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Reflection positivity, infrared bounds, Continuous symmetry breaking — I.

I. Classical Heisenberg models

Ising: $\sigma \in \{-1, 1\} = S^0$, $\underline{\sigma} = \{\sigma(x)\}_{x \in \mathbb{Z}^d}$

$$H^{(\varepsilon)}(\underline{\sigma}) = \underbrace{-\frac{J}{2} \sum_{\langle xy \rangle} \sigma(x)\sigma(y)}_{\substack{\mathbb{Z}_2 \text{ symmetry} \\ \text{(discrete!)}}} \underbrace{\left[- \varepsilon \sum_x \sigma(x) \right]}_{\text{symmetry breaking term}}$$

Gibbs measure in finite box $\Lambda \subseteq \mathbb{Z}^d$, with periodic boundary conditions

$$\mu_\Lambda^{(\varepsilon)}(\underline{\sigma}) = \left(Z_\Lambda^{(\varepsilon)} \right)^{-1} \exp - \beta H_\Lambda^{(\varepsilon)}(\underline{\sigma})$$

phase transition = breakdown of internal $\mathbb{Z}_2$ symmetry:

$d \geq 2$: $\exists\, \beta_c$: $\beta > \beta_c$

$$\lim_{\varepsilon \downarrow 0} \lim_{\Lambda \uparrow \mathbb{Z}^d} \langle \sigma(0) \rangle_\Lambda^{(\varepsilon)} = m > 0$$

$$\lim_{\varepsilon \uparrow 0} \lim_{\Lambda \uparrow \mathbb{Z}^d} \langle \sigma(0) \rangle_\Lambda^{(\varepsilon)} = -m < 0$$

beware: the order of limits is essential: $\forall \Lambda$ finite $\lim_{\varepsilon \to 0} \langle \sigma(0) \rangle_\Lambda^{(\varepsilon)} = 0$

alternative formulation:

$$r^2 = \lim_{\Lambda \uparrow \mathbb{Z}^d} \left\langle \left(\frac{\sum_{x \in \Lambda} \sigma(x)}{|\Lambda|} \right)^2 \right\rangle_\Lambda$$ $\varepsilon = 0$ <u>not</u> denoted

translation invar, periodic b.c. $$= \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \cdot \sum_{x \in \Lambda} \langle \sigma(0)\, \sigma(x) \rangle_\Lambda$$

$\beta < \beta_c$: $r^2 = 0$ normal fluctuations of $\sum_{x \in \Lambda} \sigma(x)$

$\beta > \beta_c$: $r^2 > 0$ mixture of two phases.

the two formulations are almost equivalent (Griffiths)
advantage of the second formulation: no need of "symmetry breaking term".

What happens in case of continuous internal symmetry?

$$\sigma(x) \in S^{\nu-1} \subset \mathbb{R}^\nu \qquad x \in \mathbb{Z}^d$$

$$H(\underline{\sigma}) = -\frac{J}{2} \sum_{\langle x,y \rangle} \sigma(x) \cdot \sigma(y)$$

scalar product in $\mathbb{R}^\nu$

$$\mu_\Lambda(d\underline{\sigma}) = \frac{1}{Z_\Lambda} \cdot \exp\left(-\beta H_\Lambda(\underline{\sigma})\right) \prod_{x \in \Lambda} d\sigma(x)$$

$d\sigma(x)$ uniform Lebesgue measure on $S^{\nu-1}$

$\nu = 1$: Ising; $\nu = 2$: (planar) rotator; $\nu = 3$: Heisenberg

Internal symmetry: $O(\nu)$

$O(1) = \mathbb{Z}_2$ but for $\nu \geq 2$ $O(\nu)$ is continuou

Question: the same

$$r^2 = \lim_{\Lambda \uparrow \mathbb{Z}^d} \left\langle \left| \frac{\sum_{x \in \Lambda} \sigma(x)}{|\Lambda|} \right|^2 \right\rangle_\Lambda$$

$$= \lim_{\Lambda \uparrow \mathbb{Z}^d} \left\langle \frac{\sum_{x, y \in \Lambda} \sigma(x) \cdot \sigma(y)}{|\Lambda|^2} \right\rangle_\Lambda$$

$$= \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \langle \sigma(0) \cdot \sigma(x) \rangle_\Lambda$$

