Prague, 23-25 October 1996.

Bálint Tóth (Budapest):
Reflection positivity, infrared bounds, continuous symmetry breaking

# III.

## III: Quantum Heisenberg models: Proofs.

### Proof of Mermin - Wagner:

$\Lambda \subseteq \mathbb{Z}^d$ finite box, periodic boundary conditions
I don't denote explicitly dependence on $\Lambda$, inequalities will be uniform in $\Lambda$

$$H_\varepsilon = -\frac{1}{2} \sum_{|x-y|=1} \left\{ S^+(x) S^-(y) + u\, S_3(x) S_3(y) \right\} - h \sum_x S_3(x)$$
$$\underbrace{- \frac{\varepsilon}{2} \sum_x \left( S^+(x) + S^-(x) \right)}_{U(1)\text{-symmetry breaking term}}$$

we'll apply Bogoliubov with:

$$\left.\begin{aligned} A &= \hat{S}^+(k) = \sum_{x\in\Lambda} e^{ikx} S^+(x) \\ A^* &= \hat{S}^-(-k) = \sum_{x\in\Lambda} e^{-ikx} S^-(x) \\ C &= \hat{S}_3(k) = \sum_{x\in\Lambda} e^{ikx} S_3(x) \\ C^* &= \hat{S}_3(-k) = \sum_{x\in\Lambda} e^{-ikx} S_3(x) \end{aligned}\right\} \quad k \in \Lambda^* \text{ fixed.}$$

the commutators:

$$\frac{1}{2}\{A, A^*\} = \hat{S}^+(k)\,\hat{S}^-(-k) - \sum_{x\in\Lambda} S_3(x)$$

$$[C^*, A] = \sum_{x\in\Lambda} S^+(x)$$

$$[[C, H_\varepsilon], C^*] = \sum_{x\in\Lambda}\sum_{|e|=1}(1-\cos k\cdot e)\,S^+(x)\,S^-(x+e) + \frac{\varepsilon}{2}\sum_{x\in\Lambda}\left(S^+(x)+S^-(x)\right)$$

Home Work: check these relations! (very instructive)

Some expectations:

$$\hat{C}_\varepsilon(-k) = |\Lambda|^{-1}\langle \hat{S}^+(k)\,\hat{S}^-(-k)\rangle_\varepsilon = \sum_{x\in\Lambda} e^{ikx}\langle S^+(0)\,S^-(x)\rangle_\varepsilon :$$

the correlation function

$m_\varepsilon = \langle S_3(0)\rangle_\varepsilon = \langle S_3(x)\rangle_\varepsilon$ : magnetization in the Z-direction

$\mu_\varepsilon = \langle S^+(x)\rangle_\varepsilon = \langle S^-(x)\rangle_\varepsilon$ : magnetization in the X-direction

$r_\varepsilon^2 = |\Lambda|^{-2}\sum_{x,y\in\Lambda}\langle S^+(x)\,S^-(y)\rangle_\varepsilon = |\Lambda|^{-1}\,\hat{C}_\varepsilon(k=0)$ : LRO parameter

Apply Bogoliubor:

$$\hat{c}_\varepsilon(k) - m_\varepsilon \geq \beta^{-1}(\mu_\varepsilon)^2 \cdot \left\{ \sum_{|e|=1} (1-\cos k\cdot e)\langle S^+(0)S^-(0+e)\rangle_\varepsilon + \varepsilon\mu_\varepsilon \right.$$

$$\langle S^+(0)\,S^-(e)\rangle_\varepsilon \leq \min\{\langle S^+(0)S^-(0)\rangle_\varepsilon, \langle S^-(0)S^+(0)\rangle_\varepsilon\} \leq$$

min $\leq s(s+1)$

$$\hat{c}_\varepsilon(k) - m_\varepsilon \geq \beta^{-1}(\mu_\varepsilon)^2 \{s(s+1)D(k) + \varepsilon\mu_\varepsilon\}^{-1}$$

