Gaussian Hilbert Spaces
partial, preliminary, unfinished, unpolished version
not for distribution

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1 Fock space and Gaussian Hilbert space

1.1 Bosonic Fock space

Let $\mathcal{V}$ be a real Hilbert space and

$$\mathcal{V}_0 := \mathbb{R}, \qquad \mathcal{V}_1 := \mathcal{V}, \qquad \mathcal{V}_n := \mathcal{V} \otimes \cdots \otimes \mathcal{V} \quad (n\text{-fold}).$$

For $n \in \mathbb{N}$ and $\sigma \in \mathrm{Perm}(n)$ let $R(\sigma) \in \mathcal{B}(\mathcal{V}_n)$ defined as

$$R(\sigma) v_1 \otimes \cdots \otimes v_n := v_{\sigma(1)} \otimes \cdots \otimes v_\sigma n.$$

Then $\mathrm{Perm}(n) \ni \sigma \mapsto R(\sigma) \in \mathcal{B}(\mathcal{V}_n)$ is a *unitary representation* of the symmetric group $\mathrm{Perm}(n)$ over the Hilbert space $\mathcal{V}_n$.

Define $S_n \in \mathcal{B}(\mathcal{V}_n)$ as

$$S_n := \frac{1}{n!} \sum_{\sigma \in \mathrm{Perm}(n)} R(\sigma).$$

This is an orthogonal projection on $\mathcal{V}_n$:

$$S_n = S_n^* = S_n^2.$$

Its range is

$$\mathcal{K}_n := S_n \mathcal{V}_n,$$

the *bosonic* spaces.

Assume an orthonormal basis $\left(e_j\right)_j$ in $\mathcal{V}$. Then

$$\left(e_{j_1} \otimes \cdots \otimes e_{j_n}\right)_{j_1,\dots,j_n}$$

will be the generated orthonormal basis in $\mathcal{V}_n$

Let

$$N_n := \{\underline{n} = (n_j)_j : n_j \in \{0,1,2,\dots\}, \sum_j n_j = n\}.$$

Denote for $\underline{n} \in N_n$

$$|\underline{n}\rangle := \binom{n}{\underline{n}}^{1/2} S_n\big(e_1 \otimes \cdots \otimes e_1 \otimes e_2 \otimes \cdots \otimes e_2 \otimes \dots\big)$$

where the basis element e_j appears n_j-times in the tensor product on the right hand sides.

Then the collection

$$\{|\underline{n}\rangle \in \mathcal{K}_n : \underline{n} \in N_n\}$$

forms an orthonormal basis in $\mathcal{K}_n$.

Note that if the basic space $\mathcal{V}$ is finite dimensional, with $d = \dim(\mathcal{V})$ then

$$\dim(\mathcal{K}_n) = \binom{n+d-1}{n}.$$

The (*bosonic*) *Fock space* is

$$\mathcal{K} := \overline{\oplus_{n=0}^{\infty} \mathcal{K}_n}.$$

1.2 The Gaussian Hilbert space

Let $(\Omega, \mathcal{F}, \pi)$ be a probability space and $\mathcal{H} := \mathcal{L}^2(\Omega, \mathcal{F}, \pi)$. Assume the following

Gaussian embedding of $\mathcal{V}$ in $\mathcal{H}$:
(i) There exists a map $\phi : \mathcal{V} \to \mathbb{H}$ such that for all $v \in \mathcal{V}$ the random variable $\phi(v)$ is Gaussian with

$$\mathbf{E}\big(\phi(v)\big) = 0, \qquad \mathbf{Var}\big(\phi(v)\big) = \|v\|^2.$$

(ii) The random variables $\{\phi(v) : v \in \mathcal{V}\}$ generate the sigma-algebra $\mathcal{F}$.

This is the same as saying that
(i) $\{\phi(v) : v \in \mathcal{V}\}$ are jointly Gaussian with

$$\mathbf{E}\big(\phi(v)\big) = 0, \qquad \mathbf{Cov}\big(\phi(v)\,,\,\phi(u)\big) = \langle v, u\rangle.$$

(ii) There is no random variable X which is jointly Gaussian with and independent of $\{\phi(v) : v \in \mathcal{V}\}$.