$r = 0$ no long range order

$r > 0$ existence of long range order

Griffiths: existence of LRO implies spontaneous breakdown of internal symmetry, phase transition

Results: $d = 2, 3$

① negative   Mermin & Wagner   1967

$\boxed{d = 2}$ $\forall T > 0$ $r = 0$

no long range order at any positi[ve]
temperature in two dimentions

(Dobrushin - Shlosman 1975: formulation in terms of
infinite Gibbs states

2) positive   Fröhlich, Simon & Spencer 1976

$\boxed{d \geq 3}$ $\exists T_c : \quad T < T_c \Longrightarrow \quad r > 0$

there is LRO (and phase transition)
at suff low temperatures in three
and more dimentions

Remarks

Essential difference vs Ising: $\boxed{\text{no}}$ well defined
contours (with energy proportional to the
length of contour) Peierls argument does not apply

Possibility of Kosterlitz - Thouless phase in 2d
at low temperature
(Fröhlich - Spencer)

Conventions on "Fourier transform":

$\Lambda = L_1 \times L_2 \times \cdots \times L_d$ d-dimentional discrete torus

$\Lambda^* = \left\{ \left(2\pi \frac{k_1}{L_1}, 2\pi \frac{k_2}{L_2}, \ldots, 2\pi \frac{k_d}{L_d} \right) \,\middle|\, k_i \in \left[-\frac{L_i}{2}, \frac{L_i}{2}\right) \cap \mathbb{Z}, i = 1, 2,$

the dual torus $\Lambda^* \subset [-\pi, \pi)^d$

$f : \Lambda \to \mathbb{C}$ $\quad\quad \hat{f} : \Lambda^* \to \mathbb{C}$

$$\hat{f}(p) = \sum_{x \in \Lambda} e^{ipx} f(x)$$

$$f(x) = \frac{1}{|\Lambda|} \sum_{p \in \Lambda^*} e^{-ipx} \hat{f}(p)$$

unitary betwe
$l_2(\Lambda)$ and
$l_2(\Lambda^*)$ with we
$\frac{1}{|\Lambda|}$

<u>Note</u>: $\frac{1}{|\Lambda|} \sum_{p \in \Lambda^*} \longrightarrow \frac{1}{(2\pi)^d} \int_{(-\pi,\pi)^d} dp = ⨍ dp.$

A transl. invar. lin. op. on $l_2(\Lambda)$: $(Af)(x) = \sum_y a(x-y) f$

then $\widehat{Af}(p) = \hat{a}(p) \hat{f}(p)$

in particular $A = \Delta := \Delta_{xy} = \begin{cases} -2d & |x-y| = \\ 1 & |x-y| = \\ 0 & |x-y| > 1 \end{cases}$

its Fourier transform: $\hat{\Delta} = -2D$ $\quad D : (-\pi, \pi)^d \to \mathbb{R}$

$$D(p) = \sum_{i=1}^{d} (1 - \cos p_i) = \frac{1}{2}|p|^2 + O(|$$

Proof of FSS:

The "backbone" of the proof:

$$C_{ij}^{(\Lambda)}(x) = \langle \sigma_i(0)\,\sigma_j(x)\rangle_\Lambda \underset{\text{transl. invar.}}{=} \langle \sigma_i(y)\,\sigma_j(y+x)\rangle_\Lambda$$

finite volume correlation function

$$\hat{C}_{ij}^{(\Lambda)}(p) = \sum_{x\in\Lambda} e^{ipx}\, C_{ij}^{(\Lambda)}(x) \qquad p \in \Lambda^*$$

(A) Infrared bound (the core of the proof!)

