-1} F^k\mathcal{P}\right)(\underline{\mathbf{y}}_i) \neq \emptyset\right\} \min_{i:\underline{\mathbf{y}}_i \in J_1} \left\{\widehat{\mu}\left(\left(\bigvee_{k=0}^{m-1} F^k\mathcal{P}\right)(\underline{\mathbf{y}}_i)\right)\right\}. \end{aligned}$$

By (3.16) and (3.19), for any  $\underline{\mathbf{y}} \in J_2$

$$\mu(B_{2e^{-m(\chi_\mu^s-2\varepsilon)}}(\underline{\mathbf{x}})) \geq \widehat{\mu}_{\underline{\mathbf{y}}}^{\xi^{ss}}(L_m(\underline{\mathbf{y}}))e^{m(h_s-\varepsilon)}e^{-m(h_\nu+\varepsilon)}.$$

Using (3.18), for any  $\underline{\mathbf{y}} = (\underline{\mathbf{x}}, \mathbf{i}) \in J_2$

$$\mu(B_{2e^{-m(\chi_\mu^s-2\varepsilon)}}(\underline{\mathbf{x}})) \geq \frac{1}{2}e^{-m(\chi_\mu^s-2\varepsilon)(d_s+\varepsilon)}e^{m(h_s-\varepsilon)}e^{-m(h_\nu+\varepsilon)}.$$

Hence, for any  $\underline{\mathbf{y}} \in J_2$

$$\limsup_{m \rightarrow \infty} \frac{\log \mu(B_{2e^{-m(\chi_\mu^s-2\varepsilon)}}(\underline{\mathbf{x}}))}{-m(\chi_\mu^s-2\varepsilon)} \leq d_s + \varepsilon + \frac{h_\nu - h_s}{\chi_\mu^s - 2\varepsilon} + \varepsilon + \frac{2\varepsilon}{\chi_\mu^s - 2\varepsilon}$$

Since  $\varepsilon > 0$  was arbitrary, the statement of the proposition follows.  $\square$

*Proof of Theorem 2.7.* Proposition 3.7 and Proposition 3.8 together imply that  $\mu$  is exact dimensional, moreover

$$\dim_H \mu = \frac{H(F\xi^{ss}|\xi^{ss})}{\chi_\mu^{ss}} + \frac{h_\nu - H(F\xi^{ss}|\xi^{ss})}{\chi_\mu^s}.$$

Simple algebraic manipulations show that

$$\frac{H(F\xi^{ss}|\xi^{ss})}{\chi_\mu^{ss}} + \frac{h_\nu - H(F\xi^{ss}|\xi^{ss})}{\chi_\mu^s} = \frac{h_\nu}{\chi_\mu^{ss}} + \left(1 - \frac{\chi_\mu^s}{\chi_\mu^{ss}}\right) \frac{h_\nu - H(F\xi^{ss}|\xi^{ss})}{\chi_\mu^s}.$$

The proof can be finished by applying Proposition 3.3.  $\square$

## 4. APPLICATIONS

**4.1. Hueter-Lalley Theorem.** This section is devoted to show some applications of our main theorems. In the point of view of Theorem 2.8, to prove that the Hausdorff dimension of a self-affine measure is equal to its Lyapunov dimension, one has to study the dimension of  $\nu_{ss}$  defined in (2.7). The measure  $\nu_{ss}$  is basically a self-conformal measure associated to an IFS on the projective space. If the IFS on the projective space satisfies some separation condition then one may be able to calculate its dimension. That is, what basically Hueter and Lalley [14] proved. Now we reprove their classical result.

**Theorem 4.1.** Let  $\mathcal{A} = \{A_1, A_2, \dots, A_N\}$  be a finite set of contracting, non-singular  $2 \times 2$  matrices, and let  $\Phi = \{f_i(\underline{x}) = A_i \underline{x} + \underline{t}_i\}_{i=1}^N$  be an iterated function system on the plane with affine mappings. Let  $\nu$  be a left-shift invariant and ergodic Bernoulli-probability measure on  $\Sigma^+$ , and  $\mu$  is the corresponding self-affine measure. Assume that

- (1)  $\mathcal{A}$  satisfies the dominated splitting,
- (2)  $\mathcal{A}$  satisfies the backward non-overlapping condition, i.e. there exists a backward invariant multicone  $M$  that  $A_i^{-1}(M^\circ) \subseteq M^\circ$  and  $A_i^{-1}(M^\circ) \cap A_j^{-1}(M^\circ) = \emptyset$  for every  $i \neq j$ ,
- (3)  $\mathcal{A}$  satisfies the 1-bunched property, i.e. for every  $i = 1, \dots, N$   $\alpha_1(A_i)^2 \leq \alpha_2(A_i)$ ,
- (4)  $\Phi$  satisfies the strong separation condition.

