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Exact Critical Percolation Probabilities for Site and Bond Problems in Two Dimensions*

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An exact method for determining the critical percolation probability, p_c, for a number of two-dimensional site and bond problems is described. For the site problem on the plane triangular lattice $p_c = \frac{1}{2}$. For the bond problem on the triangular, simple quadratic, and honeycomb lattices, $p_c = 2\sin(\frac{1}{18}\pi)$, $\frac{1}{2}$, $1 - 2\sin(\frac{1}{18}\pi)$, respectively. A matching theorem for the mean number of finite clusters on certain two-dimensional lattices, somewhat analogous to the duality transformation for the partition function of the Ising model, is described.

1. INTRODUCTION

PERCOLATION processes and their applications have been discussed by many authors,[1–5] and for a general introduction, reference should be made to the recent review by Frisch and Hammersley[6] who give an extensive bibliography. In this paper we shall derive some exact critical percolation probabilities for site and bond problems in two dimensions.

A study of the series expansions for the mean number of finite clusters on the plane triangular lattice leads to the discovery of a "matching" property somewhat analogous to the duality transformation for the partition function of the Ising model introduced by Kramers and Wannier[7] and interpreted geometrically by Onsager.[8] We shall introduce the series method, notice the matching property, and show that it depends essentially on a result sometimes known as Euler's Law of the Edges. We have been able to define a general class of two-dimensional lattices for which a matching property can be established. In certain special cases the property suffices to locate the critical probability. More generally we establish that the critical probabilities of certain pairs of lattices (matching pairs) are complementary. We shall locate the critical probability for one such matching pair, the bond problem on the triangular and honeycomb lattices, by a star–triangle substitution analogous to that introduced by Onsager for the corresponding Ising problem.

Apart from the theoretical interest of these exact results, a knowledge of p_c is an invaluable aid in the interpretation of power series that arise in a study of these problems.[9] We shall examine the general problem of deriving such expansions in a subsequent paper. A brief outline of the salient results in this paper has already been given.[10]

2. MEAN NUMBER OF CLUSTERS ON A FINITE GRAPH

We consider the site problem on a general linear graph G whose sites are colored at random, being black with probability p and white with probability $q = 1 - p$. For some purposes it is convenient to emphasise the symmetry of the problem and we shall then write

$$p = p_B = 1 - p_W, \tag{2.1}$$

$$q = p_W = 1 - p_B. \tag{2.2}$$

In most applications our interest in the problem will be asymmetric in that we shall consider the black sites as the primary species and refer to small p as low density and large p as high density. We shall adopt the convention of coloring a bond joining two nearest-neighbor black sites black (black bond),

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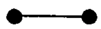
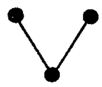
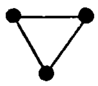
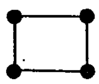
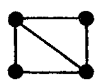
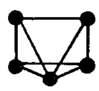
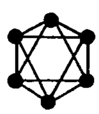
[1] S. R. Broadbent, and J. M. Hammersley, Proc. Cambridge Phil. Soc. 53, 629 (1957).
[2] V. A. Vyssotosky, S. B. Gordon, H. L. Frisch, and J. M. Hammersley, Phys. Rev. 123, 1566 (1961).
[3] H. L. Frisch, E. Sonnenblick, V. A. Vyssotosky, and J. M. Hammersley, Phys. Rev. 124, 1021 (1961).
[4] R. J. Elliot, B. R. Heap, D. J. Morgan, and G. S. Rushbrooke, Phys. Rev. Letters 5, 366 (1960).
[5] C. Domb and M. F. Sykes, Phys. Rev. 122, 77 (1961).
[6] H. L. Frisch and J. M. Hammersley, J. Soc. Ind. Appl. Math. 11, 894 (1963).
[7] H. A. Kramers and G. H. Wannier, Phys. Rev. 60, 252 (1941).
[8] See G. H. Wannier, Rev. Mod. Phys. 17, 50 (1945).
[9] M. F. Sykes and J. W. Essam, Phys. Rev. 133, A 310 (1964).
[10] M. F. Sykes and J. W. Essam, Phys. Rev. Letters 10, (1963).

Table I. Finite clusters on the octahedron.

Cluster type	Size	Number	Probability
	1	6	pq^4
	2	12	p^2q^4
	3	12	p^3q^3
	3	8	p^3q^3
	4	3	p^4q^2
	4	12	p^4q^2
	5	6	p^5q
	6	1	p^6

and that joining two nearest-neighbor white sites white (white bond). Bonds joining sites of opposite color will be called uncolored. A connected linear graph formed by sites and bonds of one color is called a cluster.

