The Alder-Wainwright argument (informal notes for friendly use)

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1 Sketch of the argument in general form

Let $t \mapsto X(t) \in \mathbb{R}^d$ be a random motion in some random environment, with stationary and ergodic increments. The random environment also evolves in time. Usually we decompose the random motion into martingale+compensator:

$$X(t) = M(t) + \int_0^t V(s)ds =: M(t) + \tilde{X}(t), \tag{1}$$

where $M(t)$ is a square integrable martingale with stationary and ergodic increments and the compensator $V(t)$ is stationary and ergodic, also square integrable. I'll call it the instantaneous velocity of the motion. We are interested in understanding the origins of the possibly superdiffusive behaviour of $X(t)$. Since $M(t)$ is anyway diffusive, we don't care much about it. The main contribution to the superdiffusive behaviour anyway comes from

$$\tilde{X}(t) := \int_0^t V(s)ds. \tag{2}$$

It is assumed that the instantaneous velocity comes from some background velocity field $U(t, x)$ as follows:

$$V(t) = U(t, X(t)). \tag{3}$$

And it is important to note that we think about a joint random dynamics $t \mapsto \big(X(t), U(t, \cdot)\big)$. The dynamics of $t \mapsto U(t, x)$ is also influenced by $X(t)$ (and vice versa). The process of the velocity field $t \mapsto U(t, \cdot)$ is usually *not stationary* while the field as seen from the position of the random motion $V(t, x) := U(t, X(t)+x)$, is. Note, that with this notation $V(t) = V(t, 0)$.

1.1 Various notations

The *correlations of the velocity field*:

$$K_{ij}(t,x) := \mathbf{E}\big(U_i(0,0)U_j(t,x) \big), \tag{4}$$

$$K(t,x) := \mathbf{E}\big(U(0,0)\cdot U(t,x) \big) = \sum_{i=1}^{d} K_{ii}(t,x). \tag{5}$$

Mind that the field $U(t,x)$ is not stationary under either time or space shifts, so

$$\mathbf{E}\big(U_i(s,y)U_j(t,x) \big) \neq K_{ij}(t-s, x-y). \tag{6}$$

The *velocity autocorrelation function*:

$$C_{ij}(t) := \mathbf{E}\big(V_i(0)V_j(t) \big), \tag{7}$$

$$C(t) := \mathbf{E}\big(V(0)\cdot V(t) \big) = \sum_{i=1}^{d} C_{ii}(t). \tag{8}$$

Mind that the velocity process $V(t)$ is assumed stationary (and ergodic), thus

$$\mathbf{E}\big(V_i(s)V_j(t) \big) = C_{ij}(t-s). \tag{9}$$

The *diffusivity*:

$$D_{ij}(t) := t^{-1}\mathbf{E}\big(X_i(t)X_j(t) \big), \tag{10}$$

$$D(t) := \mathbf{E}\big(X(t)\cdot X(t) \big) = \sum_{i=1}^{d} D_{ii}(t). \tag{11}$$

$$\tilde{D}_{ij}(t) := t^{-1}\mathbf{E}\big(\tilde{X}_i(t)\tilde{X}_j(t) \big), \tag{12}$$

$$\tilde{D}(t) := \mathbf{E}\big(\tilde{X}(t)\cdot \tilde{X}(t) \big) = \sum_{i=1}^{d} \tilde{D}_{ii}(t). \tag{13}$$

In case of superdiffusive behaviour

$$D_{ij}(t) \asymp \tilde{D}_{ij}(t) \tag{14}$$

By stationarity of $V(t)$

$$\tilde{D}_{ij}(t) = 2\int_0^t \frac{t-s}{t} C_{ij}(s)ds, \tag{15}$$

and hence

$$\tilde{D}_{ij}(t) \asymp \int^t C_{ij}(s)ds. \tag{16}$$

We shall refer to (16) as the *asymptotic form of the Green-Kubo formula.*

1.2 Scaling assumptions

I will treat separately the cases of *isotropic scaling*, when all spatial directions are scaled in the same order, respectively, the *anisotropic scaling*, when different spatial directions scale in different order. I will treat the anisotropic scaling cases only in $d = 2$ and only in the case when one of the spatial directions scales diffusively and the other one is expected to scale superdiffusively.

1.2.1 Isotropic scaling assumptions

In the isotropic case I will not care about directions, components, subscripts.