1.2.1 Wick products

Given a zero mean Gaussian random variable X its *Wick exponential* is defined as follows

$$:\exp\{X\}: = \exp\{X - \mathbf{E}\big(X^2\big)/2\}.$$

Given jointly Gaussian random variables $X = (X_1, \dots, X_k)$ and integers $n = (n_1, \dots, n_k)$ the *Wick monomial* is defined as:

$$\begin{aligned} :X_1^{n_1} \dots X_k^{n_k}: &= \left(\frac{\partial^n}{\partial t^n} :\exp\{tX\}: \right)_{t=0} \\ &= \left(\frac{\partial^{n_1+\dots+n_k}}{\partial t_1^{n_1} \dots \partial t_k^{n_k}} :\exp\{t_1 X_1 + \dots + t_k X_k\}: \right)_{t_1=\dots=t_k=0} . \end{aligned}$$

Given jointly Gaussian random variables $X = (X_1, \dots, X_k)$ and integers $n = (n_1, \dots, n_k)$ and $m = (m_1, \dots, m_k)$ the *Wick product* of two Wick monomials is defined as:

$$:(:X_1^{n_1} \dots X_k^{n_k}:)(:X_1^{m_1} \dots X_k^{m_k}:): = :X_1^{n_1+m_1} \dots X_k^{n_k+m_k}: .$$

Wick polynomials are linear combinations of Wick monomials. The Wick product extends by linearity (from Wick monomials) to Wick polynomials.

Proposition. *The Wick product (defined for Wick polynomials) is commutative, associative and distributive with respect to linear combinations. That is: given the Wick polynomials P, Q, R and the real numbers α, β, we have*

$$:PQ: = :QP:, \quad :(:PQ:)R: = :P(:QR:):, \quad :P(\alpha Q + \beta R): = \alpha :PQ: + \beta :PR: .$$

Proof. Straightforward. □

Remark: It follows that Wick monomials with only first powers suffice:

$$:X_1^{n_1} X_2^{n_2} \dots X_k^{n_k}: = :Y_1 \dots Y_n:$$

with

$$Y_{n_1+\dots n_l+r} = X_{l+1}, \qquad l = 0, \dots, k-1, \ \ r = 1, \dots, n_l.$$

Proposition. *Let $X = (X_1, \dots, X_m, Y_1, \dots, Y_n)$ be jointly Gaussian. Then*

$$\mathbf{E}\big(:X_1 \dots X_m: :Y_1 \dots Y_n: \big) = \delta_{m,n} \sum_{\sigma \in \mathrm{Perm(n)}} \prod_{i=1}^{n} \mathbf{E}\big(X_i Y_{\sigma(i)} \big)$$

Proof. Note that

$$:\exp\{tX\}: :\exp\{sX\}: = :\exp\{tX + sY\}: \exp\{tCs\},$$

where

$$C_{ij} = \mathbf{E}\big(X_i Y_j \big), \qquad i = 1, \dots, m, \ \ j = 1, \dots, n,$$

is the covariance matrix of the jointly Gaussian vectors X and Y. Thus

$$\mathbf{E}\big(:\exp\{tX\}: :\exp\{sX\}: \big) = \exp\{tCs\}.$$

The rest of the proof follows from explicit computation of

$$\left(\frac{\partial^{n+m}}{\partial s^m \partial t^n} \exp\{tCs\} \right)_{t=s=0} = \delta_{m,n} \sum_{\sigma \in \mathrm{Perm(n)}} \prod_{i=1}^{n} \mathbf{E}\big(X_i Y_{\sigma(i)} \big).$$

□

It follows that the subspaces of homogeneous Wick polynomials of degree n,

$$\mathcal{H}_n := \text{span}\{\, :\phi(v_1)\dots\phi(v_n): , v_1, \dots, v_n \in \mathcal{V}\}$$

are mutually orthogonal for $n \neq m$. Actually,

$$\mathcal{H} := \overline{\oplus_{n=0}^{\infty}\mathcal{H}_n}.$$

1.3 Unitary isomorphism between $\mathcal{K}$ and $\mathcal{H}$

Let n be fixed and $v_1, \dots, v_n \in \mathcal{V}$. Consider

$$S_n\big(v_1 \otimes \dots \otimes v_n\big) \in \mathcal{K}_n \qquad \text{and} \qquad :\phi(v_1)\dots\phi(v_n): \in \mathcal{H}_n.$$