Any particular realization R of the probability distribution on G defines two further linear graphs R_B and R_W which are, respectively, the ensemble of black and white clusters. If we denote the number of black clusters in R_B by $n(R_B)$ and the probability of occurrence of R_B by $P(R_B; G)$ then we shall define the mean number of black clusters as the weighted average

$$K(p_B; G) = \langle n(R_B) \rangle = \sum_{R_B} P(R_B; G) n(R_B), \tag{2.3}$$

and by symmetry the mean number of white clusters is

$$K(p_W; G) = \langle n(R_W) \rangle. \tag{2.4}$$

For a finite graph of N sites the 2^N realizations are enumerable and for small N the function $K(p; G)$ can readily be obtained explicitly. We illustrate the perimeter method described by Domb[11] by applying it to the octahedron. If we denote the mean number of clusters of size r (that is, with r sites) by a subscript, then

$$K = \sum_r K_r. \tag{2.5}$$

In Table I. we list all the possible connected clusters on the octahedron and group them according to size and topology together with their respective probabilities of occurrence. It will be seen that there are six possible unit clusters each with probability pq^4, and therefore

$$K_1 = 6pq^4. \tag{2.6}$$

Likewise,

$$K_2 = 12p^2q^4. \tag{2.7}$$

For clusters of size 3 there are two possible types but the classification is mutually disjoint. By collecting all the contributions from Table I and substituting in (2.5), we obtain the mean number of black clusters as a polynomial in p and q and we shall denote this polynomial by $K(p, q; G)$. We find

$$K(p, q; G) = 6pq^4 + 12p^2q^4 + 20p^3q^3 + 15p^4q^2 + 6p^4q + p^6, \tag{2.8}$$

where of course G is the octahedron. It is implicit in (2.8) that $q = 1 - p$. We thus obtain the mean number of the primary or black species by substitution[12] as a function of p only,

$$K(p; G) = 6p - 12p^2 + 8p^3 + 3p^4 - 6p^5 + 2p^6. \tag{2.9}$$

By symmetry the mean number for the secondary or white species is obtained by writing q for p in (2.9).

For some applications it is convenient to express $K(p; G)$ as a function of q; that is, to express the mean number of black clusters in terms of the probability for white sites. This substitution is particularly appropriate to an investigation of the high-density region where q is small, and a suitable variable for

[11] C. Domb, Nature **184**, 509 (1959). (Report of the Physical Society Conference on Fluctuation Phenomena and Stochastic Processes held at Birkbeck College, London.)

[12] The function $K(p; G)$ can be written as a polynomial in p and q in more than one way since these are dependent variables. We shall reserve the symbol $K(p, q; G)$ to denote the polynomial that results from application of the perimeter method and which will be fundamental to our subsequent treatment of series expansions. By reversing the roles of p and q on both sides of (2.8), the mean number of white clusters is obtained.

the derivation of series developments. We shall write

$$K(p, 1-p; G) \quad \text{as} \quad K_L(p; G) = K(p; G), \tag{2.10}$$

$$K(1-q, q; G) \quad \text{as} \quad K_H(q; G), \tag{2.11}$$

and in general we shall omit the specification of G in the brackets unless it contributes anything essential to the argument. We have chosen the subscripts because K_L is most appropriate to low densities and K_H to high densities. It is clear that for any finite graph these two functions are finite polynomials in p and q, respectively, and

$$K_L(p) = K_H(1-p), \tag{2.12}$$

$$K_H(q) = K_L(1-q). \tag{2.13}$$

For the octahedron we have

$$K_L(p) = 6p - 12p^2 + 8p^3 + 3p^4 - 6p^5 + 2p^6, \tag{2.14}$$

$$K_H(q) = 1 \qquad\qquad + 3q^4 - 6q^5 + 2q^6. \tag{2.15}$$

It will be seen that the last three coefficients in (2.14) and (2.15) are identical. A similar phenomenon is found for the corresponding functions for the icosahedron for which the last eight coefficients are identical:

$$K_L(p) = 12p - 30p^2 + 20p^3 + 12p^5 + 28p^6 - 120p^7 + 75p^8 + 80p^9 - 126p^{10} + 60p^{11} - 10p^{12}, \tag{2.16}$$

$$K_H(q) = 1 \qquad\qquad + 12q^5 + 28q^6 - 120q^7 + 75q^8 + 80q^9 - 126q^{10} + 60q^{11} - 10q^{12}. \tag{2.17}$$

This "matching" property of K_L and K_H, which remains to be defined precisely, is not found to be a general property of all finite graphs. We interpret the property in the next section.

3. MEAN NUMBER OF CLUSTERS ON AN INFINITE GRAPH

It is convenient on an infinite lattice to define the mean number of clusters per site, and we shall write for a lattice of N sites

$$K(p, q) = k(p, q)N, \tag{3.1}$$

and generally write k for K where required through all the equations of the previous section. When we apply Eq. (2.5) to an infinite crystal lattice, such as the plane triangular lattice, the summation cannot be performed. However, when p is small the mean number of very large black clusters will be very small and, following Domb, we shall suppose that the double series in p and q that replaces the right-hand side of (2.5) will converge to $k(p, q)$ for small enough p. A similar observation holds when p is close to unity, for in the limit there is only one cluster, of infinite size, which fills the whole lattice. For $q > 0$ there will be a few finite black clusters surrounded by white sites, and again we shall suppose that $k(p, q)$ converges. The problem is now characterized by the existence of a critical probability p_c above which there is a nonzero probability of a site being a member of the cluster of infinite extent. For an infinite structure, "edge effects" may be supposed negligible and we find by direct enumeration of the possible clusters on the triangular lattice:

$$k_1 = pq^6, \quad k_2 = 3p^2q^8, \quad k_3 = p^3(2q^9 + 9q^{10}), \quad k_4 = p^4(3q^{10} + 12q^{11} + 29q^{12}), \quad k_5 = p^5(6q^{11} + 21q^{12} + 66q^{13} + 93q^{14}). \tag{3.2}$$