Scaling of the displacement: We assume that $X(t)$ is of order

$$\alpha(t) = t^{\nu}(\log t)^{\gamma}. \tag{17}$$

More explicitly, assume that

$$\mathbf{P}\big(X(t) \in dx \big) \asymp \alpha(t)^{-d} \varphi(\alpha(t)^{-1}x) dx, \tag{18}$$

where $\varphi : \mathbb{R}^d \to \mathbb{R}_+$ is a density, which is regular at $x = 0$ and decays fast at $|\, x \,| \to \infty$. Its Fourier transform is

$$\hat{\varphi}(p) := \int_{\mathbb{R}^d} \varphi(x) e^{ix \cdot p} dx. \tag{19}$$

Scaling of the velocity field: We also assume that the correlations of the velocity field $U(t, x)$ scale as

$$K(t, x) \asymp \beta(t)^{-d} \psi(\beta(t)^{-1}x). \tag{20}$$

Note that under this assumption

$$\int_{\mathbb{R}^d} K(t, x) dx \asymp \text{const.} \tag{21}$$

This corresponds to some kind of *conservation of momentum* carried by the velocity field. Note, that since $\psi(x)$ is a correlation function, it is by force of positive type:

$$\hat{\psi}(p) := \int_{\mathbb{R}^d} \psi(x) e^{ix \cdot p} dx > 0. \tag{22}$$

Main scaling assumptions: We make two important assumptions

1. *Regularity of ψ and/or φ:* One of the following two regularity conditions holds: Either

$$\int_{\mathbb{R}^d} |\, \psi(x) \,| \, dx < \infty, \qquad \hat{\psi}(0) > 0, \tag{23}$$

   or

$$\hat{\varphi}(p) > 0, \qquad \overline{\lim_{p \to 0}} \, \hat{\psi}(p) > 0. \tag{24}$$

2. *Displacement scales on faster order than the velocity-field correlations:*

$$\beta(t) = \mathcal{O}(\alpha(t)). \tag{25}$$

1.2.2 Anisotropic scaling assumptions

In two dimensions, for some models we may make the following anisotropic scaling assumptions. I will denote the component of the displacement $(X(t), Y(t))$ rather than $(X_1(t), X_2(t))$.

Scaling of the displacement: We assume that $X(t)$ is of order

$$\alpha(t) = t^{\nu}(\log t)^{\gamma}, \tag{26}$$

while $Y(t)$ is assumed diffusive (of order $t^{1/2}$). More explicitly, assume that

$$\mathbf{P}\big(X(t) \in dx, Y(t) \in dy\big) \asymp \alpha(t)^{-1} t^{-1/2} \varphi(\alpha(t)^{-1}x, t^{-1/2}y) dxdy, \tag{27}$$

where $\varphi : \mathbb{R}^2 \to \mathbb{R}_+$ is a density, which is regular at $x = y = 0$ and decays fast at $|\, x\,| + |\, y\,| \to \infty$. Its Fourier transform is

$$\hat{\varphi}(p, q) := \int_{\mathbb{R}^2} \varphi(x) e^{i(xp+yq)} dxdy. \tag{28}$$

More general anisotropic scaling assumptions are also possible.

Scaling of the velocity field: We also assume that the correlations of the velocity field $U(t, x, y)$ scale as

$$K_{ij}(t, x, y) \asymp \beta(t)^{-1} \beta'(t)^{-1} \psi_{ij}(\beta(t)^{-1}x, \beta'(t)^{-1}y). \tag{29}$$

Note that the conservation law

$$\int_{\mathbb{R}^2} K_{ij}(t, x, y) dxdy \asymp \text{const.} \tag{30}$$

still holds and $\psi(x, y)$ is still of positive type:

$$\hat{\psi}_{ij}(p, q) := \int_{\mathbb{R}^2} \psi_{ij}(x, y) e^{i(xp+yq)} dxdy > 0 \quad \text{(as matrix)}. \tag{31}$$

Main scaling assumptions: We make two important assumptions

1. *Regularity of ψ and/or φ:*

$$\int_{\mathbb{R}^2} |\, \psi_{11}(x, y)\,|\, dxdy < \infty, \qquad \hat{\psi}_{11}(0, 0) > 0. \tag{32}$$

2. *Displacement scales on faster order than the velocity-field correlations:*

$$\beta(t) = \mathcal{O}(\alpha(t)), \qquad \beta'(t) = \mathcal{O}(t^{1/2}), . \tag{33}$$

2 Computations

2.1 Isotropic scaling

First, assuming condition (23):

$$\begin{aligned}
C(t) &= \mathbf{E}\big(\, V(0)\cdot V(t)\,\big) = \mathbf{E}\big(\, U(0,0)\cdot U(t,X(t))\,\big) && (34)\\
&\approx \int_{\mathbb{R}^d} K(t,x)\mathbf{P}\big(\, X(t)\in dx\,\big) && \texttt{decoupling!!!}\\
&\asymp \int_{\mathbb{R}^d} \beta(t)^{-d}\psi(\beta(t)^{-1}x)\alpha(t)^{-d}\varphi(\alpha(t)^{-1}x)dx && \texttt{scaling}\\
&= \alpha(t)^{-d}\int_{\mathbb{R}^d} \psi(x)\varphi(\beta(t)\alpha(t)^{-1}x)dx\\
&\asymp \alpha(t)^{-d}
\end{aligned}$$

Of course, the second step (decoupling) is the shaky one. In the last step we used (23) and regularity of φ at $x=0$.