Then

$$\|S_n\big(v_1 \otimes \dots \otimes v_n\big)\|^2_{\mathcal{K}_n} = \frac{1}{n!}\sum_{\sigma\in\text{Perm}(n)}\prod_{j=1}^{n}\langle v_j, v_{\sigma(j)}\rangle,$$

$$\|\ :\phi(v_1)\dots\phi(v_n):\ \|^2_{\mathcal{H}_n} = \mathbf{E}\big(\,(:\phi(v_1)\dots\phi(v_n):)^2\,\big) = \sum_{\sigma\in\text{Perm}(n)}\prod_{j=1}^{n}\langle v_j, v_{\sigma(j)}\rangle.$$

Thus,

$$\mathcal{K}_n \ni S_n\big(v_1 \otimes \dots \otimes v_n\big) \leftrightarrow (n!)^{-1/2}\ :\phi(v_1)\dots\phi(v_n): \in \mathcal{H}_n$$

extends (by linearity and polarization) to a unitary isomorphism $U_n : \mathcal{K}_n \to \mathcal{H}_n$.

1.4 Models

Finite dimension: Let $\mathcal{V} = \mathbb{R}^d$. Then one can realize

$$\begin{aligned}
&\Omega = \mathbb{R}^d,\\
&\pi(dx) = (2\pi)^{-d/2}\exp\{-x^2/2\}dx,\\
&\phi(v)(\omega) = \langle v, \omega\rangle.
\end{aligned}$$

Note, that this construction doesn't work in infinite dimension: $\mathbf{E}\big(\,\omega^2\,\big) = d$.

Basis dependent embedding: Let $\big(e_j\big)_j$ be an orthonormal basis in $\mathcal{V}$. Let $(\Omega, \mathcal{F}, \pi)$ be a probability space with $\big(\xi_j(\omega)\big)_j$ a $\mathcal{F}$-generating collection of i.i.d. standard normals and

$$\phi(v) = \sum_j \langle e_j, v\rangle \xi_j.$$

Gaussian distributions: Let $\mathcal{S} := \mathcal{S}(\mathbb{R}^d)$ be the space of rapidly decreasing test functions and $\mathcal{S}' := \mathcal{S}'(\mathbb{R}^d)$ the space of tempered distributions (Schwartz spaces). Let

$b : \mathbb{R}^d \to \mathbb{R}$ be a (smooth?) positive definite function: for any $n < \infty$, $x_1, \ldots, x_n \in \mathbb{R}^d$, $z_1, \ldots, z_n \in \mathbb{C}$

$$\sum_{i,j=1}^{n} z_i \overline{z}_j b(x_i - x_j) \geq 0,$$

or, equivalently

$$\hat{b}(p) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ipx} b(x) dx \geq 0.$$

For $u, v \in \mathcal{S}$ define the inner product

$$\langle u, v \rangle := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} u(x) b(x-y) v(y) = \int_{\mathbb{R}^d} \overline{\hat{u}(p)} \hat{v}(p) \hat{b}(p) dp,$$

and let $\mathcal{V}$ be the closure of $\mathcal{S}$ with respect to this norm. Note that

$$\mathcal{V} = \{u \in \mathcal{S}' : \|u\|^2 = \int_{\mathbb{R}^d} |\hat{u}(p)|^2 \, \hat{b}(p) dp < \infty\}.$$

By *Bochner-Minlos theorem* there exists a unique Gaussian cylinder-set measure $d\pi$ on $\mathcal{S}'$ with covariances

$$\int_{\mathcal{S}'} \omega(u) \omega(v) d\pi(\omega) = \langle u, v \rangle.$$

In this formula $u, v \in \mathcal{S}$, $\omega \in \mathcal{S}'$. Where is π concentrated?

Thus, in this model $\mathcal{V}$ is defined as above, $\Omega = \mathcal{S}'$, $\mathcal{F}$ is the sigma-algebra generated by finite cylinder sets and π is the gaussian measure on $\mathcal{S}'$ with the given covariances. The imbedding is first defined only as $\phi : \mathcal{S} \to \mathcal{L}^2(\Omega, \mathcal{F}, \pi)$ by

$$\phi(v)(\omega) := \omega(v).$$

The point is that if $v_n \in \mathcal{S}$ and $v_n \to v \in \mathcal{V}$ in the strong (norm) topology of $\mathcal{V}$ then the sequence $\phi(v_n)$ is Cauchy in $\mathcal{L}^2(\Omega, \mathcal{F}, \pi)$:

$$\mathbf{E}\big(|\phi(v_n) - \phi(v_m)|^2 \big) = \mathbf{E}\big(\phi(v_n - v_m)^2 \big) = \|v_n - v_m\|^2.$$

Define

$$\phi(v) := \lim_n \phi(v_n), \qquad \text{in} \quad \mathcal{L}^2(\Omega, \mathcal{F}, \pi).$$

But mind that if $v \in \mathcal{V} \setminus \mathcal{S}$ then pointwise π-a.s. convergence *doesn't hold.*

2 Operators

With slight abuse of notation we denote by the same symbols the operators acting on $\mathcal{K}$ and $\mathcal{H}$, *transposed unitarily* by $U : \mathcal{K} \to \mathcal{H}$.