IRB: $p \neq 0$: $\quad \frac{1}{2\beta J}\cdot\frac{1}{D(p)}\,\delta_{ij} - \hat{C}_{ij}^{(\Lambda)}(p) \geq 0$

as $\nu\times\nu$ matrix, i.e. it is positive definite (uniformly in Λ)

(B) $$1 = \langle \sigma(0)\cdot\sigma(0)\rangle_\Lambda = \sum_{i=1}^{\nu} C_{ii}^{(\Lambda)}(0)$$

$$= \frac{1}{|\Lambda|}\sum_{p\in\Lambda^*}\sum_{i=1}^{\nu}\hat{C}_{ii}^{(\Lambda)}(p)$$

$$= \underbrace{\frac{1}{|\Lambda|}\cdot\sum_{i=1}^{\nu}\hat{C}_{ii}^{(\Lambda)}(0)}_{r_\Lambda^2} + \frac{1}{|\Lambda|}\sum_{i=1}^{\nu}\sum_{p\neq 0}\hat{C}_{ii}^{(\Lambda)}(p)$$

A & B

$$\underbrace{m_\Lambda^2}_{\downarrow \; m^2} + \frac{\nu}{2\beta J} \cdot \underbrace{\frac{1}{|\Lambda|} \sum_{\substack{p \in \Lambda^* \\ p \neq 0}} \frac{1}{D(p)}}_{\downarrow} \geq 1$$

$$\int \frac{dp}{D(p)} \begin{cases} = \infty & d = 1,2 \\ < \infty & d \geq 3 \end{cases}$$

$$\boxed{m^2 \geq 1 - \frac{\nu}{2\beta J} \int \frac{dp}{D(p)}}$$

__oof of the IRB:__

$$[\Delta a](x) = \sum_{y : |y-x|=1} \left(a(y) - a(x)\right); \quad \Delta_{xy} = \begin{cases} 1 & |x-y|= \\ -2d & x=y \\ 0 & |x-y|> \end{cases}$$

(with periodic boundary conditions in Λ)

Shorthand: a, b vector valued $a = (a_1, a_2, .., a_\nu)$

$$\underline{a} \Delta \underline{b} = \sum_{xy} \Delta_{xy} \, a(x) \cdot b(y)$$

note that:

$$-\underline{a} \Delta \underline{a} = \frac{1}{2} \sum_{\langle x,y \rangle} \left(a(x) - a(y)\right)^2$$

$H_\Lambda(\underline{\sigma}) = -\frac{1}{2}\,\underline{\sigma}\,\Delta\,\underline{\sigma}$ (+ constant) | $\Delta = \Delta$ in Λ with periodic boundary conditions

let $\Lambda \ni x \longmapsto v(x) = \left(v_1(x), v_2(x), \dots, v_\nu(x)\right) \in \mathbb{R}^\nu$

be a vector field on Λ (arbitrary)

Def: $$Z_\Lambda(\underline{v}) = \int_{[S^\nu]^\Lambda} \exp\left\{\frac{\beta}{2}\,(\underline{\sigma}+\underline{v})\,\underline{\Delta}\,(\underline{\sigma}+\underline{v})\right\} \prod_{x\in\Lambda} d\sigma(x)$$

$Z_\Lambda(\underline{0}) = Z_\Lambda$ the partition function | $Z_\Lambda(\underline{v})$ depends only on the differences $v(x)-v(y)$

Theorems:

(1) $(\forall \underline{v})$: $\quad Z_\Lambda(\underline{v}) \leq Z_\Lambda(\underline{0})$

(1') $(\forall \underline{v})$: $\quad \left\langle \exp\left(\underline{\sigma}\,\Delta\,\underline{v}\right)\right\rangle_\Lambda \leq \exp\left\{-\frac{1}{2\beta}\,\underline{v}\,\Delta\,\underline{v}\right\}$

(2) $(\forall \underline{v})$: $\quad v_i(t)\,\Delta_{tx}\,\langle \sigma_i(x)\,\sigma_j(y)\rangle_\Lambda\,\Delta_{ys}\,v_j(s) \leq$

$$\leq -\frac{1}{\beta}\,\underline{v}\,\Delta\,\underline{v} = -\frac{1}{\beta}\,v_i(x)\,\Delta_{xy}\,v_i(y)$$

(2') $$M_{ij}(p) = \frac{1}{2\beta D(p)}\,\delta_{ij} - \hat{C}_{ij}(p) \qquad p\in\Lambda^*$$

is positive definite matrix

$$(1) \underset{\text{rearrangement}}{\Longleftrightarrow} (1') \underset{\substack{\text{expansion}\\ \underline{v}=\varepsilon\underline{v}}}{\Longrightarrow} (2) \underset{\substack{\text{Fourier}\\ \text{transform}}}{\Longleftrightarrow} (2')$$

Gaussian Domination

IRB

we prove (1).