We have derived further k_r and by substitution obtained the low- and high-density expansions for the number of finite black clusters as

$$k_L(p) = p - 3p^2 + 2p^3 + p^6 - p^7 + 3p^8 - 4p^9 + 9p^{10} - 15p^{11} + \cdots, \tag{3.3}$$

$$k_H(q) = \qquad\qquad q^6 - q^7 + 3q^8 - 4q^9 + 9q^{10} - 15q^{11} + \cdots. \tag{3.4}$$

These two series are valid in two separate regions (3.3) defining k_L for $p < p_c$, and (3.4) defining k_H for $p > p_c$. Again we shall suppose that these expansions converge for small values of their argument.[13] The remarkable matching of the coefficients suggests that we may write for this lattice

$$k_L(p) = \phi(p) + k_H(p), \quad (3.5)$$

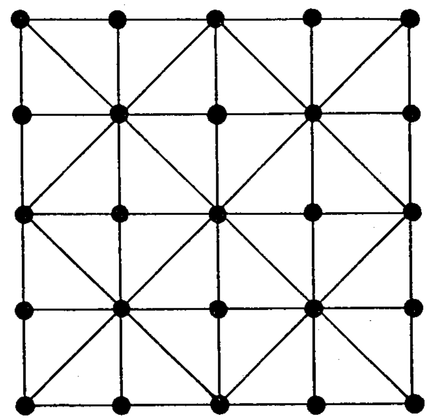
$$\phi(p) = p - 3p^2 + 2p^3. \quad (3.6)$$

We shall call $\phi(p)$ the matching polynomial, and the interpretation of (3.5) is the following:

At density p the mean number of black clusters differs from the mean number of white clusters by $\phi(p)$. (3.7)

The importance of this result lies in the observation that $\phi(p)$ is a finite polynomial. We show in Sec. 7 that the property (3.5) which we shall describe as a self-matching enables the critical probability to be located as $p_c = \frac{1}{2}$. Self-matching is a very special property confined to a very limited class of infinite lattices. It is noticed to occur on any infinite 2-dimensional lattice that is fully triangulated[14] and the fitting together of the triangular faces need not form a regular pattern. We illustrate one such lattice for which the triangles do form a regular pattern in Fig. 1(a). It is also noticed for the lattice illustrated in Fig. 1(b). This lattice, which is two-dimensional but not planar, yields a site problem that, by the well known bond-to-site transformation,[15] is isomorphic with the bond problem on the simple quadratic lattice. We shall usually find it convenient to regard bond problems as site problems on the corresponding *covering lattice*. (We give a short account of the bond-on-site transformation in Appendix II.) In this way we avoid the complication of a special notation to distinguish the two problems. For the simple quadratic bond problem we find

(a)



(b)

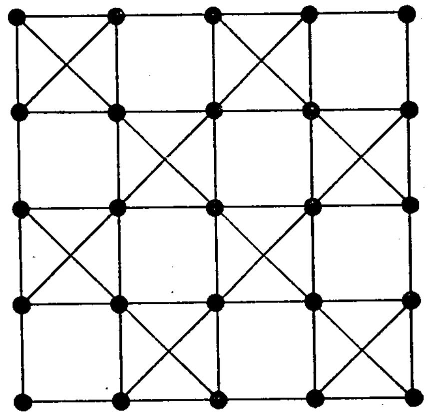


FIG. 1. Self-matching lattices. (a) Fully triangulated planar lattice; (b) covering lattice of the bond problem on the simple quadratic lattice.

$$k_L(p) = p - 3p^2 + 2p^3 + p^6 - p^7 + 3\tfrac{1}{2}p^8 - 6p^9 + 14p^{10} - 27p^{11} + 57\tfrac{1}{2}p^{12} - 118p^{13} + \cdots, \quad (3.8)$$

$$k_H(q) = q^6 - q^7 + 3\tfrac{1}{2}q^8 - 6q^9 + 14q^{10} - 27q^{11} + 57\tfrac{1}{2}q^{12} - 118q^{13} + \cdots, \quad (3.9)$$

and thus for this problem also

$$\phi(p) = p - 3p^2 + 2p^3. \quad (3.10)$$

When the mean number of clusters is expanded for the simple quadratic lattice we find

$$k_L(p) = p - 2p^2 + p^4 + p^8 - p^9 + 2p^{10} - 4p^{11} + 11p^{12} + \cdots, \quad (3.11)$$

$$k_H(q) = q^4 - q^5 + 2q^6 + 2q^8 - 3q^9 + 20q^{10} + \cdots, \quad (3.12)$$

and these two expansions do not match. However, if we also expand the mean number of clusters on the simple quadratic lattice with first- and second-neighbor bonds, we find, denoting the quantities for this case by an asterisk,

[13] It must not be assumed however that the radii of convergence of these expansions are p_c and $1 - p_c$, respectively.

[14] An infinite multiply connected planar graph all of whose finite faces are triangular.

