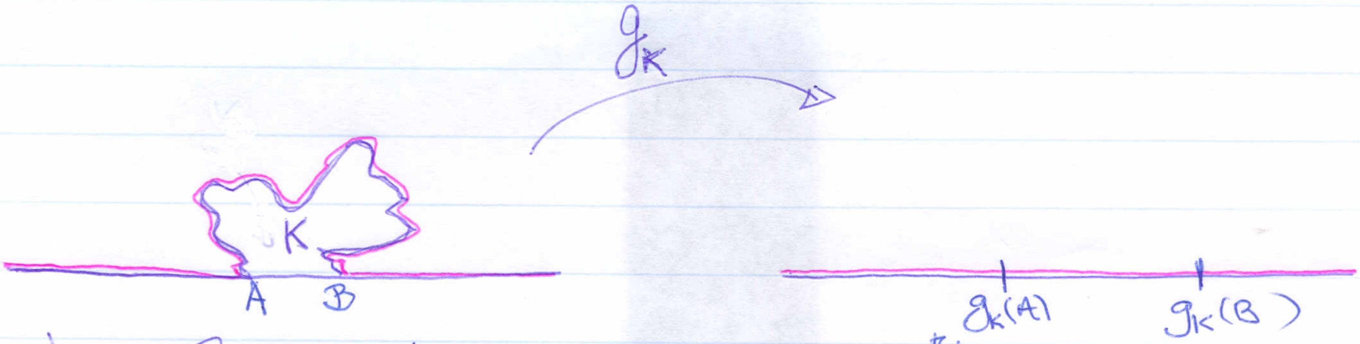
Chordal Löwner Equation (Karl Löwner 1922)

$\mathbb{H} := \{ z \in \mathbb{C} : \operatorname{Im} z > 0 \}$

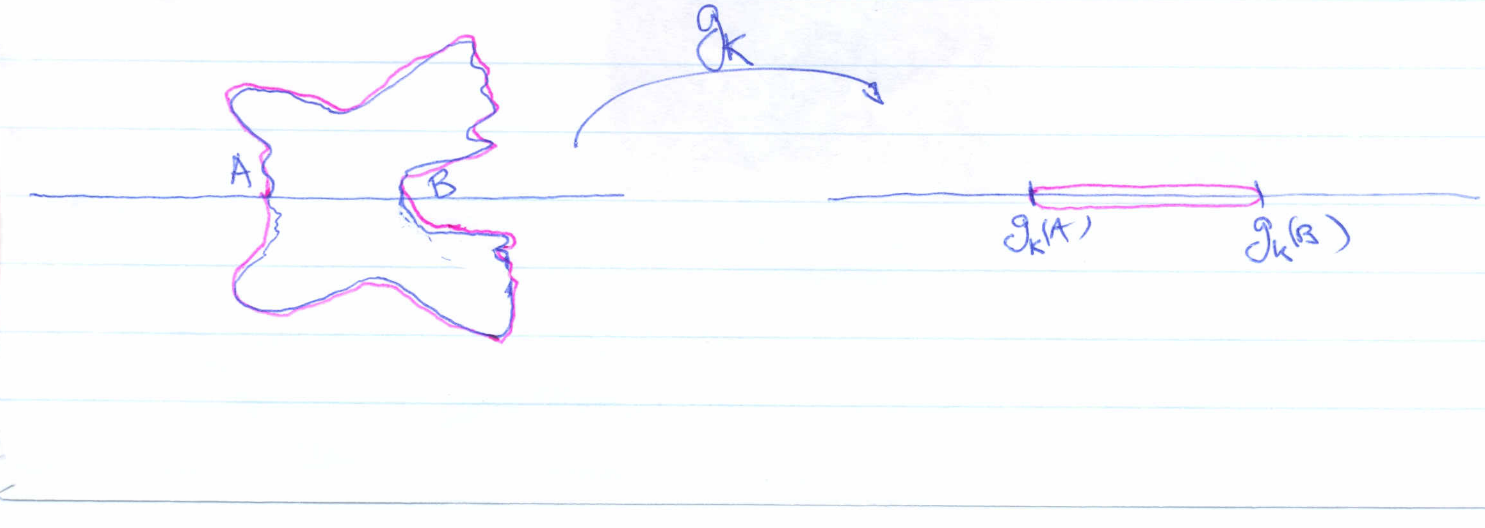
Def | $K \subset \overline{\mathbb{H}}$ compact is a $\boxed{\text{hull}}$ iff $K \cap \mathbb{H} \neq \emptyset$
both K & $\mathbb{H} \setminus K$ are simply connected



by Riemann Mapping Thm: $\exists\, g_K : \mathbb{H} \setminus K \longrightarrow \mathbb{H}$ $\underline{\text{Conformal}}$.
with $g_K(\infty) = \infty$