Fundamental Lemma (Reflection Positivity): **RP**

Let Ω be a measurable space with a finite measure $d\mu$,

$A, B, C_k, D_k : \Omega \to \mathbb{C}$ bdd. measurable functions $k = 1, 2, \ldots, \ell$.

Then:

$$\left| \iint_{\Omega\times\Omega} \mu(ds)\mu(dt) \exp\left\{ A(s) + B(t) - \frac{1}{2}\sum_{k=1}^{\ell} \left(C_k(s) - D_k(t)\right)^2 \right\} \right|^2 \leq$$

$$\iint_{\Omega\times\Omega} \mu(ds)\mu(dt) \exp\left\{ A(s) + \bar{A}(t) - \frac{1}{2}\sum_{k=1}^{\ell} \left(C_k(s) - \bar{C}_k(t)\right)^2 \right\} \cdot$$

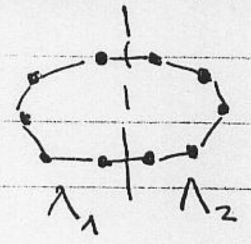
$$\iint_{\Omega\times\Omega} \mu(ds)\mu(dt) \exp\left\{ \bar{B}(s) + B(t) - \frac{1}{2}\sum_{k=1}^{\ell} \left(\bar{D}_k(s) - D_k(t)\right)^2 \right\}$$

Notes: Here we shall need it only for real functions, but the quantum analogue is essentially complex. (Right hand side is real, positive)

Proof postponed.

Proof of Gaussian Domination:

Assume Λ has even side-lengths (essential!)



$$\Lambda = \Lambda_1 \cup \Lambda_2$$

$$\Omega = \left(S^{\nu-1}\right)^{\Lambda_1} = \left(S^{\nu-1}\right)^{\Lambda_2} \qquad s = \{\sigma(x)\}_{x\in\Lambda_1} \quad t = \{\sigma(y)\}_{y\in\Lambda_2}$$

with identification according to <u>reflection</u>

$$d\mu(s) = \prod_{x\in\Lambda_1} d\sigma(x) \qquad d\mu(t) = \prod_{y\in\Lambda_2} d\sigma(y).$$

$$A(s) = A(\underline{\sigma}|_{\Lambda_1}) = -\frac{\beta}{4}\sum_{\substack{x,y\in\Lambda_1\\ \langle x,y\rangle}} \left(\sigma(x)+v(x)-\sigma(y)-v(y)\right)^2$$

$$B(t) = B(\underline{\sigma}|_{\Lambda_2}) = -\frac{\beta}{4}\sum_{\substack{x,y\in\Lambda_2\\ \langle x,y\rangle}} \left(\sigma(x)+v(x)-\sigma(y)-v(y)\right)^2$$

$$\left.\begin{aligned} C_k(s) &= C_k(\underline{\sigma}|_{\Lambda_1}) = \sqrt{\beta}\left(\sigma_i(x)+v_i(x)\right) \\ D_k(t) &= D_k(\underline{\sigma}|_{\Lambda_2}) = \sqrt{\beta}\left(\sigma_i(y)+v_i(y)\right)\end{aligned}\right\} \quad \begin{aligned} &k = (\langle x,y\rangle; i) \\ &\langle x,y\rangle \text{ on the cut-plane} \\ &i = 1,2,\dots,\nu\end{aligned}$$