Then

$$\dim_H \mu = \dim_{\text{Lyap}} \mu = \frac{h_\nu}{\chi_\mu^s} \leq 1.$$

The proof of the theorem uses the following lemma.

**Lemma 4.2.** Let  $\mathcal{A} = \{A_1, A_2, \dots, A_N\}$  be a finite set of contracting, non-singular  $2 \times 2$  matrices and let  $\nu$  be a left-shift invariant and ergodic Bernoulli-probability measure on  $\Sigma^+$ . Assume that

- (1)  $\mathcal{A}$  satisfies the dominated splitting,
- (2)  $\mathcal{A}$  satisfies the backward non-overlapping condition.

Let  $e_{ss} : \Sigma^+ \mapsto \mathbf{P}^1$  be the projection defined in Lemma 2.6. Then

$$\dim_H \nu_{ss} = \dim_H \nu \circ e_{ss}^{-1} = \frac{h_\nu}{\chi_\mu^{ss} - \chi_\mu^s},$$

where  $\chi_\mu^{ss}$  and  $\chi_\mu^s$  are the Lyapunov exponents defined in Lemma 2.4.

*Proof.* The projective space  $\mathbf{P}^1$  is equivalent to upper half unit sphere on  $\mathbb{R}^2$ . We define an iterated function system on  $\mathbf{P}^1$  by  $\mathcal{A}$  in the natural way, i.e.

$$\tilde{A}_i : \theta \in \mathbf{P}^1 \mapsto \text{sgn}((A_i^{-1}\theta)_2) \frac{A_i^{-1}\theta}{\|A_i^{-1}\theta\|},$$

where  $\text{sgn}((A_i\theta)_2)$  denotes the signum of the second coordinate of the vector  $A_i\theta$ . By [5, Lemma 3.2], the IFS  $\tilde{\mathcal{A}} = \{\tilde{A}_1, \dots, \tilde{A}_N\}$  is uniformly contracting on  $M$ , where  $M$  is the backward invariant multicone with non-overlapping condition. Hence, the measure  $\nu_{ss}$  is the invariant measure associated to the IFS  $\tilde{\mathcal{A}}$ , and

$$\dim_H \nu_{ss} = \lim_{r \rightarrow 0^+} \frac{\log \nu_{ss}(B_r(\theta))}{\log r} \text{ for } \nu_{ss}\text{-a.e } \theta,$$

where  $B_r(\theta)$  denotes the ball with radius  $r$  centered at  $\theta$  according to the spherical distance. Since  $\mathcal{A}$  satisfies the backward non-overlapping condition

$$\dim_H \nu_{ss} = \lim_{n \rightarrow \infty} \frac{\log \nu_{ss}(\tilde{A}_{i_1} \circ \dots \circ \tilde{A}_{i_n}(M))}{\log \text{diam}(\tilde{A}_{i_1} \circ \dots \circ \tilde{A}_{i_n}(M))} \text{ for } \nu\text{-a.e } \mathbf{i},$$

where  $\text{diam}(\cdot)$  denotes the diameter of a set according to the spherical distance. It is easy to see that for any  $\theta_1, \theta_2 \in \mathbf{P}^1$  with distance less than  $\pi/2$

$$\frac{\det(A_i^{-1})}{2\|A_i^{-1}(\theta_1)\|\|A_i^{-1}(\theta_2)\|} \leq d(\tilde{A}_i(\theta_1), \tilde{A}_i(\theta_2)) \leq \frac{2\det(A_i^{-1})}{\|A_i^{-1}(\theta_1)\|\|A_i^{-1}(\theta_2)\|}$$

for any  $i \in \Sigma^*$ . Thus

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \text{diam}(\tilde{A}_{i_1} \circ \dots \circ \tilde{A}_{i_n}(M)) = \chi_\mu^s - \chi_\mu^{ss} \text{ for } \nu\text{-a.e } \mathbf{i}.$$

On the other hand

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \nu_{ss}(\tilde{A}_{i_1} \circ \cdots \circ \tilde{A}_{i_n}(M)) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \nu([i_1, \dots, i_n]) = -h_\nu \text{ for } \nu\text{-a.e } \mathbf{i}.$$

The statement follows by taking the ratio of the previous two limits.  $\square$

*Proof of Theorem 4.1.* By 1-bunched property,  $\chi_\mu^{ss} \leq 2\chi_\mu^s$ . Hence, by using Lemma 4.2

$$1 \geq \dim_H \nu_{ss} = \frac{h_\nu}{\chi_\mu^{ss} - \chi_\mu^s} \geq \frac{h_\nu}{\chi_\mu^s} = \dim_{\text{Lyap}} \mu.$$

Thus, applying Theorem 2.8 we have that  $\dim_{\text{Lyap}} \mu = \dim_H \mu$ .  $\square$

**Corollary 4.3** (Hueter-Lalley,[14]). *Let  $\mathcal{A} = \{A_1, A_2, \dots, A_N\}$  be a finite set of contracting, non-singular  $2 \times 2$  matrices, and let  $\Phi = \{f_i(\underline{x}) = A_i \underline{x} + \underline{t}_i\}_{i=1}^N$  be an iterated function system on the plane with affine mappings and denote  $\Lambda$  the attractor of the IFS  $\Phi$ . With the assumptions (1)-(4) of Theorem 4.1*

$$\dim_H \Lambda = \dim_B \Lambda = s \leq 1,$$

where  $s$  is the unique root of the pressure function  $P(s)$ , defined in (1.1).