1, take $|\Lambda|^{-1}\sum_{k\in\Lambda^*\setminus\{0\}} \cdots$ of both sides:

$$\langle S^+(0)S^-(0)\rangle_\varepsilon - r_\varepsilon^2 - m_\varepsilon \geq \cancel{\frac{\mu_\varepsilon^2}{\beta}\cdot\frac{1}{|\Lambda|}\sum_{k\in\Lambda^*\setminus\{0\}}\{s(s+1)D(k)+\varepsilon}$$

$$\geq \frac{\mu_\varepsilon^2}{\beta}\cdot\frac{1}{|\Lambda|}\sum_{k\in\Lambda^*\setminus\{0\}}\{s(s+1)D(k) + \varepsilon\mu_\varepsilon\}^{-1}$$

left hand side $\leq s(s+1) - r_\varepsilon^2 \leq s(s+1)$

, take the thermodynamical limit $\Lambda\uparrow\mathbb{Z}^d$:

$$s(s+1) \geq \overline{\lim_{\Lambda\uparrow\mathbb{Z}^d}} \frac{\mu_\varepsilon^2}{\beta} \int \{s(s+1)D(k) + \varepsilon\mu_\varepsilon\}^{-1} dk$$

$$\geq \beta^{-1}\left\{\overline{\lim_{\Lambda\uparrow\mathbb{Z}^d}}\mu_\varepsilon^2\right\} \int\{s(s+1)D(k) + \varepsilon s\}^{-1} dk$$

3, take the $\varepsilon \downarrow 0$ limit.

$$s(s+1) \geq \beta^{-1} \left\{ \overline{\lim_{\varepsilon \downarrow 0}}\ \overline{\lim_{\Lambda \uparrow \mathbb{Z}^d}}\ \mu_\varepsilon^2 \right\} . \underbrace{\int \frac{1}{s(s+1)\, D(k)}\, dk}_{= \infty \text{ in } d = 1,2}$$

<u>Conclusion</u>: $\left. \begin{array}{ll} \text{in} & d = 1,2 \\ \text{if} & \beta < \infty \end{array} \right\}$ then $\overline{\lim_{\varepsilon \downarrow 0}}\ \overline{\lim_{\Lambda \uparrow \mathbb{Z}^d}}\ \mu_\varepsilon^2 = 0$

<u>no</u> spontaneous symmetry breaking at positive temperature.

(and <u>no</u> off diagonal LRO)

$\blacksquare$ Mermin - Wagner

# Proof of Dyson – Lieb – Simon (with improvements by Kubo-Kishi, Kennedy-Lieb-Shastri)

$$H_\Lambda = -\frac{1}{2} \sum_{|x-y|=1} \{ S_1(x) S_1(y) + S_2(x) S_2(y) + u\, S_3(x) S_3(y) \}$$

$$- h \sum_x S_3(x)$$

LRO - parameter:

$$r^2 = \lim_{\Lambda \uparrow \mathbb{Z}^d} |\Lambda|^{-2} \langle \sum_{x,y \in \Lambda} S_1(x) S_1(y) \rangle =$$

$$= \lim_{\Lambda \uparrow \mathbb{Z}^d} |\Lambda|^{-2} \langle \sum_{x,y \in \Lambda} S_2(x) S_2(y) \rangle$$

Classical reasoning $\boxed{\text{fails}}$ due to quantum ground state fluctuations !!!

let
$$\begin{cases} u = 1 & \text{i.e. isotropic ferromagnet} \\ h = 0 & \text{no transversal field.} \\ s = \tfrac{1}{2} & \text{spin - one - half.} \end{cases}$$

$$c(x) = \langle \vec{S}(0) \cdot \vec{S}(x) \rangle \qquad \hat{c}(k) = \sum_x e^{ikx} c(x).$$

assume the (classical) IRB: $\hat{c}(k) \le \frac{\text{Const}}{\beta} \frac{1}{D(k)}$

then:

$$\left| \langle \vec{S}(0)\cdot\vec{S}(0)\rangle - \langle \vec{S}(0)\cdot\vec{S}(x)\rangle \right| = | C(0) - C(x) | =$$

$$= \left| \sum_{\substack{k\in\Lambda^* \\ k\neq 0}} \left(1-e^{ikx}\right) \hat{C}(k) \right| \leq$$

$$\leq \frac{\text{const}}{\beta} \sum_{\substack{k\in\Lambda^* \\ k\neq 0}} \left|1-e^{ikx}\right| \frac{1}{D(k)} \longrightarrow 0$$

as $\beta \to \infty$, $\Lambda$ fixed.

<u>But</u>: $\langle \vec{S}(0)\cdot\vec{S}(0)\rangle - \langle \vec{S}(0)\vec{S}(x)\rangle =$

for $x \neq 0$

$$\frac{3}{4} - \langle \vec{S}(0)\cdot\vec{S}(x)\rangle \geq \frac{1}{2} \quad \text{in any state}$$

Thus: the classical IRB $\hat{C}(k) \leq \frac{\text{const}}{\beta}\cdot\frac{1}{D(k)}$

<u>can't hold</u>.

