Prague, 23-25 October 1996

Bálint Tóth (Budapest):

Reflection Positivity, Infrared Bounds, Continuous symmetry breaking

II.

II. Quantum Heisenberg models: intro. & results
Quantum correlation inequalities.

General framework of quantum statistical physics:

$\mathcal{H}$: separable Hilbert space (over $\mathbb{C}$)
H : self-adjoint lin. op. "the Hamiltonian"
$e^{-\beta H}$ trace class
physical observables: self adjoint (bdd) operators

$$Z(\beta) = tr\left(\exp\{-\beta H\}\right), \quad \langle A \rangle = \left(Z(\beta)\right)^{-1} tr\left(A \exp\{-\beta H\}\right)$$

in our concrete case $\mathcal{H}_\Lambda$ will be finite dimensional (as long as $\Lambda \subset \mathbb{Z}^d$ is finite) $\Rightarrow$ no problem with boundedness etc.

SU(2), su(2) & representations:

$SU(2) = \{ 2\times 2$ unitary matrices with $\det = 1 \}$
Covering group of $SO(3)$

$su(2) = \{ 2\times 2$ self-adjoint matrices with $tr = 0 \}$
tangent Lie-alg of $SU(2)$

" $SU(2) = \exp\{i\, su(2)\}$ "

spin representations of $su(2)$: $2s+1$ – dim. $s = \frac{1}{2}, 1, \frac{3}{2}, \dots$

3 generators: S_1, S_2, S_3 $\quad (S_i^* = S_i)$
determined (uniquely – up to unitary equivalence) by the commutation relations:

$$\begin{cases} [S_\alpha, S_\beta] = i\,\epsilon_{\alpha\beta\gamma} S_\gamma \\ S_1^2 + S_2^2 + S_3^2 = s(s+1)\,I \end{cases}$$

alternatively
3 generators S_+, S_-, S_3 $\quad (S_- = S_+^*, S_3 =$

$$\begin{cases} [S_3, S_\pm] = \pm S_\pm \quad , \quad [S_+, S_-] = 2S_3 \\ S_-S_+ + S_3(I + S_3) = s(s+1)\,I \end{cases}$$

related by: $S_\pm = S_1 \pm i S_2$, $S_1 = \frac{S_+ + S_-}{2}$, $S_2 = \frac{S_+ - S}{2i}$

examples:
$s = \frac{1}{2}$ fundamental representation

$S_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; $S_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$; $S_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ Pauli matrices

$s = 1$

$S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$; $S_2 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$; $S_3 = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\vec{v} \in \mathbb{R}^3 \qquad \vec{v} = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}\ , \quad S_{\vec{v}} = v_1 S_1 + v_2 S_2 + v_3 S_3$

$$[S_{\vec{u}}, S_{\vec{v}}] = i\, S_{\vec{u} \times \vec{v}}$$

Consequence : $e^{i S_{\vec{v}}}\, S_{\vec{u}}\, e^{-i S_{\vec{v}}} = S_{(\Omega_{\vec{v}} \vec{u})}$

<u>The quantum Heisenberg model:</u> spin s $(= \frac{1}{2}, 1, \frac{3}{2}, \cdots)$ fixe

$$\Lambda \subset \mathbb{Z}^d \qquad (+ \text{periodic boundary conditions})$$

<u>even length</u>.

$$\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathbb{C}^{2s+1}$$

notation : A operator over $\mathbb{C}^{2s+1}$: $A(x) = I \otimes \cdots \otimes I \otimes A \otimes I \otimes \cdots \otimes$

at site <u>x</u>.

the Hamiltonian :

$$H_\Lambda = -\frac{1}{2} \sum_{\substack{x,y \in \Lambda \\ |x-y|=1}} \left\{ S_1(x) S_1(y) + S_2(x) S_2(y) + u\, S_3(x) S_3(y) \right\}$$

$$- h \sum_{x \in \Lambda} S_3(x)$$

"XX~~Z~~ - model"

$u > 0$ ferromagnetic , $u = +1$ isotropic

$u < 0$ antiferromagnetic ; $u = -1$ isotropic

$u = 0$ XY - model .