The *grade number operator* is

$$N : \mathcal{K}(\mathcal{H}) \to \mathcal{K}(\mathcal{H}), \quad N \restriction_{\mathcal{K}_n(\mathcal{H}_n)} = nI \restriction_{\mathcal{K}_n(\mathcal{H}_n)} .$$

2.1 Creation and annihilation

2.1.1 Acting on Fock space

Let now $f \in \mathcal{V}$. We define $a^*(f) : \mathcal{K}_n \to \mathcal{K}_{n+1}$ and $a(f) : \mathcal{K}_n \to \mathcal{K}_{n-1}$, $n = 0, 1, \dots$ by

$$a^*(f) S_n\big(v_1 \otimes \dots \otimes v_n\big) := (n+1)^{1/2} S_{n+1}\big(f \otimes v_1 \otimes \dots \otimes v_n\big),$$
$$a(f) S_n\big(v_1 \otimes \dots \otimes v_n\big) := n^{1/2} \sum_{j=1}^{n} \langle f, v_j \rangle S_{n-1}\big(v_1 \otimes \dots \otimes v_{j-1} \otimes v_{j+1} \otimes \dots \otimes n_n\big).$$

and extended by linearity. It is not difficult to check that these are indeed mutually adjoint: $a^*(f) = a(f)^*$, indeed. Denote

$$N(f) := a^*(f) a(f).$$

2.1.2 Acting on Gaussian space

Now

$$a^*(f) : \mathcal{H}_n \to \mathcal{H}_{n+1}, \qquad a(f) : \mathcal{H}_n \to \mathcal{H}_{n-1},$$

act as follows:

$$a^*(f) \;:\!\phi(v_1) \dots \phi(v_n)\!:\; = \;:\!\phi(f)\phi(v_1) \dots \phi(v_n)\!:\;,$$
$$a(f) \;:\!\phi(v_1) \dots \phi(v_n)\!:\; = \sum_{j=1}^{n} \langle f, v_j \rangle \;:\!\phi(v_1) \dots \phi(v_{j-1}) \phi(v_{j+1}) \dots \phi(v_n)\!:\; .$$

It is again easy to check that

$$a^*(f) + a(f) = \phi(f) \cdot$$

(That is: multiplication by $\phi(f)$ on $\mathcal{L}^2(\Omega, \pi)$.)

2.2 Second quantization

2.2.1 Acting on Fock space

Given a linear operator A on $\mathcal{V}$ define the operators $\Gamma(A)$ and $d\Gamma(A)$ on $\mathcal{K}$ as follows

$$\Gamma(A) S_n\big(v_1 \otimes \dots \otimes v_n\big) = S_n\big(A v_1 \otimes \dots \otimes A v_n\big)$$
$$d\Gamma(A) S_n\big(v_1 \otimes \dots \otimes v_n\big) = \sum_{j=1}^{n} S_n\big(v_1 \otimes \dots \otimes A v_j \otimes \dots \otimes v_n\big).$$

Clearly

$$\Gamma(AB) = \Gamma(A)\Gamma(B), \qquad \Gamma(A^*) = \Gamma(A)^*, \qquad \|\Gamma(A) \restriction_{\mathcal{K}_n}\| = \|A\|^n,$$
$$d\Gamma(A+B) = d\Gamma(A) + d\Gamma(B) \qquad d\Gamma(A^*) = d\Gamma(A)^*, \qquad \|d\Gamma(A) \restriction_{\mathcal{K}_n}\| = n\|A\|.$$

Note that

$$\Gamma(A)\mathbb{1} = \mathbb{1}, \qquad d\Gamma(A)\mathbb{1} = 0,$$

and

$$\Gamma(0) = (\mathbb{1}, \cdot)\mathbb{1}, \quad d\Gamma(I) = N, \quad \Gamma(\exp\{A\}) = \exp\{d\Gamma(A)\}.$$