Modified – more sophisticated – strategy is needed,

an IRB will be proved (in some cases)

for the <u>Duhamel</u> correlation fct-s instead of

the ordinary correlation functions.

## Sketch :

(A) Backbone of proof / reduction to IRB.

U(2) symmetry: [we do it for isotropic HAF ($u=-1$) and HF ($u=$
with no transversal field $h=0$

this restriction simplifies computations but it is not essential.

(B) Proof of IRB : works for $u \leq 0$ & $h=0$
(i.e. antiferromagnetic coupling and no transversal field)
this restriction is essential

## (A) Backbone of proof / reduction to IRB.

$$H_\Lambda = -\frac{1}{2}\sum_{|x-y|=1} \left\{ S_1(x)S_1(y) + S_2(x)S_2(y) \mp S_3(x)S_3(y) \right\}$$

$-$ for AF , $+$ for F

in the sequel : upper sign for AF
lower sign for F.

correlation fct :

$$(x) = \langle S_1(0)S_1(x)\rangle = \langle S_2(0)S_2(x)\rangle = (\mp 1)^{|x|}\langle S_3(0)S_3(x)\rangle$$

Duhamel correlation fct.:

$$b(x) = \left(S_1(0), S_1(x)\right) = \left(S_2(0), S_2(x)\right) = (\mp 1)^{|x|} \left(S_3(0), S_3(x)\right)$$

Double-commutator correlation fct (Bogoliubov corr. fct.)

$$g(x) = \langle [S_1(0), [H_\Lambda, S_1(x)]] \rangle = \langle [S_2(0), [H_\Lambda, S_2(x)]] \rangle =$$

$$(\mp 1)^{|x|} \langle [S_3(0), [H_\Lambda, S_3(x)]] \rangle$$

$$\hat{c}(k) = \sum_x e^{ikx} c(x) = \frac{1}{|\Lambda|} \langle \hat{S}_1(k) \hat{S}_1(-k) \rangle$$

$$\hat{b}(k) = \sum_x e^{ikx} b(x) = \frac{1}{|\Lambda|} \left(\hat{S}_1(k), \hat{S}_1(k)\right)$$

$$\hat{g}(k) = \sum_x e^{ikx} g(x) = \frac{1}{|\Lambda|} \langle [\hat{S}_1(-k), [H_\Lambda, \hat{S}_1(k)]] \rangle$$

<u>Apply Falk-Bruch</u> with $A = \hat{S}_1(k)$, $A^* = \hat{S}_1(-k)$

in the notation the proof of F-B $\quad c = |\Lambda| \hat{c}(k), \quad b = |\Lambda| \hat{b}(k), \quad g = |\Lambda| \cdot \hat{g}(k)$

<u>F-B</u>:

$$\hat{c}(k) \leq \frac{\beta \cdot \hat{g}(k)}{4} \Phi\left(\frac{4 \hat{b}(k)}{\beta \cdot \hat{g}(k)}\right)$$

these computations would be more complicated without the restrictive assumptions.

2) Compute the double commutators:

$$[S_1(0), [H_\Lambda, S_1(x)]] = \begin{cases} \sum_{|e|=1} \{ S_2(0) S_2(e) \mp S_3(0) S_3(e) \} & x = 0 \\ -\{ S_3(0) S_3(x) \mp S_2(0) S_2(x) \} & |x-0| = 1 \\ 0 & |x-0| > 1 \end{cases}$$

HomeWork: check it! (very instructive!)

assume cubic $\Lambda$ – not essential, but simplifies computations.

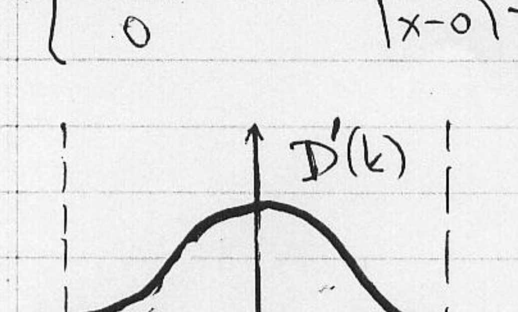
let $t = \langle S_1(0) S_1(e) \rangle = \langle S_2(0) S_2(e) \rangle = \mp \langle S_3(0) S_3(e) \rangle$