[not unique: two real parameters left free]

by Schwarz reflection g_K extends analytically to



Hence

$$g_K(z) = a_{-1} z + a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots \quad \text{near } z=\infty.$$

with $a_{-1} > 0$, $a_j \in \mathbb{R}$ $j = 0, 1, 2, \ldots$

Canonical choice: $a_{-1} = 1$, $a_0 = 0$

["hydrodynamic normalization"]

$$g_K(z) = z + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots$$

$\exists!$ conformal mapping $g_K : \mathbb{H} \setminus K \to \mathbb{H}$

$g_K(\infty) = \infty$

with expansion:

$$g_K(z) = z + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots \quad \text{near } z = \infty$$

The coefficients $a_j = a_j(K)$ $j = 1, 2, \ldots$ are real.

$K \subseteq L \Rightarrow J := g_K(L \setminus K)$ is a hull and

$$g_J \circ g_K = g_L$$

in particular: $a_1(L) = a_1(K) + a_1(J)$

$a_1(K) > 0$ "capacity of K"

Proof & probabilistic meaning:

$B_t :=$ Brownian motion on $\mathbb{H}$

$\tau = \tau_K := \inf\{t : B_t \in K \cup \mathbb{R}\} < \infty$ a.s.

$\varphi_K : \mathbb{H} \setminus K \to \mathbb{R}$ $\varphi_K(z) = \operatorname{Im}(g_K(z) - z)$

φ_K is harmonic and bounded

$\Longrightarrow \varphi_K(B_t)$ is a bdd. martingale

By Doob:

$$\operatorname{Im}(g_K(z) - z) = \mathbb{E}_z\big(\underbrace{\operatorname{Im} g_K(B_\tau)}_{=0} - \operatorname{Im} B_\tau\big)$$

$$= -\mathbb{E}_z(\operatorname{Im} B_\tau)$$

$z = iy$, $y \to \infty$

$$\boxed{a_1(K) = \lim_{y\to\infty}\{y \cdot \mathbb{E}_{iy}(\operatorname{Im} B_{\tau_K})\}}$$

Lemme (to be used later).

$K_\varepsilon \subseteq D_\varepsilon = \{z : |z| < \varepsilon\}$, $\varepsilon \downarrow 0$

$z = x+iy$ fixed. $y > 0$

$$\mathrm{Im}\left(g_{K_\varepsilon}(z) - z\right) = - a_1(K_\varepsilon)\frac{y}{x^2+y^2}\left(1+o(1)\right)$$

$\varepsilon \downarrow 0.$

$$= a_1(K_\varepsilon)\,\mathrm{Im}\,\frac{1}{z}\left(1+o(1)\right).$$

Proof: $\tau_\varepsilon = \tau_{K_\varepsilon} := \inf\{t : B_t \in K_\varepsilon \cup \mathbb{R}\}$

$\sigma_\varepsilon := \inf\{t : B_t \in D_\varepsilon\}$

$$\mathrm{Im}\left(z - g_{K_\varepsilon}(z)\right) = \mathbb{E}_z\left(\mathrm{Im}\, B_{\tau_\varepsilon}\right) =$$

$$= \underbrace{\mathbb{E}_z\left(\mathrm{Im}\, B_{\tau_\varepsilon} \,\middle|\, \sigma_\varepsilon \le \tau_\varepsilon\right)}_{\substack{\| \\ \frac{\pi}{4}\frac{a_1(K_\varepsilon)}{\varepsilon}(1+o(1)) \\ \text{indep of } z}} \underbrace{\mathbb{P}_z\left(\sigma_\varepsilon \le \tau_\varepsilon\right)}_{\substack{\| \\ \underbrace{\frac{4}{\pi}\,\varepsilon\cdot\frac{y}{x^2+y^2}}_{\text{explicit comput.}}(1+o(1))}}$$

$K \mapsto a_1(K)$ is strictly increasing & continuous (in some strong sense?).

scaling of g_K: let $\alpha > 0$

$$g_{\alpha K}(z) = \alpha\, g_K(z/\alpha)$$

translations: $b \in \mathbb{R}$

$$g_{K+b}(z) = g_K(z-b) + b$$

in particular: $a_1(\alpha K) = \alpha^2 a_1(K)$

[also follows from the previous derivation]

Löwner for simple curves:

%

see the same for "locally growing domains"

$\gamma : (0,T) \to \mathbb{H}$ simple continuous curve $T \leq \infty$,

$$\gamma(0) := \lim_{t \downarrow 0} \gamma(t) = 0$$

$$\gamma(T) := \lim_{t \uparrow T} \gamma(t) = \infty$$

$K_t := \gamma[0,t]$

reparametrize so that $a_1(K_t) = 2t$, $T = \infty$ ✓

, K_t "pivotally growing domains":

$>s \Rightarrow K_t \supsetneq K_s$

$\bigcap_{\varepsilon} \overline{K_{t+\varepsilon} \quad K_t} = \{\gamma(t)\}$ $\quad \gamma(t) \in \partial K_t$

$t \mapsto \gamma(t)$ continuous $\underline{\text{on } \partial K_t}$
[no crossings!]