*Proof.* It is easy to see that the assumptions (1)-(4) of Theorem 4.1 are inherited to the higher iterations, i.e. for any  $n \geq 1$  the IFS  $\Phi^n = \{f_{\underline{i}}\}_{|\underline{i}|=n}$  and the set of matrices  $\mathcal{A}^n = \{A_{\underline{i}}\}_{|\underline{i}|=n}$  satisfy the assumptions (1)-(4).

Let us define a monotone decreasing sequence  $\{s_n\}_{n=1}^\infty$  such that  $s_n$  are the unique solution of the equations

$$\sum_{|\underline{i}|=n} \alpha_1(A_{\underline{i}})^{s_n} = 1.$$

We define the left-shift invariant Bernoulli measure  $\nu_n$  with probability vector  $(\alpha_1(A_{\underline{i}})^{s_n})_{|\underline{i}|=n}$  and let  $\mu_n$  be the associated self-affine measure. Then by Theorem 4.1 and (2.2), for every  $n \geq 1$

$$1 \geq \dim_H \mu_n = \frac{h_{\nu_n}}{\chi_{\mu_n}^s} \geq \frac{s_n}{1 + \frac{C}{n \log \alpha_{\max}}},$$

where  $\alpha_{\max} = \max_i \alpha_1(A_i)$ . Hence  $\lim_{n \rightarrow \infty} s_n = s \leq 1$ . Moreover, by [7, Proposition 5.1]

$$s \geq \overline{\dim_B \Lambda} \geq \dim_H \Lambda \geq \lim_{n \rightarrow \infty} \dim_H \mu_n = s,$$

which had to be proven.  $\square$

**4.2. Triangular matrices.** The other way to study the dimension of  $\nu_{ss}$  is to handle the overlaps of the associated IFS on the projective space. Since this IFS is very difficult to handle in general, we focus on a special family of self-affine sets. Let us assume that the matrices in  $\mathcal{A}$  are lower triangular, i.e.

$$A_i = \begin{bmatrix} a_i & 0 \\ b_i & c_i \end{bmatrix}, \quad (4.1)$$

where  $0 < |a_i|, |c_i| < 1$  for every  $i = 1, \dots, N$ . Using [10, Theorem 2.5], the subadditive pressure function  $P(s)$  defined in (1.1) can be written in a simpler form, i.e

$$P(s) = \begin{cases} \log \max \left\{ \sum_{i=1}^N |a_i|^s, \sum_{i=1}^N |c_i|^s \right\} & \text{if } 0 \leq s < 1 \\ \log \max \left\{ \sum_{i=1}^N |a_i||c_i|^{s-1}, \sum_{i=1}^N |c_i||a_i|^{s-1} \right\} & \text{if } 1 \leq s < 2 \\ \log \sum_{i=1}^N (|a_i||c_i|)^{s/2} & \text{if } s \geq 2. \end{cases} \quad (4.2)$$



In the case of triangular matrices, the calculation of Lyapunov-exponents of a self-affine measure  $\mu$  with probability vector  $(p_1, \dots, p_N)$  is much simpler. That is,

$$\chi_\mu^{ss} = \max \left\{ - \sum_{i=1}^N p_i \log |a_i|, - \sum_{i=1}^N p_i \log |c_i| \right\} \text{ and } \chi_\mu^s = \min \left\{ - \sum_{i=1}^N p_i \log |a_i|, - \sum_{i=1}^N p_i \log |c_i| \right\}.$$

**Lemma 4.4.** *The set  $\mathcal{A} = \{A_1, A_2, \dots, A_N\}$  of contracting, non-singular  $2 \times 2$  lower-triangular matrices in the form (4.1), satisfies the dominated splitting if*

$$\text{either } |a_i| > |c_i| \text{ for every } i = 1, \dots, N \text{ or } |a_i| < |c_i| \text{ for every } i = 1, \dots, N.$$

The proof of the lemma is straightforward by Lemma 2.3.

In the case of triangular matrices the study of the dimension of self-affine set can be tracked back to study the dimension of some self-similar measure. In the first case of Lemma 4.4, the projected measure in Theorem 2.7 is a self-similar measure. In the second case, the measure  $\nu_{ss}$  is a self-similar measure. We introduce here a condition, which guarantees according to the recent result of Hochman [13, Theorem 1.1] that the dimension of a self-similar measure is the quotient of the entropy and Lyapunov-exponent.