2.2.2 Acting on Gaussian space

$$\Gamma(A) \ {:}\phi(v_1)\dots\phi(v_n){:} \ = {:}\phi(Av_1)\dots\phi(Av_n){:}\,,$$

$$d\Gamma(A) \ {:}\phi(v_1)\dots\phi(v_n){:} \ = \sum_{j=1}^{n} \ {:}\phi(v_1)\dots\phi(Av_j)\dots\phi(v_n){:} \ .$$

Note also that for $A \in \mathcal{B}(\mathcal{V})$

$$\Gamma(A) \ {:}\exp\{\phi(v)\}{:} = {:}\exp\{\phi(Av)\}{:} \ .$$

2.3 Commutation relations

The following commutation relations hold:

$$[a(f), a(g)] = [a^*(f), a^*(g)] = 0, \qquad [a(f), a^*(g)] = \langle f, g\rangle I.$$

$$[\,\mathrm{d}\Gamma(A), a^*(f)] = a^*(Af), \qquad [\,\mathrm{d}\Gamma(A), a(f)] = -a(A^*f).$$

2.4 Ornstein-Uhlenbeck processes

Theorem. *Let $C \in \mathcal{B}(\mathcal{V})$.*
(i) If $\|C\| \leq 1$ then $\Gamma(C)$ acting on $\mathcal{H}$ is positivity preserving.
(ii) If $\|C\| < 1$ then $\Gamma(C)$ acting on $\mathcal{H}$ is positivity improving.

Remark: Together with $\Gamma(C)\mathbb{1} = \mathbb{1}$, this means that if $C \in \mathcal{B}(\mathcal{V})$ is a contraction, then $\Gamma(C)$ is a Markovian transition operator, with stationary measure π. Or, if G is the infinitesimal generator of a one parameter contraction semigroup $t \mapsto \exp\{tC\}$ on $\mathcal{V}$ then $d\Gamma(G)$ is the infinitesimal generator of a (stationary) Markovian semigroup $t \mapsto \exp\{td\Gamma(G)\}$ on $\mathcal{H}$.

Proof. Let $n \in \mathbb{N}$ and, for $F \in \mathcal{S}(\mathbb{R}^n)$ denote the direct and inverse Fourier transforms in $\mathbb{R}^n$ as follows:

$$\hat{F}(p) = (2\pi)^{n/2} \int_{\mathbb{R}^n} e^{-ip\cdot x} F(x)dx,$$

$$\check{F}(x) = (2\pi)^{n/2} \int_{\mathbb{R}^n} e^{ip\cdot x} F(p)dp.$$

Consider $F(\phi(v)) := F(\phi(v_1), \ldots, \phi(v_n)) \in \mathcal{H}$ and write it as

$$\begin{aligned} F(\phi(v)) &= (2\pi)^{n/2} \int_{\mathbb{R}^n} \exp\{ip \cdot \phi(v)\} \hat{F}(p) dp \\ &= (2\pi)^{n/2} \int_{\mathbb{R}^n} :\exp\{ip \cdot \phi(v)\}: \exp\{-\frac{1}{2} p \cdot \langle v, v \rangle \cdot p\} \hat{F}(p) dp \end{aligned}$$

Using the identity

$$\Gamma(C) :\exp\{\phi(v)\}: = :\exp\{\phi(Cv)\}:,$$

by linearity we get

$$\begin{aligned} \Gamma(C) F(\phi(v)) &= (2\pi)^{n/2} \int_{\mathbb{R}^n} :\exp\{ip \cdot \phi(Cv)\}: \exp\{-\frac{1}{2} p \cdot \langle v, v \rangle \cdot p\} \hat{F}(p) dp \\ &= (2\pi)^{n/2} \int_{\mathbb{R}^n} \exp\{ip \cdot \phi(Cv)\} \exp\{-\frac{1}{2} p \cdot \langle v, (I - C^* C)) v \rangle \cdot p\} \hat{F}(p) dp. \end{aligned}$$