$|e| = 1$      this would not hold without th[e] restrictive assumptions

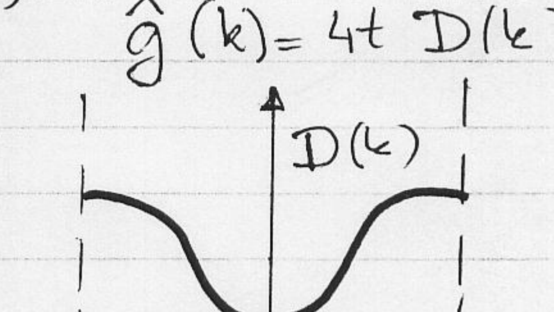
then

$$g(x) = \begin{cases} 4dt & |x-0| = 0 \\ \pm 2t & |x-0| = 1 \\ 0 & |x-0| > 1 \end{cases} ; \qquad \hat{g}(k) = 4t \underbrace{(2d - D(k))}_{D'(k)} \quad \text{A[F]}$$

$$\hat{g}(k) = 4t\, D(k) \quad \text{F}$$




anti ferro




ferro.

③ Infrared bound for the Duhamel correlation fct

$$\hat{b}(k) \leq \frac{1}{2\beta} \frac{1}{D(k)}$$

Proved for $u \leq 0$, $h = 0$ only (essential restric

but expected to hold in general (possibly with

$\frac{K}{2\beta} \cdot \frac{1}{D(k)}$ ) proof postponed

④ Put things together:

$$\hat{c}(k) \underset{F\text{-}B}{\leq} \frac{\beta \hat{g}(k)}{4} \phi\left(\frac{4 \hat{b}(k)}{\beta \hat{g}(k)}\right)$$

$$\underset{IRB}{\leq} \frac{\beta \hat{g}(k)}{4} \phi\left(\frac{2}{\beta^2 D(k) \hat{g}(k)}\right)$$

$$\underset{\text{computation.}}{=} \begin{cases} \beta t D'(k) \, \phi\left(\frac{1}{2\beta^2 t D(k) D'(k)}\right) & \text{AF (proved} \\ \beta t D(k) \, \phi\left(\frac{1}{2\beta^2 t D^2(k)}\right) & \text{F (presum} \end{cases}$$

same in the thermodyn. limit $\Lambda \uparrow \mathbb{Z}^d$

ground state expectations : $\lim_{\beta\to\infty} \langle \cdots \rangle_\beta$

## ⑤ LRO in the ground state AF

(the ground states of the F are trivial)

Remark : as long as $\Lambda$ is finite the HAF in $\Lambda$ has a **unique** g.s. (Lieb & Mattis)

$$\beta \to \infty : \quad \hat{c}(k) \le \sqrt{\frac{t}{2}} \left( \frac{D'(k)}{D(k)} \right)^{1/2}$$

Sa: $t = \quad c(e) = \frac{1}{2d} \cdot \sum_{|e|=1} c(e) = \frac{1}{|\Lambda|} \sum_{k \in \Lambda^*} \left( 1 - \frac{D(k)}{d} \right) \hat{c}(k)$

$$= \underbrace{\frac{1}{|\Lambda|} \hat{c}(k=0)}_{r_\Lambda^2} + \frac{1}{|\Lambda|} \cdot \sum_{\substack{k \in \Lambda^* \\ k \neq 0}} \left( 1 - \frac{D(k)}{d} \right) \hat{c}(k)$$

$\forall k: \ \hat{c}(k) \ge 0$

$$\le r_\Lambda^2 + \frac{1}{|\Lambda|} \sum_{\substack{k \in \Lambda^* \\ k \neq 0}} \left( 1 - \frac{D(k)}{d} \right)_+ \hat{c}(k)$$

$$\le r_\Lambda^2 + \frac{1}{|\Lambda|} \cdot \sum_{\substack{k \in \Lambda^* \\ k \neq 0}} \left( 1 - \frac{D(k)}{d} \right)_+ \cdot \sqrt{\frac{t}{2}} \left( \frac{D'(k)}{D(k)} \right.$$

in the Thermodynamic limit :

$$t \le r^2 + \sqrt{\frac{t}{2}} \underbrace{ ⨍ \left( 1 - \frac{D(k)}{d} \right)_+ \left( \frac{D'(k)}{D(k)} \right)^{1/2} dk}_{I = I(d)}$$

$$r^2 \geq 2\sqrt{\frac{t}{2}}\left(\sqrt{\frac{t}{2}} - \frac{1}{2} I\right) \qquad \circledast$$