If instead of (23) we assume (24), then in the previous computation, after the scaling step we take the Fourier transforms:

$$\begin{aligned}
C(t) &\asymp \int_{\mathbb{R}^d} \beta(t)^{-d}\psi(\beta(t)^{-1}x)\alpha(t)^{-d}\varphi(\alpha(t)^{-1}x)dx && \texttt{scaling} \quad (35)\\
&= \int_{\mathbb{R}^d} \hat{\psi}(\beta(t)p)\hat{\varphi}(\alpha(t)p)dp\\
&= \alpha(t)^{-d}\int_{\mathbb{R}^d} \hat{\psi}(\beta(t)\alpha(t)^{-1}p)\hat{\varphi}(p)dp\\
&\asymp \alpha(t)^{-d}
\end{aligned}$$

Now, in the last step we have used (24).

Conclusions in the isotropic scaling

From the scaling assumption (18) it follows that

$$\tilde{D}(t) \asymp t^{-1}\alpha(t)^2. \tag{36}$$

On the other hand, using the Green-Kubo formula (16) and the computations in (34) we get

$$\tilde{D}(t) \asymp \int^t \alpha(s)^{-d}ds. \tag{37}$$

The only choices of the scaling function $\alpha(t)$ consistent with both (36) and (37), are

$$\mathbf{d=1}: \qquad \nu = \frac{2}{3}, \quad \gamma = 0, \qquad D(t) \asymp t^{1/3}, \tag{38}$$

$$\mathbf{d=2}: \qquad \nu = \frac{1}{2}, \quad \gamma = \frac{1}{4}, \qquad D(t) \asymp (\log t)^{1/2}, \tag{39}$$

$$\mathbf{d=3}: \qquad \nu = \frac{1}{2}, \quad \gamma = 0, \qquad D(t) \asymp 1. \tag{40}$$

2.2 Anisotropic scaling

I will compute the asymptotics of $C_{11}(t)$

$$\begin{aligned} C_{11}(t) &= \mathbf{E}\big(\,V_1(0)\cdot V_1(t)\,\big) = \mathbf{E}\big(\,U_1(0,0)\cdot U_1(t,X(t),Y(t))\,\big) && (41)\\ &\approx \int_{\mathbb{R}^2} K_{11}(t,x,y)\mathbf{P}\big(\,X(t)\in dx, Y(t)\in dy\,\big) && \texttt{decoupling!!!}\\ &\asymp \int_{\mathbb{R}^2} \beta(t)^{-1}\beta'(t)^{-1}\psi(\beta(t)^{-1}x,\beta'(t)^{-1}y)\alpha(t)^{-1}t^{-1/2}\varphi(\alpha(t)^{-1}x,t^{-1/2}y)dxdy && \texttt{scaling}\\ &= \alpha(t)^{-1}t^{-1/2}\int_{\mathbb{R}^2}\psi(x,y)\varphi(\beta(t)\alpha(t)^{-1}x,\beta'(t)t^{-1/2}y)dxdy\\ &\asymp \alpha(t)^{-1}t^{-1/2} \end{aligned}$$

Same remarks/comments are in order as in the isotropic case.

Conclusions in the anisotropic scaling

From the scaling assumption (27) it follows that

$$\tilde{D}_{11}(t) \asymp t^{-1}\alpha(t)^2. \tag{42}$$

On the other hand, using the Green-Kubo formula (16) and the computations in (41) we get

$$\tilde{D}_{11}(t) \asymp \int^t \alpha(s)^{-1}t^{-1/2}ds. \tag{43}$$

We are in $d=2$. The only choice of the scaling function $\alpha(t)$ consistent with both (42) and (43), is

$$\mathbf{d=2}: \qquad \nu = \frac{1}{2}, \quad \gamma = \frac{1}{3}, \qquad D(t) \asymp (\log t)^{2/3}. \tag{44}$$

3 Examples

3.1 Second class particle in ASEP (anisotropic scaling)

We start the system ASEP+SCP from the stationary distribution — as seen from the position of the SCP, corresponding to $\rho = 1/2$. $\big(\eta_t(x)\big)_{x\in\mathbb{Z}^d}$ is the ASEP. $X(t)$ is the displacement of the SCP. Then

$$V(t) = \sum_v p(v)v\big(1 - \eta_t(X(t)+v)\big) - \sum_v p(v)v\eta_t(X(t)-v) \tag{45}$$
$$= \langle v\rangle - \sum_v p(v)v\big(\eta_t(X(t)+v) + \eta_t(X(t)-v)\big)$$

where

$$\langle v\rangle := \sum_v p(v)v. \tag{46}$$

The velocity field is

$$U(t,x) = \langle v\rangle - \sum_v p(v)v\big(\eta_t(x+v) + \eta_t(x-v)\big), \tag{47}$$

with correlations

$$K_{ij}(t,x) = 4\langle v_i\rangle\langle v_j\rangle\mathbf{Cov}\big(\,\eta_0(0)\,,\,\eta_t(x)\,\big) + \text{small error}. \tag{48}$$