<u>important remark</u> : ferro and antiferro are not equivalent, even if $h = 0$

Internal symmetries: G group of symmetries

$$G \ni g \mapsto U_g \quad \text{unitary repr. over } \mathcal{H}.$$

such that: $(\forall g \in G)$: $U_g H U_g^* = H$.

<u>Symmetries of the Heisenberg model:</u>

$|u| \neq 1$: $SU(1)$ generated by $\mathcal{S}_3 = \sum_x S_3(x)$

$$\left[\sum_x S_3(x),\ H\right] = 0 \qquad \text{continuous}$$

$|u| = 1,\ h = 0$: $SU(2)$ generated by $\mathcal{S}_i = \sum_x S_i(x)$ $i=1,$

$$\left[\sum_x S_i(x), H\right] = 0 \qquad \text{continuous}$$

$|u| \neq 1,\ h = 0$: $R = \exp \frac{i\pi}{2} \sum_x S_1(x)$ $\qquad \mathbb{Z}_2$

$$R H R^* = H \qquad \text{discrete}$$

altogether:
$h \neq 0$: $SU(1)$
$h = 0,\ |u| \neq 1$: $SU(1) \rtimes \mathbb{Z}_2$
$h = 0,\ |u| = 1$: $SU(2)$

We are interested in the <u>breakdown of continuous symmetry</u>, LRO in the 1-2 (X-Y) plane

$\left\langle \frac{\sum_x S_i(x)}{\Lambda} \right\rangle_\Lambda = 0$ by symmetry $\qquad$ <u>Question:</u> $r^2 = \lim_{|\Lambda| \uparrow \mathbb{Z}^d} \left\langle \left(\frac{\sum S_i(x)}{\Lambda} \right)^2 \right\rangle \overset{=}{>} 0$

Results:

① Mermin-Wagner (1966): $d = 1,2;\ T > 0 \Rightarrow r^2 = 0$

no long range order at positive temperature in $d = 1,2$

2) Dyson-Lieb-Simon, (1978) improvements: Kennedy-Lieb-Shastry
Kubo-Kishi (1988)

$d \geq 3$, $h = 0$, $u \leq 0$ $\forall\ s \geq \frac{1}{2}$

$(\exists\, T_c > 0)$: $T < T_c \Longrightarrow m^2 > 0$

3) ground state in $d = 2$ (ferro - trivial)

$d = 2$, $h = 0$, $u \leq 0$ $s \geq 1$
at $T = 0$ $r^2 > 0$.

4) breakdown of discrete symmetry
Kennedy (1985): $u > 1$ at suff low temperature behaves as classical Ising.

Open
1) $d = 2$, $T > 0$: Kosterlitz-Thouless phase
2) $d \geq 3$; $T > 0$: $u \leq 0$ $h \neq 0$ (transversal field)
$u \geq 0$ (in partic. $u = 1$), $h =$
3) $d = 2$, $h = 0$, $u \leq 0$; $\underbrace{s = \frac{1}{2}}_{T=0}$ (in partic. $u = -1$
4) $d = 1$, $h = 0$, $T = 0$: the Haldane phase.

Quantum correlation inequalities:

(Bogoliubov, Roepstorff, Falk-Bruch)

$\mathcal{H}$ Hilbert space, H self-adjoint Hamiltonian (such that $e^{-\beta H}$ is trace class)

$A, B, C, \ldots$ bdd. linear operators

$$Z = Z(\beta) = tr\left(e^{-\beta H}\right)$$

$$\langle A \rangle = Z^{-1}\, tr\left(A e^{-\beta H}\right) \qquad \text{"average"}$$

$$(A, B) = Z^{-1} \int_0^1 ds\, tr\left(e^{-\beta(1-s)H} A^* e^{-\beta s H} B\right)$$