[15] J. W. Essam and M. E. Fisher, J. Math. Phys. 2, 609 (1961).

$$k_L^*(p) = p - 4p^2 + 4p^3 + (-p^4 + p^4) - p^5 + 2p^6 + 2p^8 - 3p^9 + 20p^{10} + \cdots, \quad (3.13)$$

$$k_H^*(q) = \qquad q^8 - q^9 + 2q^{10} - 4q^{11} + 11q^{12} + \cdots, \quad (3.14)$$

and again these expansions do not match. However, it seems that (3.14) matches (3.11) and (3.13) matches (3.12). This suggests that we may write

$$k_L(p) = \phi(p) + k_H^*(p), \quad (3.15)$$

$$k_L^*(p) = \phi^*(p) + k_H(p), \quad (3.16)$$

$$\phi(p) = p - 2p^2 + p^4, \quad (3.17)$$

$$\phi^*(p) = p - 4p^2 + 4p^3 - p^4. \quad (3.18)$$

The statement equivalent to (3.7) is now

At density p the mean number of black clusters on the simple quadratic lattice differs from the mean number of white clusters on the simple quadratic lattice with first and second neighbors by $\phi(p)$. (3.19)

We show in Sec. 7 that this property which we shall call a cross-matching enables the critical probabilities of the matching pair p_c, p_c^* to be related by

$$p_c + p_c^* = 1. \quad (3.20)$$

We shall show in a subsequent paper that the coefficients in the series expansions of k_L and k_H can be related to certain enumerative problems on the lattice. By a closer examination of these enumerative problems, the matching properties illustrated in this section can be proved to hold term by term. In the following sections we develop a proof of the matching theorems that is more direct and derive them from certain simple results of the theory of linear graphs.

4. APPLICATION OF THE THEORY OF LINEAR GRAPHS TO THE SITE PROBLEM ON THE ICOSAHEDRON

In this section we establish the matching polynomial for the icosahedron as

$$12\phi(p) = 12p - 30p^2 + 20p^3 - 1. \quad (4.1)$$

We have already obtained this polynomial in Sec. 2 by a method that requires a complete enumeration of all possible clusters.

An account of the theory of linear graphs should be sought in the literature and, in particular, in the book of Berge.[16] We simply recall here by means of an example the results we require and illustrate our terminology. For a precise treatment, reference should be made to Berge, Chap. 4 (Cyclomatic Index), and Chap. 21 (Euler's Law of the Edges). In Fig. 2 we illustrate a typical planar linear graph (G). It has 12 sites, 11 bonds, and 2 finite faces. There is also an infinite face and there are 3 connected components. Denoting the number of sites by s, of bonds by b, of finite faces by f, and of the total number of faces including the infinite face by F $(= f + 1)$, and the number of connected components by n, then the *cyclomatic index* of the graph is defined to be

$$C(G) = b - s + n. \quad (4.2)$$

In our example $C(G) = 11 - 12 + 3 = 2$. The definition is not restricted to planar graphs, but for these we have an important result often known as Euler's Law of the Edges, which in its modern form states that, for a planar graph, the cyclomatic index is equal to the number of finite faces. For a planar graph we can thus write

$$f = b - s + n, \quad (4.3)$$

and we shall use this result in the form

$$n = s - b + F - 1. \quad (4.4)$$

In Fig. 3. we draw the icosahedron as a planar graph. It has one infinite (triangular) face and 19 finite triangular faces. The faces are all polygons but for a more general planar graph we have seen that this is not necessarily the case. For example, in the particular realization in which the four sites A, B, C, D are black and all the others white, the planar graph R_B has one finite face which is not a simple polygon. We shall call the sites adjacent to the face contour sites of that face, and the bonds joining two contour sites the contour bonds. In our example R_B, the points A, B, C, D are contour sites and AB, BD, AC, BD are contour bonds. To exploit the result (4.4) we require the following important Lemma:

For any realization R_B on the icosahedron, every face is either empty or contains one and only one connected component (white cluster) of R_W. (4.5)

For the present we shall regard this lemma as proved by an examination of all the possible realizations. A more general result, of which the present lemma is a particular case, is proved in Appendix I.

[16] C. Berge, *Theorie des Graphes et ses Applications* (Dunod Cie., Paris, 1958).

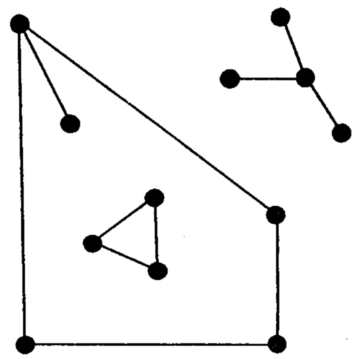


FIG. 2. Typical planar linear graph with 12 sites, 11 bonds, 2 finite faces, and 3 connected components.

Applying Euler's Law to the two graphs R_B and R_W we have from (4.4)

$$n_B = s_B - b_B + F_B - 1, \tag{4.6}$$

$$n_W = s_W - b_W + F_W - 1. \tag{4.7}$$

To apply the lemma, we observe that it follows that the number of white clusters is equal to the number of faces of R_B (black faces) that are not empty. Denoting the number of empty faces by $F_B(0)$, we can thus write

$$n_B = s_B - b_B + F_B(0) - 1 + n_W. \tag{4.8}$$

This result holds for all realizations, and we can therefore write the average sign through it and substitute

$$\begin{gathered}\langle s_B\rangle = 12p;\ \ \langle b_B\rangle = 30p^2;\ \ \langle n_B\rangle = K(p),\\ \langle n_W\rangle = K(q);\qquad \langle F_B(0)\rangle = 20p^3.\end{gathered} \tag{4.9}$$

The last entry results from the observation that there are 20 faces (all triangles and including the infinite face) that could be empty. We obtain

$$K(p) = 12p - 30p^2 + 20p^3 - 1 + K(q). \tag{4.10}$$

Thus the difference between the mean number of black and white clusters is the matching polynomial (4.1).