**Definition 4.5.** *For a self-similar IFS  $\phi = \{g_i(x) = \beta_i x + \gamma_i\}_{i=1}^N$  on the real line let*

$$d(g_{\underline{i}}, g_{\underline{j}}) := \begin{cases} \infty & \text{if } \beta_{\underline{i}} \neq \beta_{\underline{j}} \\ |g_{\underline{i}}(0) - g_{\underline{j}}(0)| & \text{if } \beta_{\underline{i}} = \beta_{\underline{j}}. \end{cases} \text{ and} \\ \Delta_n := \min \left\{ d(g_{\underline{i}}, g_{\underline{j}}) : \underline{i} \neq \underline{j} \in \{1, \dots, N\}^n \right\}.$$

We say that the IFS  $\phi$  satisfies the Hochman-condition if

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \log \Delta_n < +\infty.$$

**Theorem 4.6.** *Let  $\mathcal{A} = \{A_i\}_{i=1}^N$  be a finite set of triangular matrices of type (4.1) and let  $\Phi = \{f_i(\underline{x}) = A_i \underline{x} + (t_i, q_i)\}_{i=1}^N$  be the IFS on the plane.*

- (1)  $|a_i| > |c_i|$  for every  $i = 1, \dots, N$ ,
- (2)  $\Phi$  satisfies the strong separation condition,
- (3) the self-similar IFS  $\phi = \{g_i(x) = a_i x + t_i\}_{i=1}^N$  satisfies the Hochman-condition

then for every  $\mu$  self-affine measure

$$\dim_H \mu = \dim_{\text{Lyap}} \mu. \quad (4.3)$$

Moreover,

$$\dim_H \Lambda = \dim_B \Lambda = \min \{s_1, s_2\}, \quad (4.4)$$

where  $\Lambda$  is the attractor of  $\Phi$  and  $s_1$  and  $s_2$  are the unique solutions of the equations

$$\sum_{i=1}^N |a_i|^{s_1} = 1 \text{ and } \sum_{i=1}^N |a_i| |c_i|^{s_2-1} = 1. \quad (4.5)$$

*Proof.* Let  $\nu = \{p_1, \dots, p_N\}^{\mathbb{N}}$  be an arbitrary Bernoulli-measure on  $\Sigma^+$  and  $\mu$  be the corresponding self-affine measure.

Condition (1) implies by Lemma 4.4 that the set  $\mathcal{A}$  of matrices satisfies the dominated splitting and  $e_{ss}(\mathbf{i})$ , defined in Lemma 2.3, is equal to the subspace parallel to the  $y$ -axis for every  $\mathbf{i} \in \Sigma^+$ .

Hence, the transversal measure  $\mu_1^T \equiv \mu^T$ , defined in (2.4), is a self-similar measure with the IFS  $\phi = \{g_i(x) = a_i x + t_i\}_{i=1}^N$ , namely

$$\mu^T = \sum_{i=1}^N p_i \mu^T \circ g_i^{-1}.$$

By condition (3) and [13, Theorem 1.1],

$$\dim_H \mu^T = \min \left\{ \frac{h_\nu}{\chi_\mu^s}, 1 \right\}.$$

Thus,  $\dim_H \mu^T = \min \{\dim_{\text{Lyap}} \mu, 1\}$ . By condition (2), applying Theorem 2.7 and (2.6) we get (4.3).

To prove (4.4), first let us observe that condition (1) implies that the root of the subadditive pressure (4.2) is the minimum of the solutions of the equations (4.5). Then we get the upper bound by [7, Proposition 5.1]. The lower bound follows by choosing the measure  $\nu$  according to the probability vector  $\{|a_1|^{s_1}, \dots, |a_N|^{s_1}\}$  or  $\{|a_1||c_1|^{s_2-1}, \dots, |a_N||c_N|^{s_2-1}\}$ .  $\square$

Now we turn to the second case of Lemma 4.4.

**Theorem 4.7.** *Let  $\mathcal{A} = \{A_i\}_{i=1}^N$  be a finite set of triangular matrices of type (4.1) and let  $\Phi = \{f_i(\underline{x}) = A_i \underline{x} + (t_i, q_i)\}_{i=1}^N$  be the IFS on the plane. Moreover, let  $\nu$  be a left-shift invariant and ergodic Bernoulli-probability measure on  $\Sigma^+$ , and  $\mu$  is the corresponding self-affine measure. Suppose that*