"Duhamel two-point function"

(A, B) is a scalar product on $\mathcal{B}(\mathcal{H})$

$(A^*, B^*) = (B, A) = \overline{(A, B)}$: compatible with the complex structure of $\mathcal{B}(\mathcal{H})$

$$\langle A \rangle = \frac{1}{Z} \frac{\partial}{\partial \lambda} \mathrm{Tr}\left(\exp\{-\beta H + \lambda A\}\right)\Big|_{\lambda=0}$$

$$(A, B) = \frac{1}{Z} \frac{\partial^2}{\partial \lambda\, \partial \mu} \mathrm{Tr}\left(\exp\{-\beta H + \lambda A^* + \mu B\}\right)\Big|_{\lambda=\mu=0}$$

(prove with Trotter's formula)

Two identities:

1) $$\frac{d}{ds} \mathrm{Tr}\left(e^{-\beta(1-s)H} \; A^* \; e^{-\beta s H} B\right) =$$

$$\beta \, \mathrm{Tr}\left(e^{-\beta(1-s)H} \, [H, A^*] \, e^{-\beta s H} B\right) =$$

$$\beta \, \mathrm{Tr}\left(e^{-\beta(1-s)H} \, A^* e^{-\beta s H} \, [B, H]\right)$$

2) $$\frac{d^2}{ds^2} \mathrm{Tr}\left(e^{-\beta(1-s)H} \; A^* \, e^{-\beta s H} B\right) =$$

$$\beta^2 \, \mathrm{Tr}\left(e^{-\beta(1-s)H} \, [H, A^*] \, e^{-\beta s H} \, [B, H]\right)$$

fix a bdd. op. A, we prove inequalities for (A, A).

~~Co~~ consider the function $k : [0,1] \to \mathbb{R}_+$

$$k(s) = Z^{-1} \cdot \mathrm{Tr}\left(e^{-\beta(1-s)H} A^* e^{-\beta s H} A\right).$$

<u>Bogolinbov's inequality:</u>

$k''(s) \geq 0 \implies k$ is convex

$$\left\langle \frac{A^*A + AA^*}{2} \right\rangle = \frac{1}{2}\left(k(0) + k(1)\right) \geq \int_0^1 k(s)\,ds = (A, A)$$

by def. of k — convexity of k — by def of k

(A, B) is a scalar product, so ... (next page)

$$(A,A) = \sup_{B\neq 0} \frac{|(A,B)|^2}{(B,B)} \geq \sup_{C:[C,H]\neq 0} \frac{|(A,[C,H])|^2}{([C,H],[C,H])}$$

$$(A,[C,H]) = Z^{-1}\int_0^1 \mathrm{Tr}\left(e^{-\beta(1-s)H} A^* e^{-\beta s H} [C,H]\right) ds$$

use identity ① $\longrightarrow$ $$= Z^{-1}\int_0^1 \frac{1}{\beta}\cdot\frac{d}{ds} \mathrm{Tr}\left(e^{-\beta(1-s)H} A^* e^{-\beta s H} C\right)$$

$$= \frac{1}{\beta}\langle [C,A^*]\rangle = \frac{1}{\beta}\overline{\langle [A,C^*]\rangle}$$

similarly:

$$([C,H],[C,H]) = \frac{1}{\beta}\langle [C,[C,H]^*]\rangle =$$

$$\frac{1}{\beta}\langle [[C,H],C^*]\rangle$$

altogether:

$$\left\langle \frac{A^*A+AA^*}{2}\right\rangle \geq \sup_{C:[C,H]\neq 0} \frac{|\langle [A,C^*]\rangle|^2}{\beta\langle [[C,H],C^*]\rangle}$$