The method of this section can be applied without modification to the tetrahedron and the octahedron for which the lemma holds. For the tetrahedron

$$4\phi(p) = 4p - 6p^2 + 4p^3 - 1. \tag{4.11}$$

For any finite section of the plane triangular lattice the lemma will be found to hold except for the infinite face. The general treatment of the next section enables a proper account of the infinite face to be taken if required, but for our present purposes we assume, as it is certainly reasonable to do, that, as we require the mean number per site, we may neglect these edge effects. We then have at once as $N \to \infty$

$$\langle s_B/N\rangle = p, \quad \langle b_B/N\rangle = 3p^2, \quad \langle F_B(0)/N\rangle = 2p^3,$$

and therefore for this lattice,[17]

$$\phi(p) = p - 3p^2 + 2p^3, \tag{4.12}$$

as already surmised in Sec. 3.

We observe that this polynomial vanishes at $p = 0$, $\frac{1}{2}$, and 1—a result that is readily understood from (3.7).

5. GENERAL MATCHING PROPERTY FOR DECORATED MOSAICS

We shall use the term mosaic[18] to describe a planar graph or infinite planar lattice which is connected and has no articulation points. Such a graph has finite faces whose contour bonds form non-self-intersecting polygons. For such a graph the infinite face also has a polygonal contour, and we shall use the term face without qualifications to include the infinite face.

We now choose a mosaic M (the parent mosaic) and "decorate" it by drawing in all the possible diagonals on some of its polygonal faces, inside those faces, to form a new graph L (which is not necessarily planar). We shall call the operation of drawing in all the possible diagonals of a selected face close packing, and it results in the polygonal cluster formed by the n_c contour sites becoming the complete graph[16] of n_c sites or a close-packed cluster of n_c sites.[19] The graph L will be called a *decorated mosaic*.

We define the matching graph[20] L^* of the graph L to be the decorated mosaic graph which results from close packing all those faces of the parent mosaic M of L which were not close-packed to form L.[21] It follows from the symmetry of the defi-

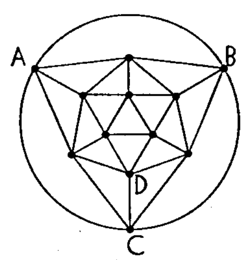


FIG. 3. The Icosahedron drawn as a planar graph.

[17] On an infinite lattice we shall write ϕ for the matching polynomials per site.

[18] A mosaic is simply a multiply-connected planar graph. We shall find the word mosaic conveniently short.

[19] C. Domb, Phil. Mag. Suppl. 9, 149 (1960).

[20] We anticipate in these definitions the results of previous sections.

[21] Triangular faces remain invariant under close packing.

nition that if L^* is the matching graph of L then L is also the matching graph of L^*. We shall call L and L^* a matching pair.

We shall take it as obvious that no bond of L crosses any bond of L^*, and the converse. For the infinite face the diagonals that result from close packing will be supposed drawn in the infinite face. A mosaic M can usually give rise to a variety of matching pairs. For example, the mosaic formed by the infinite simple quadratic lattice yields two important matching pairs[22]:

(i) L = Simple quadratic lattice. (5.1)
L^* = Simple quadratic matching lattice or simple quadratic lattice with first- and second-neighbor bonds.

(ii) L = Simple quadratic lattice in which alternate squares are replaced by tetrahedra [Sec. 3, Fig. 1 (b)]. (5.2)
$L^* = L$.

In this last example the two lattices are topologically identical and will be described as self-matching. (For our present purpose of examining percolation on graphs all of whose sites are occupied with equal probability, this definition will suffice. A slightly more detailed examination is required if the probabilities are not distributed equally and the sites are therefore labeled).

At this juncture we notice that all bond problems on mosaic lattices without multiple bonds can be made to correspond by the well known bond-to-site transformation to a site problem on a "covering lattice" which is a decorated mosaic. We notice further that the covering lattices of a lattice and its dual lattice form a matching pair of decorated mosaics. The proof of this is elementary.

As an example, the bond problem on the simple quadratic lattice is isomorphic with the site problem on the lattice L of (5.2). The property $L^* = L$ corresponds to the self-dual property of the simple quadratic lattice. We further illustrate these properties in Appendix II.

For a particular realization of the probability distribution on the parent mosaic M and the corresponding realizations on L and L^*, we have four graphs,

R_B and R_W on L, R_B^* and R_W^* on L^*,

and these graphs are not necessarily planar. However any realization, R_B say, will be made up of connected components some of which may contain subsets of points forming complete close-packed graphs. We shall extend our definition of a face to include these complete graphs as faces (close-packed faces). It is clear that these complete graphs contain no white sites and are therefore always empty.

[22] We neglect the infinite face.

We now state an important property of a matching pair of decorated mosaics.

Any two white sites in a face of R_B on L are connected on R_W^* on L^* and conversely if two white sites are connected on R_W^* on L^* then they both lie in the same face of R_B on L. (5.3)

The reader will readily satisfy himself of the truth of this statement by drawing a few examples. We relegate the rather tedious proof to the Appendix I.