5b :

$$\frac{s(s+1)}{3} = \langle S_1(0) S_1(0) \rangle = C(0) = \frac{1}{|\Lambda|} \cdot \sum_{k \in \Lambda^*} \hat{C}(k)$$

$$= \underbrace{\frac{1}{|\Lambda|} \hat{C}(k=0)}_{r_\Lambda^2} + \frac{1}{|\Lambda|} \sum_{\substack{k \in \Lambda^* \\ k \neq 0}} \hat{C}(k)$$

$$\leq r_\Lambda^2 + \frac{1}{|\Lambda|} \cdot \sum_{\substack{k \in \Lambda^* \\ k \neq 0}} \sqrt{\frac{t}{2}} \cdot \left(\frac{D'(k)}{D(k)}\right)^{1/2}$$

in the thermodynamic limit

$$\frac{s(s+1)}{3} \leq r^2 + \sqrt{\frac{t}{2}} \underbrace{\int \left(\frac{D'(k)}{D(k)}\right)^{1/2} dk}_{J = J(d)}$$

$$r^2 \geq \frac{s(s+1)}{3} - \sqrt{\frac{t}{2}} J \qquad \circledast\circledast$$

$\circledast$ & $\circledast\circledast \Longrightarrow r^2 > 0$ guaranteed if $\frac{3}{2} I J < s(s+1)$

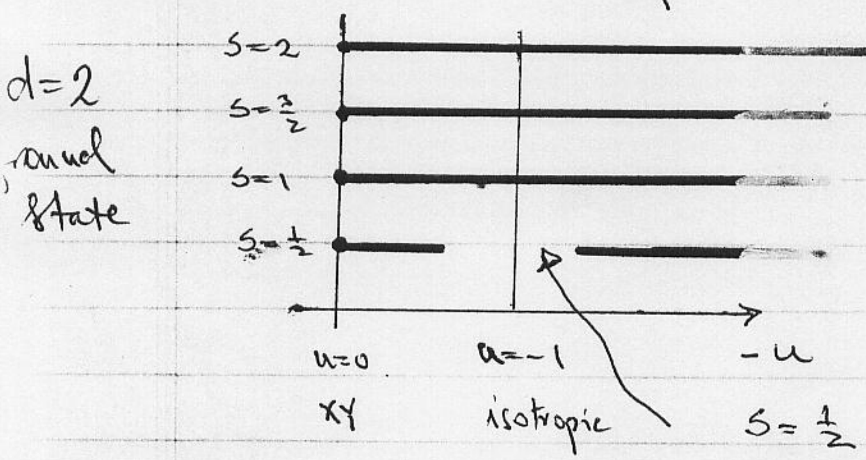
| | $I$ | $J$ | $\frac{3}{2} I \cdot J$ |
|---|---|---|---|
| $d=2$ | 0.690 | 1.443 | $1.494 < 1\,(1+1)$ |
| $d=3$ | 0.350 | 1.157 | $0.607 < \frac{3}{4} = \frac{1}{2}\left(\frac{1}{2}+1\right)$ |

Conclusion:

- in two dimensions the isotropic HAF exhibits Néel order in the ground state (zero temp.) for $s = 1, \frac{3}{2}, 2, \dots$
  $s = \frac{1}{2}$ remains open

- in three (and more) dimensions the isotropic HAF exhibits Néel order in the ground state (zero temperature) for any spin $s = \frac{1}{2}, 1, \frac{3}{2}, \dots$

Remark: the anisotropic HAF is more subtle




—— : existence of g.s. Néel order proved

$u \in (-1.78, -0.13)$
open
Kubo-Kishi

in $d \geq 3$ the g.s. exhibits Néel order for

$$\forall s = \tfrac{1}{2}, 1, \tfrac{3}{2}, \dots \quad \text{and} \quad \forall u \leq 0 .$$

$\square$ ground state

## ⑥ Stability of the ground state LRO for low (but positive) temperatures, in $d \geq 3$: a dominated convergence argument

$$\Gamma_\beta^2 = \frac{s(s+1)}{3} - \int \hat{C}_\beta(k)\, dk$$

$$\hat{C}_\beta(k) \longrightarrow \hat{C}_\infty(k) \quad \text{as } \beta \nearrow \infty$$

$$\hat{C}_\beta(k) \leq \begin{cases} \beta t\, D'(k)\, \phi\left(\dfrac{1}{2\beta^2 t\, D(k)\, D'(k)}\right) & \underset{(\text{proved})}{AF} \\ \beta t\, D(k)\, \phi\left(\dfrac{1}{2\beta^2 t\, D^2(k)}\right) & \underset{(\text{presumed})}{\underline{\underline{F}}} \end{cases}$$

$$\sup_{\beta \geq 1} \beta \cdot x\, \phi\left(\frac{1}{\beta^2 y}\right) \leq a \frac{x}{\sqrt{y}} + b \frac{x}{y}$$

use $\phi(y) \leq a\sqrt{y} + b \cdot y$.