- (1)  $|a_i| < |c_i|$  for every  $i = 1, \dots, N$ ,
- (2)  $\Phi$  satisfies the strong separation condition,
- (3) the self-similar IFS  $\phi = \left\{ g_i(x) = \frac{a_i}{c_i}x - \frac{b_i}{c_i} \right\}_{i=1}^N$  satisfies the Hochman-condition,
- (4)  $\frac{h_\nu}{\chi_\mu^{ss} - \chi_\mu^s} \geq \min \left\{ 1, \frac{h_\nu}{\chi_\mu^s} \right\}$

then

$$\dim_H \mu = \dim_{\text{Lyap}} \mu. \quad (4.6)$$

Moreover, if  $\mathcal{A}$  satisfies the 1-bunched property, i.e.  $|a_i| \geq |c_i|^2$  then

$$\dim_H \Lambda = \dim_B \Lambda = \min \{s_1, s_2\}, \quad (4.7)$$

where  $\Lambda$  is the attractor of  $\Phi$  and  $s_1$  and  $s_2$  are the unique solutions of the equations

$$\sum_{i=1}^N |c_i|^{s_1} = 1 \text{ and } \sum_{i=1}^N |c_i||a_i|^{s_2-1} = 1. \quad (4.8)$$

We note if  $\dim_{\text{Lyap}} \mu \leq 1$  then condition (4) is basically the 1-bunched property, defined in Theorem 4.1. However, if  $\dim_{\text{Lyap}} \mu > 1$  then condition (4) is much relaxed and holds if  $\dim_{\text{Lyap}} \mu$  is sufficiently large, for example if  $h_\nu/\chi_\mu^{ss} \geq 1$ .

**Lemma 4.8.** *Let  $\mathcal{A} = \{A_i\}_{i=1}^N$  be a finite set of matrices of type (4.1) and let us suppose that  $|a_i| < |c_i|$  for every  $i = 1, \dots, N$ . Then the slopes of strong stable directions, defined in Lemma 2.3, form a self-similar set of IFS  $\phi = \left\{ g_i(x) = \frac{a_i}{c_i}x - \frac{b_i}{c_i} \right\}_{i=1}^N$ . In particular, for every  $\mathbf{i} = (i_0, i_1, \dots) \in \Sigma^+$  the subspace  $e_{ss}(\mathbf{i})$  is parallel to the vector  $v(\mathbf{i}) = (1, \vartheta(\mathbf{i}))^T$ , where*

$$\vartheta(\mathbf{i}) = - \sum_{n=0}^{\infty} \frac{b_{i_n} a_{i_{n-1}} \cdots a_{i_0}}{c_{i_n} c_{i_{n-1}} \cdots c_{i_0}}.$$

*Proof.* By simple algebraic calculations show that

$$A_{i_0}v(\mathbf{i}) = a_{i_0}v(\sigma\mathbf{i}).$$

The statement follows by Lemma 2.4 and Lemma 2.6.  $\square$

An immediate consequence of Lemma 4.8 that for any Bernoulli measure on  $\Sigma^+$

$$\dim_H \nu_{ss} = \dim_H \nu \circ \vartheta^{-1}, \quad (4.9)$$

where  $\nu_{ss}$  is defined in (2.7).

*Proof of Theorem 4.7.* First, let us observe that condition (3) with (4.9) and [13, Theorem 1.1] imply that

$$\dim_H \nu_{ss} = \min \left\{ 1, \frac{h_\nu}{\chi_\mu^{ss} - \chi_\mu^s} \right\}.$$

By Lemma 4.4, condition (1) implies that the IFS  $\Phi$  satisfies the dominated splitting, and together with conditions (2) and (4) by using Theorem 2.8, (4.6) follows.

To prove (4.7), first let us observe that condition (1) implies that the root of the subadditive pressure (4.2) is the minimum of the solutions of the equations (4.8). One can check that the 1-bunched property implies that condition (4) holds for any self-similar measure. Hence, the lower bound follows by choosing the measure  $\nu$  according to the probability vector  $\{|c_1|^{s_1}, \dots, |c_N|^{s_1}\}$  or  $\{|c_1||a_1|^{s_2-1}, \dots, |c_N||a_N|^{s_2-1}\}$ . The upper bound follows by [7, Proposition 5.1].  $\square$

**4.3. An example.** Finally, we consider a concrete family of self-affine sets inspired by the example of Falconer and Miao, [10, Figure 1]. Let  $\Phi_c = \{f_1, \dots, f_6\}$  be a parameterized family of IFSs on the plane given by the functions

$$\begin{aligned} f_1(\underline{x}) &= \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & c \end{bmatrix} \underline{x} + \begin{bmatrix} \frac{1}{3} \\ 0 \end{bmatrix}, & f_2(\underline{x}) &= \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & c \end{bmatrix} \underline{x} + \begin{bmatrix} \frac{1}{3} \\ 1-c \end{bmatrix}, \\ f_3(\underline{x}) &= \begin{bmatrix} \frac{1}{3} & 0 \\ \frac{1}{2}-c & c \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}, & f_4(\underline{x}) &= \begin{bmatrix} \frac{1}{3} & 0 \\ \frac{1}{2}-c & c \end{bmatrix} \underline{x} + \begin{bmatrix} \frac{2}{3} \\ 0 \end{bmatrix}, \\ f_5(\underline{x}) &= \begin{bmatrix} \frac{1}{3} & 0 \\ c-\frac{1}{2} & c \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ \frac{1}{2}-c \end{bmatrix}, & f_6(\underline{x}) &= \begin{bmatrix} \frac{1}{3} & 0 \\ c-\frac{1}{2} & c \end{bmatrix} \underline{x} + \begin{bmatrix} \frac{2}{3} \\ 1-c \end{bmatrix}, \end{aligned}$$

where  $0 < c < 1/2$ . Denote  $\Lambda_c$  the attractor of  $\Phi_c$ , see Figure 2.