This means that

$$\Gamma(C) F(\phi(v)) = \big(H * F\big)(\phi(Cv))$$

where $H(x) := H(x_1, \ldots, x_n)$ is the Gaussian

$$H(x) = \det(2\pi D)^{-1/2} \exp\{-\frac{1}{2} x \cdot D^{-1} \cdot x\},$$

and $D = \langle v, (I - C^* C)) v \rangle$ is the matrix

$$D_{k,l} = \langle v_k, (I - C^* C) v_l \rangle.$$

If D is not invertible, then approximate. □

Finite dimension: Let $\mathcal{V} = \mathbb{R}^d$ with the representation

$$\Omega = \mathbb{R}^d, \qquad d\pi(x) = (2\pi)^{-d/2} \exp\{-|x|^2/2\} dx.$$

The most general form of infinitesimal generator of a contraction semigroup over $\mathcal{V} = \mathbb{R}^d$ is written in matrix form as

$$G = -S + A, \qquad S_{i,j} = s_j \delta_{i,j}, \qquad A_{i,j} = \sum_{k=1}^{d} \sum_{l=k+1}^{d} a_{k,l} \big(\delta_{k,i} \delta_{l,j} - \delta_{k,j} \delta_{l,i}\big),$$

where $s_j \geq 0$, $a_{k,l} \in \mathbb{R}$. Then $d\Gamma(G)$ acting on $\mathcal{H} = \mathcal{L}^2(\mathbb{R}^d, \pi(dx))$ is

$$d\Gamma(G) = \sum_{j=1}^{d} s_j \left(\frac{1}{2} \frac{\partial^2}{\partial x_j^2} - x_j \frac{\partial}{\partial x_j} \right) + \sum_{k=1}^{d} \sum_{l=k+1}^{d} a_{k,l} \left(x_k \frac{\partial}{\partial x_l} - x_l \frac{\partial}{\partial x_k} \right).$$

This is the infinitesimal generator of a general Ornstein-Uhlenbeck process on $\mathbb{R}^d$.

Pure point spectrum: Let $\mathcal{V}$ be general Hilbert space and assume that the infinitesimal generator is self adjoint and has a pure point spectrum:

$$G = G^* < 0, \qquad Ge_k = -g_k e_k, \quad 0 < g_1 \leq g_2 \leq \ldots$$

for an orthonormal basis $\big(e_j\big)_j$ in $\mathcal{V}$. Let $X_k(t)$ be independent stationary 1d Ornstein-Uhlenbeck processes with covariances

$$\mathbf{E}\big(\, X_k(t) X_l(s) \,\big) = \delta_{k,l} e^{-g_k|t-s|}.$$

For $t \in \mathbb{R}$ let

$$\xi(t, v) := \mathcal{L}^2\text{-} \lim_{n\to\infty} \sum_{k=1}^{n} \langle e_k, v \rangle X_k(t).$$

Then $t \mapsto \xi(\cdot, t)$ is the infinite-dimensional Ornstein-Uhlenbeck process whose infinitesimal generator is $d\Gamma(G)$. Its covariances are

$$\mathbf{E}\big(\, \xi(s, u) \xi(t, v) \,\big) = \sum_{k=1}^{\infty} \langle e_k, u \rangle \langle e_k, v \rangle e^{-g_k|t-s|}.$$

Distribution valued Ornstein-Uhlenbeck processes: Let $b : \mathbb{R}^d \to \mathbb{R}$ be a (smooth?) positive definite function, as before. For $u, v \in \mathcal{S}$ define the inner product

$$\langle u, v \rangle := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} u(x) b(x-y) v(y) = \int_{\mathbb{R}^d} \overline{\hat{u}(p)} \hat{v}(p) \hat{b}(p) dp,$$

and let $\mathcal{V}$ be the closure of $\mathcal{S}$ with respect to this norm. Then

$$\mathcal{V} = \{ u \in \mathcal{S}' : \|u\|^2 = \int_{\mathbb{R}^d} |\hat{u}(p)|^2 \, \hat{b}(p) dp < \infty \}.$$

The Laplacian acts on $\mathcal{V}$:

$$\Delta u(x) = \frac{\partial^2 u}{\partial x^2}(x), \qquad \widehat{\Delta u}(p) = -p^2 \hat{u}(p).$$

The generalized Ornstein-Uhlenbeck process with infinitesimal generator $d\Gamma(\Delta)$ is the $\mathcal{S}'$-valued Gaussian process $\xi(t, x)$ with covariances

$$\begin{aligned} \mathbf{E}\big(\, \xi(t, x) \xi(s, y) \,\big) &= \int_{\mathbb{R}^d} e^{ip(y-x)} \hat{b}(p) e^{-p^2|t-s|} dp \\ &= \int_{\mathbb{R}^d} (4\pi|t-s|)^{-\frac{d}{2}} e^{-\frac{(x-z)^2}{4|t-s|}} b(y-z) dz. \end{aligned}$$