For a decorated mosaic, Euler's Law of the Edges is not immediately applicable since the graph is not planar. In general, a realization R_B on a decorated mosaic will contain some close-packed faces. The result (4.4) will hold for the graph formed by unpacking all the close-packed faces of R_B. If we close-pack the faces in turn, each face of α points will increase the number of bonds by $\frac{1}{2}\alpha(\alpha-3)$ without alteration of the total number of faces. We must therefore write

$$n = s - b + \sum_F \tfrac{1}{2}(\alpha^2 - 3\alpha + 2) + F - 1, \quad (5.4)$$

where the summation is taken over all the faces that are close-packed and F is the number of faces that are not close-packed. We shall make the substitution

$$F = F(0) + F', \quad (5.5)$$

where $F(0)$ is the number of empty faces (i.e., contain no white sites) that are not close-packed and write the resulting rather cumbersome expression as

$$n = \Phi + F'. \quad (5.6)$$

This equation is the modified form of Euler's Law of the Edges applicable to realizations on decorated mosaics.

6. GENERAL MATCHING THEOREM FOR A DECORATED MOSAIC

We now combine the results (5.3) and (5.6) of the previous section for the four graphs

R_B and R_W on L, R_B^* and R_W^* on L^*,

and the argument is simply a generalization of that used for the icosahedron in Sec. 4. Applying first Euler's Law (5.6) we have

$$n_B = \Phi_B + F'_B, \quad (6.1)$$

$$n_B^* = \Phi_B^* + F'^*_B, \quad (6.2)$$

TABLE II. Matching polynomials for the more usual lattices.

Triangular lattice (self-matching)	$\phi(p) = p - 3p^2 + 2p^3$
Simple quadratic lattice	$\phi(p) = p - 2p^2 + p^4$
Simple quadratic matching lattice (S.Q. with first- and second-neighbor bonds)	$\phi(p) = p - 4p^2 + 4p^3 - p^4$
Honeycomb lattice	$\phi(p) = \frac{1}{2}(2p - 3p^2 + p^6)$
Honeycomb matching lattice (honeycomb with first-, second- and third-neighbor bonds)	$\phi(p) = \frac{1}{2}(2p - 12p^2 + 20p^3 - 15p^4 + 6p^5 - p^6)$
Kagomé lattice (the covering lattice of the honeycomb bond problem)	$\phi(p) = \frac{1}{3}(3p - 6p^2 + 2p^3 + p^6)$
Kagomé matching lattice (the covering lattice of the triangular bond problem)	$\phi(p) = \frac{1}{3}(3p - 15p^2 + 22p^3 - 15p^4 + 6p^5 - p^6)$
Kagomé covering lattice (covering lattice of the Kagomé bond problem)	$\phi(p) = \frac{1}{6}(6p - 18p^2 + 14p^3 - 3p^4 + p^6)$
Matching lattice for Kagomé covering lattice (covering lattice of the dice lattice bond problem)	$\phi(p) = \frac{1}{6}(6p - 21p^2 + 22p^3 - 12p^4 + 6p^5 - p^6)$
Covering lattice for the simple quadratic bond problem (self-matching)	$\phi(p) = p - 3p^2 + 2p^3$

and from the matching property (5.3), we have

$$n_W^* = F'_B, \tag{6.3}$$

$$n_W = F'^*_B. \tag{6.4}$$

By substitution of (6.3) in (6.1), and (6.4) in (6.2),

$$n_B = \Phi_B + n_W^*, \tag{6.5}$$

$$n_B^* = \Phi_B^* + n_W. \tag{6.6}$$

On averaging over all the possible realizations and writing

$$\langle\Phi_B\rangle = N\phi(p), \tag{6.7}$$

$$\langle\Phi_B^*\rangle = N\phi^*(p), \tag{6.8}$$

we obtain

$$K(p; L) = N\phi(p) + K(q; L^*), \tag{6.9}$$

$$K(p; L^*) = N\phi^*(p) + K(q; L), \tag{6.10}$$

or

$$K_L(p; L) = N\phi(p) + K_H(p; L^*), \tag{6.11}$$

$$K_L(p; L^*) = N\phi^*(p) + K_H(p; L). \tag{6.12}$$

The result is conveniently expressed in words:

At density p the mean number of black clusters on L differs from the mean number of white clusters on L by $\phi(p)$.*

The polynomial $\phi(p)$ we have called the matching polynomial of L. It follows from the relations (2.12) and (2.13) that for a matching pair

$$\phi(p) = -\phi^*(1 - p). \tag{6.13}$$

The matching polynomial is readily obtained from the definitive equation

$$N\phi(p) = \langle s - b + F(0) + \sum \tfrac{1}{2}(\alpha^2 - 3\alpha + 2) - 1\rangle. \tag{6.14}$$

For example, for the simple quadratic covering lattice (5.2) we have, working per site, $\langle s/N\rangle = p$ $\langle b/N\rangle = 3p^2$, and the possible empty faces are a quadrilateral ($\frac{1}{2}N$) which has expectation $\frac{1}{2}p^4$, a tetrahedron ($\frac{1}{2}N$) which has weight 3 and therefore contributes $1\frac{1}{2}p^4$, or a triangle which does not occur in a tetrahedron and this has expectation $2p^3 - 2p^4$. Thus

$$\phi(p) = p - 3p^2 + 2p^3, \tag{6.15}$$

and we notice that for this lattice, which is self matching,

$$\phi(1 - p) = -p + 3p^2 - 2p^3; \tag{6.16}$$

as required by (6.13).

We summarize in Table II the matching polynomials for the more usual lattices.