The "small error" consists of four terms like

$$\sum_v\sum_u p(v)v_i p(u)u_j\big(\mathbf{Cov}\big(\,\eta_0(\pm u)\,,\,\eta_t(x\pm v)\,\big) - \mathbf{Cov}\big(\,\eta_0(0)\,,\,\eta_t(x)\,\big)\big), \tag{49}$$

which clearly disappear under scaling. Note that the covariance matrix K_{ij} is asymptotically *rank one*. This leads to the anisotropic scaling in $d \geq 2$.

Beside these we have the *fundamental identity*

$$\mathbf{Cov}\big(\,\eta_0(0)\,,\,\eta_t(x)\,\big) = \mathbf{P}\big(\,X(t) = x\,\big). \tag{50}$$

$\mathbf{d=1}$: Using the fundamental identity (50) we get

$$\beta(t) = \alpha(t), \qquad \psi(x) = \phi(x). \tag{51}$$

Plugging these into (34) we conclude (38).

$\mathbf{d=2}$: We are in the anisotropic case. Say, $\langle v\rangle$ points in the horizontal direction (of subscript 1). Then – using again the fundamental relation (50) – we get:

$$\beta(t) = \alpha(t), \quad \beta'(t) = t^{1/2}, \quad \psi(x,y) = \varphi(x,y) = \varphi(x)e^{-y^2/2}. \tag{52}$$

Plugging these into (41) we conclude (44).

3.2 Diffusion in curl-MFGF field and Random Walk on Square Ice (isotropic scaling)

We are in $d = 2$. Let $U : \mathbb{R}^2 \to \mathbb{R}^2$ be the zero mean Gaussian vector field with covariances

$$\mathbf{E}\big(U_i(x) \big) = 0, \qquad \mathbf{E}\big(U_i(x)U_j(y) \big) = K_{ij}(y - x), \tag{53}$$

where $K_{ij}(x)$ has the Fourier transform

$$\hat{K}_{ij}(p) = (2\delta_{i,j} - 1)\frac{p_{\underline{i}}p_{\underline{j}}}{|\,p\,|^2}e^{-|\,p\,|^2/2}, \tag{54}$$

where we have denoted

$$\underline{i} := 3 - i, \qquad i = 1, 2. \tag{55}$$

This corresponds to the vector field $U(x)$ being *the curl of the massless free Gaussian field, smeared out by convolution with* $e^{-|\,x\,|^2/2}$. Note that the field $U(x)$ is *divergence free.*

Let $X(t)$ be the diffusion in the random drift field $U(x)$:

$$dX(t) = dB(t) + U(X(t))dt. \tag{56}$$

Since the vector field $U(x)$ is divergence free, the *random environment seen from the moving particle*

$$V(t, x) := U(t, X(t) + x) \tag{57}$$

is stationary (and ergodic) in time.

The problem is isotropic. Since the velocity field $U(x)$ does not change with time, from start we know that

$$\beta(t) \equiv 1, \qquad \psi_{ij}(x) = K_{ij}(x). \tag{58}$$

Note that

$$|\,K_{ii}(x)\,| \sim |\,x\,|^{-2} \tag{59}$$

and thus, the condition (23) doesn't hold. Nevertheless, we do the computations (35) and apply condition (24). We get

$$\begin{aligned} C(t) &= \int_{\mathbb{R}^2} \hat{K}(p)\hat{\varphi}(\alpha(t)p)dp \\ &= \alpha(t)^{-2} \int_{\mathbb{R}^2} \hat{V}(\alpha(t)^{-1}p)\hat{\varphi}(p)dp \\ &\asymp \alpha(t)^{-2}. \end{aligned} \tag{60}$$

This leads to the conclusion (39).

The *random walk on square ice* is the analogous discrete space-time RWRE problem. Take the isotropic six-vertex (or: square ice) model and let $X(n)$ be the n.n. RWRE which jumps with probability $\frac{1}{4} + \varepsilon$, respectively, $\frac{1}{4} - \varepsilon$ $(0 < \varepsilon < \frac{1}{4})$ in, respectively, against the direction of orientation of the edges. Everything seems to be very similar to the previous diffusion in RE.

3.3 True Self-Avoiding Walk and Self-Repelling Brownian Polymer (isotropic scaling)

See ....