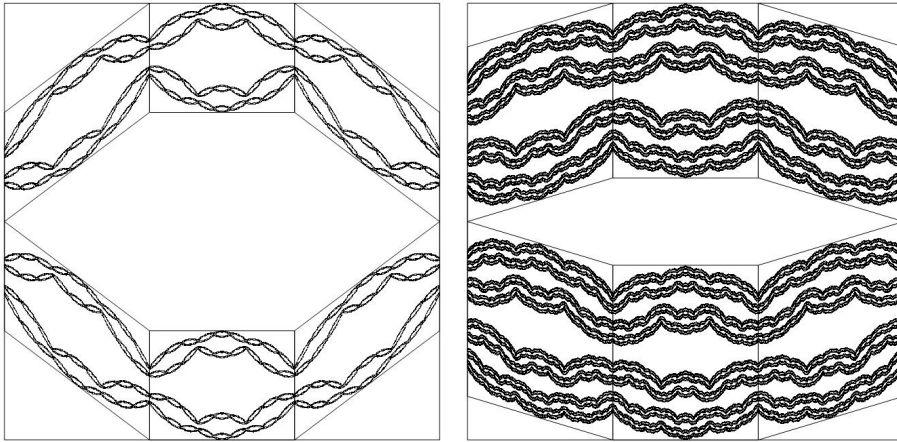


FIGURE 2. The attractors  $\Lambda_c$  of IFSs  $\Phi_c$  with parameters  $c = 0.25$  and  $c = 0.4$ .

**Theorem 4.9.** For every  $0 < c < \frac{1}{3}$

$$\dim_H \Lambda_c = \dim_B \Lambda_c = 1 - \frac{\log 2}{\log c}, \quad (4.10)$$

and there exists a set  $\mathcal{C} \subseteq (\frac{1}{3}, \frac{1}{2})$  such that  $\dim_P \mathcal{C} = 0$  and

$$\dim_H \Lambda_c = \dim_B \Lambda_c = 2 + \frac{\log 2c}{\log 3} \text{ for every } c \in \left(\frac{1}{3}, \frac{1}{2}\right) \setminus \mathcal{C}. \quad (4.11)$$

The box dimension of  $\Lambda_c$  is already known for every  $c \in (0, \frac{1}{2})$  by [8, Corollary 5].

*Proof.* Denote  $\mathcal{S} = \{1, \dots, 6\}$  the set of symbols and let  $\tilde{\mathcal{S}}^n := \mathcal{S}^n \setminus \{4, 6\}^n$ .

Observe that the IFS  $\Phi_c$  satisfies the open set condition but not the strong separation condition. However, the IFS  $\tilde{\Phi}_c^n$  given by  $\tilde{\Phi}_c^n = \{f_i\}_{i \in \tilde{\mathcal{S}}^n}$  satisfies SSC for every  $n \geq 1$  and  $0 < c < \frac{1}{2}$ . Denote the attractor of  $\tilde{\Phi}_c^n$  by  $\tilde{\Lambda}_{n,c}$ . For every  $n \geq 1$  let  $\tilde{\Sigma}_n = (\tilde{\mathcal{S}}^n)^{\mathbb{N}}$  be the symbolic space and  $\nu^{(n)}$  let be the uniform Bernoulli measure on  $\tilde{\Sigma}_n$  and  $\tilde{\mu}_{n,c}$  the corresponding self-affine measure supported on  $\tilde{\Lambda}_{n,c}$ .

First, let us consider the case  $0 < c < \frac{1}{3}$ . Then by Lemma 4.4 the IFS  $\tilde{\Phi}_c^n$  satisfies the dominated splitting and by Lemma 2.3 and Lemma 2.4, the strong stable directions are parallel to the  $y$ -axis. Hence, the transversal measure  $\tilde{\mu}_{n,c}^T$ , defined in (2.4), is a self-similar measure with uniform probabilities, satisfying SSC. Thus,

$$\dim_H \tilde{\mu}_{n,c}^T = \frac{\log(3^n - 1)}{\log 3^n}.$$

Applying Theorem 2.7, we get

$$\begin{aligned} \dim_H \Lambda &\geq \lim_{n \rightarrow \infty} \dim_H \tilde{\Lambda}_{n,c} \geq \\ &\lim_{n \rightarrow \infty} \dim_H \tilde{\mu}_{n,c} = \lim_{n \rightarrow \infty} \frac{\log(6^n - 2^n)}{-\log c^n} + \left(1 - \frac{\log 3^n}{-\log c^n}\right) \frac{\log(3^n - 1)}{\log 3^n} = 1 - \frac{\log 2}{\log c}, \end{aligned}$$

which proves (4.10).