7. CRITICAL PERCOLATION PROBABILITIES

The matching property (6.11–12) makes it possible to derive a property of the critical percolation probabilities of certain infinite lattices. Suppose first that the lattice is self-matching. Then from (6.9)

$$k(p;L)=\phi(p)+k(q;L). \tag{7.1}$$

Since $\phi(p)$ is a finite polynomial, its behavior is nonsingular. We shall suppose, without offering proof, that for real p ($0\leq p\leq 1$) the function K is singular at $p=p_c$, but nowhere else. This is to be expected in the light of exact results for closely related problems, and in particular, for percolation problems on lattices of the Bethe type for which K has been given exactly.[15] Now following closely the method of Kramers and Wannier in their derivation of the critical temperature of the Ising model we argue as follows.

From (7.1), if K is singular at p_c then it is also singular at $1-p_c$, and if there is only one singularity these must be identical points, or

$$p_c=\tfrac{1}{2}. \tag{7.2}$$

This establishes two important percolation probabilities as $\frac{1}{2}$—that for the site problem on the triangular lattice and that for the bond problem on the simple quadratic lattice. The result (7.2) holds for any fully triangulated lattice.

For a matching pair we observe that if $k(L)$ is singular at p_c, then $k(L^*)$ is singular at $1-p_c$. Thus if $k(L^*)$ has only one singularity at p_c^*, we must have

$$p_c+p_c^*=1, \tag{7.3}$$

or the critical points are complementary.

To determine p_c, p_c^* we need a second relation. We have been able to find such a relation in only one case—the matching pair formed by the bond problem on the honeycomb and triangular lattices. To derive this relation we shall depart from our practice hitherto and study this pair directly as a bond problem. (This is not essential but is visually simpler.) We suppose that the bonds of the triangular lattice are occupied with probabilities ξ, η, ζ, along the usual three directions and that those of the honeycomb are occupied with probabilities x, y, z so orientated that ξ crosses x on the dual, etc. By an obvious extension of the preceding argument, if we assume there is only one critical locus for the honeycomb

$$\Psi(x,y,z)=0, \tag{7.4}$$

then the corresponding locus for the triangular lattice must be

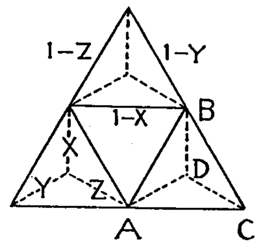


FIG. 4. Star–triangle overlapping of the triangular and honeycomb lattices with the probabilities as assigned in the text.

$$\Psi(1-\xi,1-\eta,1-\zeta)=0. \tag{7.5}$$

We can satisfy this condition in the treatment that follows by supposing the bonds of the triangular lattice to be occupied with the complementary probabilities $1-x$, $1-y$, $1-z$, respectively.

In Fig. 4 we draw the well known star–triangle overlapping of the two lattices. We calculate for one individual "star–triangle" the probabilities (a) of A being connected to neither B nor C; (b) of A being connected to B but not C; (c) of A being connected to C but not B; (d) of A being connected to both B and C. We find

	Honeycomb	Triangular	
(a)	$1-xy-yz+xyz$	xz	(7.6)
(b)	$xy(1-z)$	$xy(1-z)$	
(c)	$zy(1-x)$	$zy(1-x)$	
(d)	xyz	$1-xy-yz-xz+2xyz$	

We notice that the conditions (b) and (c) are satisfied with equal probabilities on the two lattices as a result of our choice of complementary probabilities. We can obtain equality for both the remaining conditions if we select x, y, z to satisfy

$$1-xy-yz-xz+xyz=0. \tag{7.7}$$

On this locus the connectivity of each individual star–triangle will be identical. Thus the occurrence of an infinite cluster on one lattice would imply such an occurrence on the other, and by (7.5) and (7.4) these are mutually exclusive events except on the critical locus. Thus (7.7) is the critical locus and by substitution we obtain it as a function of the probabilities on the triangular lattice:

$$1-\xi-\eta-\zeta+\xi\eta\zeta=0. \tag{7.8}$$

From (7.8) by setting $\zeta=0$ we obtain for the simple quadratic bond problem with two different probabilities at right angles the locus[23]

$$\xi+\eta=1. \tag{7.9}$$

[23] The result for the asymmetric simple quadratic may be obtained directly by exploiting the self-dual property but, as remarked earlier, the matching lattice must be carefully defined and is the same lattice with the roles of ξ and η reversed.

For this case the matching polynomial factorizes as

$$\phi(\xi, \eta) = (1 - \xi - \eta)(\xi + \eta - 2\xi\eta), \qquad (7.10)$$

and this vanishes along the critical locus. Thus K is continuous there. The result can be established for any self-matching lattice.

For the symmetric triangular lattice we obtain from (7.8) the cubic

$$1 - 3p + p^3 = 0, \qquad (7.11)$$

which has only one root between 0 and 1 and yields

$$\begin{aligned} p_c &= 2 \sin \tfrac{1}{18}\pi = 0.347296 \quad \text{(triangular)}, \\ p_c &= 0.652704 \quad \text{(honeycomb)}. \end{aligned} \qquad (7.12)$$

We notice that these results apply equally to the isomorphic site problem on the Kagomé lattice for which $p_c = 0.652704$.