hence

$$\hat{C}_\beta(k) \leq \left\{ \begin{array}{l} \underbrace{a\sqrt{\frac{t}{2}}}_{\leq a'} \sqrt{\dfrac{D'(k)}{D(k)}} + \dfrac{b}{2} \cdot \dfrac{1}{D(k)} \quad \text{proved} \\ \underbrace{a\sqrt{\frac{t}{2}}}_{\leq a'} + \dfrac{b}{2} \cdot \dfrac{1}{D(k)} \quad \text{presumed} \end{array} \right\}$$

dominating fct integrable in $d \geq 3$

$\Rightarrow$ $\Gamma_\beta^2 \longrightarrow \Gamma_\infty^2$ in $d \geq 3$ as $\beta \to \infty$

## Conclusion:

in $d \geq 3$

$$\Gamma_\beta^2 \longrightarrow \Gamma_{g.s.}^2 \quad \text{as } \beta \to \infty$$

$\square$ DLS proved modulo IRB. $\square$

## B) Proof of IRB:

$$H_\Lambda = -\frac{1}{2} \sum_{|x-y|=1} \{ S_1(x) S_1(y) + S_2(x) S_2(y) + u \cdot S_3(x) S_3(y) \} - h \sum_x S_3(x)$$

$$= \frac{1}{4} \sum_{|x-y|=1} \{ (S_1(x) - S_1(y))^2 + (S_2(x) - S_2(y))^2 + u\,(S_3(x) - S_3(y))^2 \}$$

$$- \sum_x \{ h\, S_3(x) + d\, (S_1^2(x) + S_2^2(x) + u\, S_3^2(x)) \}$$

$v : \Lambda \to \mathbb{R}$ a "scalar field" on $\Lambda$

$$H_\Lambda(\underline{v}) = \frac{1}{4} \sum_{|x-y|=1} \{ (S_1(x) + v(x) - S_1(y) - v(y))^2 + (S_2(x) - S_2(y))^2$$

$$+ u\,(S_3(x) - S_3(y))^2 \}$$

$$- \sum_x \{ h\, S_3(x) + d\, (S_1^2(x) + S_2^2(x) + u\, S_3^2(x)) \}$$

$$\left( HF : \frac{\partial^2}{\partial \lambda \partial \mu} \lg \operatorname{Tr} \left( \exp(-\beta H + \lambda A^* + \mu B) \right) \right) \Big|_{\lambda=\mu=0} = ?$$

$$Z_\Lambda(\underline{v}) = \operatorname{Tr}\left(\exp - \beta H_\Lambda(\underline{v})\right)$$

$$Z_\Lambda(\underline{0}) = Z_\Lambda .$$

Simple computation yields:

Λ gráfon, periodikus peremfelt.

$$\Delta_{x,z_1} = \begin{cases} -2d & |x-z_1| = 0 \\ 1 & |x-z_1| = 1 \\ 0 & |x-z_1| > 1 \end{cases}$$

$$\frac{\partial^2 Z_\Lambda}{\partial v(x)\, \partial v(y)} \Bigg|_{\underline{v}=0} = \sum_{z_1, z_2 \in \Lambda} \Delta_{x,z_1}\, b(z_1 - z_2)\, \Delta_{z_2 y} + \frac{1}{\beta} \Delta_{x,y}$$

Similar to the classical case, <u>but</u> instead of the ordinary correlation fct the Duhamel two-point fct appears in the formula

<u>Theorem</u>: If $\boxed{u \leq 0 \text{ and } h = 0}$ then

① $(\forall \underline{v})$ : $Z_\Lambda(\underline{v}) \leq Z_\Lambda(0)$ (**G**aussian **D**omination)

② $$-\frac{\partial^2 Z_\Lambda}{\partial v(x)\, \partial v(y)} \Bigg|_{\underline{v}=0}$$ is positive definite (as matrix indexed by $x, y \in \Lambda$)

②' $\forall p \in \Lambda^*$ : $\hat{b}(p) \leq \frac{1}{2\beta} \frac{1}{D(p)}$ **(IRB)**

xpansion ① $\Longrightarrow$ ② $\Longleftrightarrow$ ②' Fourier we prove ①:

## Fundamental Lemma (Reflection Positivity) **RP**

Let $m \in \mathbb{N}$ and

$I$, $A$, $B$, $C_k, D_k$ $k = 1, 2, \dots, \ell$ $m \times m$ complex matrices

$\uparrow$ identity.