Now we turn to the case  $\frac{1}{3} < c < \frac{1}{2}$ . Lemma 4.4 implies that the IFS  $\Phi$  satisfies again the dominated splitting, moreover, by Lemma 2.3 and Lemma 4.8, the strong stable directions can be given by the IFS  $\phi_c$  of similarities

$$g_1(x) = g_2(x) = \frac{1}{3c}x, \quad g_3(x) = g_4(x) = \frac{1}{3c}x + \frac{2c-1}{2c}, \quad \text{and } g_5(x) = g_6(x) = \frac{1}{3c}x + \frac{1-2c}{2c}.$$

Thus by (4.9), the distribution  $\tilde{\nu}_{n,c}^{ss}$  of strong stable directions of the IFS  $\tilde{\Phi}_c^n$  are given by the IFS  $\tilde{\phi}_c^n = \{g_i\}_{i \in \tilde{\mathcal{S}}^n}$  with the uniform Bernoulli measure on  $\tilde{\Sigma}_n$ . Applying [13, Theorem 1.8] we get that for every  $n \geq 1$  there exists a set  $\mathcal{C}_n$  with  $\dim_P \mathcal{C}_n = 0$  such that

$$\dim_H \tilde{\nu}_{n,c}^{ss} = \min \left\{ 1, \frac{-2^n \frac{2^n-1}{6^n-2^n} \log \frac{2^n-1}{6^n-2^n} - (3^n - 2^n) \frac{1}{3^n-1} \log \frac{1}{3^n-1}}{\log(3c)^n} \right\} \text{ for every } c \in \left(\frac{1}{3}, \frac{1}{2}\right) \setminus \mathcal{C}_n.$$

For sufficiently large  $n$ , we apply Theorem 2.8, and therefore

$$\begin{aligned} \dim_H \Lambda &\geq \lim_{n \rightarrow \infty} \dim_H \tilde{\Lambda}_{n,c} \geq \\ &\lim_{n \rightarrow \infty} \dim_H \tilde{\mu}_{n,c} = \lim_{n \rightarrow \infty} 1 + \frac{\log(6^n - 2^n) - (-\log c^n)}{\log 3^n} = 2 + \frac{\log 2c}{\log 3} \text{ for every } c \in \left(\frac{1}{3}, \frac{1}{2}\right) \setminus \mathcal{C}, \end{aligned}$$

where  $\mathcal{C} = \bigcup_{n=1}^{\infty} \mathcal{C}_n$ , which proves (4.11).  $\square$

5. APPENDIX

**Lemma 5.1** (Marstrand, [20]). *Let  $m$  be a probability measure on  $\mathbb{R}^2$  and let  $\lambda$  be a measure on  $[0, \pi)$  such that  $\dim_H \lambda \geq \min \{1, \dim_H m\}$ . For a  $\theta \in [0, \pi)$  denote  $\text{proj}_\theta$  the orthogonal projection onto the line perpendicular to the vector  $(\sin \theta, \cos \theta)$ . Then for every  $\varepsilon > 0$  there exists a set  $A_\varepsilon \subseteq [0, \pi)$  such that  $\lambda(A_\varepsilon) > 0$  and*

$$\dim_H(\text{proj}_\theta)_*m \geq \min \{1, \dim_H m\} - \varepsilon \text{ for every } \theta \in A_\varepsilon. \quad (5.1)$$

*Proof.* Let us denote  $\min \{1, \dim_H m\}$  by  $s$ . Since  $\dim_H \lambda \geq s$  then using (1.3) and Egorov's Theorem for every  $\varepsilon > 0$  there exists a set  $A_\varepsilon$  and  $C > 0$  such that  $\lambda(A_\varepsilon) > 0$  and for every  $\theta \in A_\varepsilon$  and  $r > 0$

$$\lambda((\theta - r, \theta + r)) \leq Cr^{s-\varepsilon/2}.$$

Moreover, without loss of generality we may assume that  $A_\varepsilon$  is bounded away from 0 and  $\pi$ , i.e. there exists a constant  $c > 0$  s.t.  $\text{dist}(\theta, \{0, \pi\}) > c$  for every  $\theta \in A_\varepsilon$ . Let  $\lambda' := \lambda|_{A_\varepsilon} / \lambda(A_\varepsilon)$  be the restricted and normalized measure. It is easy to see that there exists a constant  $c' > 0$  such that for any interval  $I \subseteq [0, \pi)$

$$\lambda'(I) \leq c'|I|^{s-\varepsilon/2}.$$

We prove that for almost every point w.r.t  $\lambda'$  (5.1) holds. For simplifity we denote  $(\text{proj}_\theta)_*m$  by  $m_\theta$ . By Frostman's Lemma [9, Theorem 4.13] it is enough to show that

$$\mathcal{I} := \iiint \frac{1}{|x - y|^{s-\varepsilon}} dm_\theta(x) dm_\theta(y) d\lambda'(\theta) < +\infty.$$