8. CONCLUSIONS

We have defined a class of two-dimensional lattices (decorated mosaics) and proved a matching theorem which relates the mean number of black clusters on such a mosaic to the mean number of white clusters on another mosaic, the matching mosaic. The mean number of clusters for a matching pair is related by the results (6.9–12) which can be summarized symmetrically as

$$k(p; L) - \tfrac{1}{2}\phi(p) = k(q; L^*) - \tfrac{1}{2}\phi^*(q). \qquad (8.1)$$

When the matching pair are identical the critical probability is a $\frac{1}{2}$, and we have established this result for the site problem on the plane triangular lattice and the bond problem on the simple quadratic lattice.

We have also found that the mean number of clusters is continuous at p_c for a self-matching lattice. We shall examine the continuity of higher derivatives of K, and the continuity of K for cross-matching lattices in a subsequent paper.

The class of decorated mosaics for which (8.1) applies includes all bond problems on multiply-connected planar graphs without multiple bonds. This can be proved separately by repeating the arguments of this paper directly for the bond problem on a graph and its dual. We have preferred to work on the covering lattice so as to include the more general class of matching decorated mosaics, some of which do not correspond to bond problems.

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APPENDIX I. PROOF OF THE MATCHING PROPERTY (5.3)

Any individual face of R_B on L is an area which we shall regard as made up of smaller units of area which can be classified as follows:

(1) Primary units, which are faces of the parent mosaic, and which arise from undecorated faces of M occurring in faces of R_B on L.

(2) Secondary units, which correspond to decorated faces of M or fractions thereof.

Each unit will have a polygonal contour which we shall call its edge, and we shall refer to the bonds and sites that form the polygonal boundary as its edge bonds and edge sites. These edge bonds can be black, white or uncolored for a particular realization R_B, R_W on L. Edge bonds that are white or uncolored for a particular realization will be called *internal* edges. We shall assume without proof that if two units A and B are units of the same face of R_B, then it is possible to find at least one path from A to B in the plane which crosses only internal edges. (This is equivalent to the elementary definitive property that a face has a connected interior).

We now show that it follows that any white edge site of A is connected to any white edge site of B on R_W^*. It will suffice to prove the result for two adjacent units (i.e., units having at least one internal edge in common).

In Fig. 5 let PQ be a common internal edge of A and B. Then at least one of P and Q must be white. Suppose P is white. Now suppose that X and Y are white edge sites of A and B, respectively. (We shall assume X $\neq$ P, Y $\neq$ P; if X = P and or Y = P then the proof is shorter).
Now

(1) if A is a primary unit, then on L^* this face will be close-packed and therefore X connected to P on L^*. Since both X and P are white, X is connected to P on R_W^* on L^*.

(2) If A is a secondary unit, then X must be connected to P along the edge sites of A. For if not then there is at least one black site along each of the two possible routes, and since the unit is secondary, these black sites must be connected on L which vitiates our assumption that A is a unit of one face. The connection along the edge will not be destroyed

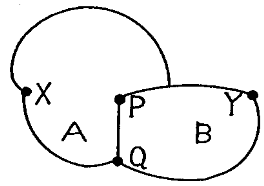

FIG. 5. Two adjacent units of area with a common internal edge PQ.

by "unpacking" the face.[24] Therefore X is connected to P on L^*. Likewise, Y is connected to P on L^*, and therefore X is connected to Y on L^*; and since X, Y, P and any sites employed on secondary units are white, X is connected to Y on R_W^*.

Thus any two white sites in a face of R_B on L are connected on R_W^* on L^*.

To establish the converse we notice that if two white sites X, Y are in different faces of R_B on L and are connected on R_W^* on L^* then at least one bond on R_W^* must cross a black contour. But no bond of L^* crosses any bond of L.

Thus if two white sites are connected on R_W^* on L^* they both lie in the same face of R_B on L.

This completes the proof of (5.3).

APPENDIX II. ILLUSTRATION OF THE BOND-TO-SITE TRANSFORMATION AND THE MATCHING CLUSTER FOR THE TRIANGULAR PRISM

The bond-to-site transformation introduced by Fisher and Essam[15] has also been described by Fisher,[25] who introduced the term covering lattice, and also by Dean[26] and Hammersley and Frisch.[6] The covering graph for the bond problem on a graph is constructed by replacing each bond of the graph by a site (placed at its center) and linking these sites together by sufficient new bonds to ensure that if two bonds of the original graph meet, then the corresponding two sites of the covering graph are joined by a direct bond and vice versa. We illustrate the transformation in Fig. 6.

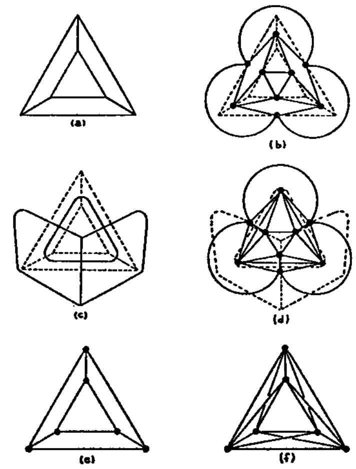

FIG. 6. Illustration of the bond-to-site transformation and the matching cluster for the triangular prism. (a) Bond problem on prism; (b) covering cluster (site problem); (c) bond problem on dual of prism; (d) covering cluster of dual of prism [this is the matching cluster for (b)]; (e) site problem on prism; (f) matching cluster for the site problem on prism.

[24] This is true because the edge bonds of the secondary face used for the connection are also bonds of the parent mosaic.

[25] M. E. Fisher, J. Math. Phys. 4, 620 (1961).

[26] P. Dean, Proc. Cambridge Phil. Soc. 59, 397 (1963).