RP Lemma $\Rightarrow \quad |Z_\Lambda(\underline{v})|^2 \le Z_\Lambda(\underline{v}_1)\cdot Z_\Lambda(\underline{v}_2)$

where $\quad \underline{v}_i\big|_{\Lambda_i} = \underline{v}\big|_{\Lambda_i} \quad \underline{v}_i\big|_{\Lambda_j} = \mathrm{Refl}\left(\underline{v}\big|_{\Lambda_i}\right) \quad \begin{aligned} i&=1,2\\ j&=2,1\end{aligned}$

choose $\underline{v}^*$: 1/ $Z_\Lambda(\underline{v}^*) = \max_{\underline{v}} Z_\Lambda(\underline{v})$

2/ $\#\{\langle x,y\rangle \mid v^*(x) \neq v^*(y)\} =$ possib[le]
minimum among maximizers.
$= \delta^*$

If $\delta^* = 0 \Rightarrow \underline{v}^* = \underline{0}$ is maximizer, done!

If $\delta^* > 0$: cut Λ with a mid-plane through
an edge $\langle x,y\rangle$ with $v^*(x) \neq v^*(y)$

RP Lemma $|Z_\Lambda(\underline{v}^*)|^2 \leq Z_\Lambda(\underline{v}_1^*) \cdot Z_\Lambda(\underline{v}_2^*)$

$\Downarrow$

$\underline{v}_1^*$ and $\underline{v}_2^*$ are also maximizers

but

$\min\{\#\{\langle x,y\rangle \mid v_1^*(x) \neq v_1^*(y)\}, \#\{\langle x,y\rangle \mid v_2^*(x) \neq v_2^*(y)\}\}$

$< \{\langle x,y\rangle \mid v^*(x) = v^*(y))$

strictly! by construction

Contradiction !!

$\Rightarrow \delta^* = 0$ and G.D. is proved $\boxtimes$ GD.

Proof of RP Lemma: assume $l = 1$ (no loss)

Fourier — change order of integration — Schwarz — change order of integration (back) — Fourier (back):

use: $e^{-\frac{1}{2}a^2} = \int_{-\infty}^{\infty} \frac{d\xi}{2\pi} e^{-\frac{\xi^2}{2}} e^{i\xi a}$ $a \in \mathbb{C}$.

$$\left| \iint_{\Omega\times\Omega} \mu(ds)\mu(dt) \exp\left\{A(s)+B(t)-\frac{1}{2}\left(C(s)-D(t)\right)^2\right\} \right|^2 =$$

$$\left| \iint_{\Omega\times\Omega} \mu(ds)\mu(dt) \exp\left\{A(s)+B(t)\right\} \int_{-\infty}^{\infty} \frac{d\xi}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} \exp\left\{i\xi\left(C(s)-D(t)\right)\right\} \right|^2$$

$$\left| \int_{-\infty}^{\infty} \frac{d\xi}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} \left(\int_{\Omega} \mu(ds) \exp\left\{A(s)+i\xi C(s)\right\}\right)\left(\int_{\Omega} \mu(dt) \exp\left\{B(t)-i\xi D(t)\right\}\right)\right.$$

$$\int_{-\infty}^{\infty} \frac{d\xi}{\sqrt{2\pi}} \left(\int_{\Omega} \mu(ds) \exp\left\{A(s)+i\xi\, C(s)\right\}\right)\left(\int_{\Omega} \mu(dt) \exp\left\{\bar{A}(s)-i\xi\, \bar{C}(s)\right\}\right)$$

$$\int \frac{d\xi}{\sqrt{2\pi}} \left(\int_{\Omega} \mu(ds) \exp\left\{\bar{B}(s)+i\xi\, \bar{D}(s)\right\}\right)\left(\int_{\Omega} \mu(dt) \exp\left\{B(t)-i\xi\, D(t)\right\}\right) =$$

$$\iint_{\Omega\times\Omega} \mu(ds)\mu(dt) \exp\left\{A(s)+\bar{A}(t)-\frac{1}{2}\left(C(s)-\bar{C}(t)\right)^2\right\} \cdot$$

$$\iint_{\Omega\times\Omega} \mu(ds)\mu(dt) \exp\left\{\bar{B}(s)+B(t)-\frac{1}{2}\left(\bar{D}(s)-D(t)\right)^2\right\}.$$

$\boxtimes$ RP