then:

$$\left| \mathrm{Tr} \exp\left\{ A \otimes I + I \otimes B - \frac{1}{2} \sum_{k=1}^{\ell} \left( C_k \otimes I - I \otimes D_k \right)^2 \right\} \right|^2 \leq$$

$$\mathrm{Tr}\left( \exp\left\{ A \otimes I + I \otimes \bar{A} - \frac{1}{2} \sum_{k=1}^{\ell} \left( C_k \otimes I - I \otimes \bar{C}_k \right)^2 \right\} \right) \cdot$$

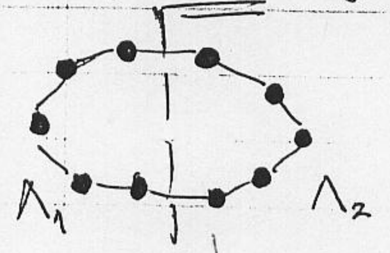
$$\mathrm{Tr}\left( \exp\left\{ \bar{B} \otimes I + I \otimes B - \frac{1}{2} \sum_{k=1}^{\ell} \left( \bar{D}_k \otimes I - I \otimes D_k \right)^2 \right\} \right)$$

where $\bar{A}, \dots$ is the complex conjugate (not the adjoint !!!) of $A, \dots$

proof postponed.

**Proof of Gaussian Domination**: very similar to FSS proo[f]

$\Lambda$ has <u>even</u> side-lengths




$\Lambda = \Lambda_1 \cup \Lambda_2$, $\mathcal{H}_\Lambda = \mathcal{H}_{\Lambda_1} \otimes \mathcal{H}_{\Lambda_2}$ with natural identification of reflected sites.

egy d-dimenziós tórusz, kettévágtam

try to apply RP with:

$$A = -\frac{\beta}{2} \sum_{\substack{x,y \in \Lambda_1 \\ |x-y|=1}} \left\{ \left( S_1(x) + v(x) - S_1(y) - v(y) \right)^2 + \left( S_2(x) - S_2(y) \right)^2 + u \left( S_3(x) - S_3(y) \right)^2 \right\}$$

$$- \sum_{x \in \Lambda_1} \left\{ h\, S_3(x) + d \left( S_1^2(x) + S_2^2(x) + u\, S_3^2(x) \right) \right\}$$

$B$ = same with $\Lambda_2$ instead of $\Lambda_1$

$k = (\langle x,y \rangle, \alpha)$ $\quad \langle x,y \rangle$ on the boundary, $\alpha = 1,2,3$

$$C_k = \sqrt{\tfrac{\beta}{2}} \left( S_1(x) + v(x) \right) \qquad D_k = \sqrt{\tfrac{\beta}{2}} \left( S_1(y) + v(y) \right)$$

$$\sqrt{\tfrac{\beta}{2}}\, S_2(x) \qquad \sqrt{\tfrac{\beta}{2}}\, S_2(y)$$

$$\sqrt{\tfrac{\beta}{2}} \sqrt{u}\, S_3(x) \qquad \sqrt{\tfrac{\beta}{2}} \sqrt{u}\, S_3(y)$$

RP Lemma can be applied iff all these operators have <u>jointly real representation</u>

But $(S_1, S_2, S_3)$ have (real, real, imaginary) representations (or mixed) only !!!

$\Longrightarrow$ <u>$h = 0$ and $u \leq 0$ is imposed.</u>

The rest of the proof is essentially identical to the classical case, FSS.

$\square$ GD

<u>Comments:</u>

the RP method is algebraic, and, consequently stiff:

- the geometry (torus with even side lengths) is essential
- transversal magnetic field excluded
- ferromagnetic interaction excluded

These are not physically motivated restrictions

Main open problem in this field:

IRB without RP.

## Proof of RP Lemma:

Trotter's formula: $\exp(A+B) = \lim_{n\to\infty}\left(\exp\frac{A}{n}\cdot\exp\frac{B}{n}\right)^n$

assume $l=1$ ($l>1$ essentially the same, with more complicated formulas)