Applying Fubini's Theorem we have

$$\begin{aligned} \mathcal{I} &= \iiint \frac{1}{|\text{proj}_\theta(\underline{x}) - \text{proj}_\theta(\underline{y})|^{s-\varepsilon}} d\lambda'(\theta) dm(\underline{x}) dm(\underline{y}) = \\ &= \iint \frac{1}{\|\underline{x} - \underline{y}\|^{s-\varepsilon}} \int \frac{1}{\left(\frac{|\text{proj}_\theta(\underline{x}) - \text{proj}_\theta(\underline{y})|}{\|\underline{x} - \underline{y}\|}\right)^{s-\varepsilon}} d\lambda'(\theta) dm(\underline{x}) dm(\underline{y}). \end{aligned}$$

Applying some algebraic manipulation we have for every  $\underline{x} \neq \underline{y} \in \mathbb{R}^2$

$$\int \frac{1}{\left(\frac{|\text{proj}_\theta(\underline{x}) - \text{proj}_\theta(\underline{y})|}{\|\underline{x} - \underline{y}\|}\right)^{s-\varepsilon}} d\lambda'(\theta) \leq 2^{s-\varepsilon} \sum_{n=0}^{\infty} 2^{n(s-\varepsilon)} \lambda' \left( \left\{ \theta : \frac{|\text{proj}_\theta(\underline{x}) - \text{proj}_\theta(\underline{y})|}{\|\underline{x} - \underline{y}\|} \leq \frac{1}{2^n} \right\} \right).$$

Since the set  $\left\{ \theta : \frac{|\text{proj}_\theta(\underline{x}) - \text{proj}_\theta(\underline{y})|}{\|\underline{x} - \underline{y}\|} \leq \frac{1}{2^n} \right\}$  is contained in at most two intervals with length  $c''/2^n$  we get

$$\int \frac{1}{\left(\frac{|\text{proj}_\theta(\underline{x}) - \text{proj}_\theta(\underline{y})|}{\|\underline{x} - \underline{y}\|}\right)^{s-\varepsilon}} d\lambda'(\theta) \leq 2^{s-\varepsilon} \sum_{n=0}^{\infty} 2^{n(s-\varepsilon)} \left(\frac{c''}{2^n}\right)^{s-\varepsilon/2} < +\infty$$

Since  $\iint \frac{1}{\|\underline{x} - \underline{y}\|^{s-\varepsilon}} dm(\underline{x}) dm(\underline{y}) < +\infty$ , (5.1) follows.  $\square$

**Lemma 5.2** (Maker, [19]). *Let  $T : X \mapsto X$  be an endomorphism on  $X \subset \mathbb{R}^d$  compact set and let  $m$  be a  $T$ -invariant ergodic measure. Moreover, let  $h_{n,k} : X \mapsto \mathbb{R}$  be a family of functions s.t.  $\sup_{n,k} h_{n,k} \in L^1(m)$  and  $\lim_{n-k \rightarrow \infty} h_{n,k} = h$  in  $L^1(m)$  sense and  $m$ -almost everywhere, where  $h \in L^1(m)$ . Then*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} h_{n,k}(T^k x) = \int h(x) dm(x) \text{ for } m\text{-a.e } x \in X.$$

The proof is a slight modification of Shannon-McMillan-Breiman Theorem in [3].

*Proof.* Since

$$\frac{1}{n} \sum_{k=0}^{n-1} h_{n,k}(T^k x) = \frac{1}{n} \sum_{k=0}^{n-1} h(T^k x) + \frac{1}{n} \sum_{k=0}^{n-1} (h_{n,k}(T^k x) - h(T^k x))$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} h(T^k x) = \int h(x) dm(x) \text{ for } m\text{-a.e } x \in X,$$

it is enough to show that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} |h_{n,k}(T^k x) - h(T^k x)| = 0 \text{ for } m\text{-a.e } x \in X.$$

Let  $H_N(x) := \sup_{n-k \geq N} |h_{n,k}(x) - h(x)|$ . Then

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} |h_{n,k}(T^k x) - h(T^k x)| \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-N} H_N(T^k x) + \frac{1}{n} \sum_{k=n-N}^{n-1} H_0(T^k x) = \int H_N(x) dm(x),$$

for  $m$ -a.e  $x$ . Since  $H_N \in L^1$  and  $H_N \rightarrow 0$  for  $m$ -almost everywhere as  $N \rightarrow \infty$ , the statement of the theorem follows.  $\square$

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