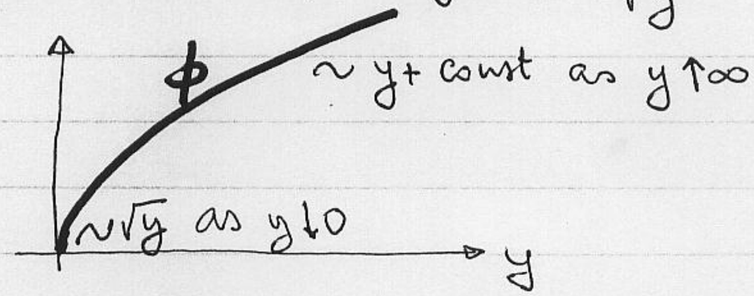
Bogoliubov

<u>Remark</u>: actually k is log-convex, since $k''k - (k')^2 \geq 0$. From this a stronger inequality (Roepstorff's ineq.) is obtained. Bogoliubov is suff. for our purposes.

Falk - Bruch / Dyson - Lieb - Simon inequality :

A a bdd op. (fixed)

$\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+ \qquad \phi(y) = \sqrt{y} \coth \frac{1}{\sqrt{y}}$



$$\phi(y) = \sqrt{y}\left(1 + O\left(\exp - \tfrac{1}{\sqrt{y}}\right)\right) \quad \text{as } y \downarrow 0$$

$$\phi(y) = y + \tfrac{1}{3} + O\left(\tfrac{1}{y}\right) \qquad \text{as } y \uparrow \infty$$

ϕ is concave

$$\frac{1}{2}\langle \{A^*, A\}\rangle \leq \beta \frac{\langle [A^*,[H,A]]\rangle}{4} \cdot \phi\left(\frac{4\,(A,A)}{\beta \langle [A^*,[H,A]]\rangle}\right)$$

Falk - Bruch / Dyson - Lieb - Simon

Proof: (assume pure point spectrum of H, for simplicity)

$= \frac{1}{2} \langle \{A^*, A\}\rangle = \frac{1}{2}\left(k(0) + k(1)\right)$

$= (A, A) = \int_0^1 k(s)\, ds$

$g = \beta \langle [A^*, [H, A]]\rangle = k'(1) - k'(0)$

$$k(s) = Z^{-1} \sum_{m,n} |A_{m,n}|^2 e^{-\beta \epsilon_n} e^{\beta(\epsilon_n - \epsilon_m)s}$$

$$= \int_{\mathbb{R}} e^{st}\, d\mu(t)$$ where μ is a positive measure on $\mathbb{R}$!

$$c = \frac{1}{2}\Big(k(0) + k(1)\Big) = \int_{\mathbb{R}} \frac{1+e^t}{2}\, d\mu(t)$$

$$b = \int_0^1 k(s)\, ds = \int_{\mathbb{R}} \frac{e^t - 1}{t}\, d\mu(t)$$

$$\beta \cdot g = k'(1) - k'(0) = \int_{\mathbb{R}} t\,(e^t - 1)\, d\mu(t)$$

define $$d\nu(t) = \left[\int_{\mathbb{R}} t\,(e^t-1)\, d\mu(t)\right]^{-1} t\,(e^t - 1)\, d\mu(t)$$

this is a probability measure on $\mathbb{R}$

$$\frac{4c}{\beta g} = \int_{\mathbb{R}} \frac{2}{t} \coth \frac{t}{2}\, d\nu(t)$$

$$\frac{4b}{\beta g} = \int_{\mathbb{R}} \frac{4}{t^2}\, d\nu(t)$$

Jensen

$$\phi\left(\frac{4b}{\beta g}\right) = \phi\left(\int_{\mathbb{R}} \frac{4}{t^2}\, d\nu(t)\right) \geq \int_{\mathbb{R}} \phi\left(\frac{4}{t^2}\right) d\nu(t) =$$

$$\int_{\mathbb{R}} \frac{2}{t} \coth\frac{t}{2}\, d\nu(t) = \frac{4c}{\beta g}$$

Q.E.D. (Falk-Bruch

<u>Remark</u>: the inequality is sharp!

saturated by $[A, A^*] = I$, $H = A^*A$

(the Harmonic oscillator)