$\mathrm{Tr}$ = trace on $\mathbb{C}^n\otimes\mathbb{C}^m$

$\mathrm{tr}$ - trace on $\mathbb{C}^m$

$$\left|\mathrm{Tr}\left(\exp\left\{A\otimes I + I\otimes B - \frac{1}{2}(C\otimes I - I\otimes D)^2\right\}\right)\right|^2 \overset{①}{=}$$

$$\lim_{n\to\infty}\left|\mathrm{Tr}\left(\exp\left\{\frac{1}{n}A\otimes I\right\}\exp\left\{\frac{1}{n}I\otimes B\right\}\exp\left\{-\frac{1}{2n}(C\otimes I - I\otimes D)^2\right\}\right)^n\right|^2 \overset{②}{=}$$

$$\lim_{n\to\infty}\left|\int_{-\infty}^{\infty}\cdots\int\frac{d\xi_1}{\sqrt{2\pi}}\cdots\frac{d\xi_n}{\sqrt{2\pi}}\,e^{-\frac{\xi_1^2}{2}}\cdots e^{-\frac{\xi_n^2}{2}}\times\right.$$

$$\left.\mathrm{Tr}\prod_{r=1}^{n}\left(\exp\left\{\frac{1}{n}A\otimes I\right\}\cdot\exp\frac{1}{n}\{I\otimes B\}\exp\left\{\frac{i\xi_r}{\sqrt{n}}(C\otimes I - I\otimes D)\right\}\right)\right|^2 \overset{③}{=}$$

$$\lim_{n\to\infty}\left|\int_{-\infty}^{\infty}\cdots\int\frac{d\xi_1}{\sqrt{2\pi}}\cdots\frac{d\xi_n}{\sqrt{2\pi}}\,e^{-\frac{\xi_1^2}{2}}\cdots e^{-\frac{\xi_n^2}{2}}\times\right.$$

$$\left.\mathrm{tr}\prod_{r=1}^{n}\left(\exp\left\{\frac{1}{n}A\right\}\cdot\exp\left\{\frac{i\xi_r}{\sqrt{n}}C\right\}\right)\mathrm{tr}\prod_{r=1}^{n}\left(\exp\left\{\frac{1}{n}B\right\}\exp\left\{-\frac{i\xi_r}{\sqrt{n}}D\right\}\right)\right|$$

$$\overset{④}{\leq}$$

Schwartz - egyenlőtlenség



$$\overset{④}{\leq} \lim_{n\to\infty} \int_{-\infty}^{\infty}\cdots\int \frac{d\xi_1}{\sqrt{2\pi}} \cdots \frac{d\xi_n}{\sqrt{2\pi}} e^{-\frac{\xi_1^2}{2}} \cdots e^{-\frac{\xi_n^2}{2}} \times$$

$$tr \prod_{r=1}^{n} \left( \exp\left\{\frac{1}{n} A\right\} \exp\left\{\frac{i\xi_r}{\sqrt{n}} C\right\}\right) tr \prod_{i=1}^{n} \exp\left\{\frac{1}{n}\bar{A}\right\} \exp\left\{-\frac{i\xi_r}{\sqrt{n}}\bar{C}\right\}\Big)$$

$$\int_{-\infty}^{\infty}\cdots\int \frac{d\xi_1}{\sqrt{2\pi}} \cdots \frac{d\xi_n}{\sqrt{2\pi}} e^{-\frac{\xi_1^2}{2}} \cdots e^{-\frac{\xi_n^2}{2}}$$

$$tr \prod_{r=1}^{n} \left( \exp\left\{\frac{1}{n} \bar{B}\right\} \exp\left\{-\frac{i\xi_r}{\sqrt{n}} \bar{D}\right\}\right) tr\left( \prod_{r=1}^{n} \exp\left\{\frac{1}{n} B\right\} \exp\left\{\frac{i\xi_r}{\sqrt{n}} D\right\}\right)$$

$$\overset{\tilde{3}}{=} \cdots \overset{\tilde{2}}{=} \cdots \overset{\tilde{1}}{=}$$

$$\mathrm{Tr} \exp\left\{ A\otimes I + I\otimes \bar{A} - \frac{1}{2}\left(C\otimes I - I\otimes \bar{C}\right)^2 \right\} \cdot$$

$$\mathrm{Tr} \exp\left\{ \bar{B}\otimes I + I\otimes B - \frac{1}{2}\left(\bar{D}\otimes I - I\otimes D\right)^2 \right\}$$

①: Trotter ; ②: $\exp\{-\frac{1}{2}M^2\} = \int_{-\infty}^{\infty} \frac{d\xi}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} \exp\{i\xi M\}$ ;

③: $\mathrm{Tr}\left(M\otimes N\right) = tr M \cdot tr N$ , ④: Schwarz

<u>Remark</u> We can't put $M^*$ instead of $\bar{M}$: the order of the product in Trotter would change $\rightleftarrows$. Non-commutative effect.

□ RP