$_0 = \{0\}$ $\quad \gamma(t) \to \infty$ as $t \to \infty$.

parametrized so that $a_1(K_t) = 2t$

$$T_z = \inf\{t : z \in K_t\} = \inf\{t : \gamma(t) = z\}$$

$$g_t = g_{K_t} : \underbrace{\mathbb{H}\setminus K_t}_{\overset{\parallel}{\{z \in \mathbb{H} \,;\, \cancel{z}\, T_z > t\}}} \to \mathbb{H}$$

Prop ① g_t extends continuously to γ_t &

$t \mapsto U(t) := g_t(\gamma(t)) \in \mathbb{R}$
is continuous

2) for $z \in \mathbb{H}$, $t \in [0, T_z)$

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U(t)}, \quad g_0(z) = z$$

$z \in \mathbb{H}$: $T_z = \inf\{t : z \in K_t\}$

denote $g_t = g_{K_t} : \underbrace{\mathbb{H}\setminus K_t}_{\substack{\| \\ \{z \in \mathbb{H} \,:\, T_z > t\}}} \to \mathbb{H}$.

Proposition: ① g_t extends continuously to $\gamma(t)$ and $U(t) := g_t(\gamma(t))$ is continuous

② For $z \in \mathbb{H}$, $t \in [0, T_z)$

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U(t)}, \quad g_0(z) = z.$$

Proof: suff. to prove $\partial_t \,\mathrm{Im}\left(g_t(z) - z\right) = \mathrm{Im}\,\dfrac{2}{g_t(z) - U(t)}$ $\circledast$

[Indeed $\circledast \Rightarrow$

$$\mathrm{Im}\left(g_t(z) - z\right) = \int_0^t \mathrm{Im}\,\frac{2}{g_s(z) - U(s)}\,ds + \underbrace{\mathrm{Im}\,\cancel{\left(g_0(z) - z\right)}}_{=0}$$

$$= \mathrm{Im}\int_0^t \frac{2}{g_s(z) - U(s)}\,ds$$

$\Rightarrow$ [by Cauchy-Riemann]

$$\mathrm{Re}\left(g_t(z)-z\right) = \mathrm{Re}\int_0^t \frac{2}{g_s(z)-U(s)}\,ds + \underbrace{C_t}_{\in\mathbb{R}},\ \text{indep. of } z$$

$z \to \infty$: $g_t(z) - z \longrightarrow 0$

$\left(g_s(z)-U(s)\right)^{-1} \longrightarrow 0$ unif $s \in [0,t]$

$$\Rightarrow \mathrm{Re}\left(g_t(z)-z\right) = \int_0^t \mathrm{Re}\,\frac{2}{g_s(z)-U(s)}\,ds$$

$$\Rightarrow \partial_t \mathrm{Re}\left(g_t(z)-z\right) = \mathrm{Re}\,\frac{2}{g_s(z)-U(s)} \quad]$$

proof of ($*$):

sufficient to prove for $t = 0$, $U(0) = \gamma(0) = 0$

[due to the "semigroup? property": (and real translation invariance)

$u > t$: $K_{u,t} := g_t(\gamma[t,u])$

$h_{u,t} := g_{K_{u,t}}$

$u \geq t \geq s \geq 0$:

$h_{us} = h_{ut} \circ h_{ts}$]

+ real translations ($*$) follows from Lemma on page (3b)

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U(t)} \qquad g_0(z) = z$$

$$f_t(z) := g_t(z) - U(t)$$

$$\partial_t f_t(z) = \frac{2}{f_t(z)} - \dot{U}(t) \; ; \; f_0(z) = z$$

$$f_t(z) = z + \int_0^t \frac{2}{f_s(z)} \, ds - U(t)$$

Scaling $\alpha > 0$ (still deterministic) $\tilde{K}_t = \alpha^{-1} K_{\alpha^2 t}$

dilation: the only conformal maps $\mathbb{H} \circlearrowleft$ which fix 0 and ∞

reparametrization so that $a_1(\tilde{K}_t) = t$

$$\tilde{g}_t(z) := g_{\tilde{K}_t}(z) = \alpha^{-1} g_{\alpha^2 t}(\alpha z)$$

$$\partial_t \tilde{g}_t(t) = \alpha \left[\partial_t g\right]_{\alpha^2 t}(\alpha z) =$$

$$\frac{2\alpha}{g_{\alpha^2 t}(\alpha z) - U(\alpha^2 t)} = \frac{2}{\tilde{g}_t(z) - \alpha^{-1} U(\alpha^2 t)}$$

$$\boxed{\partial_t \tilde{g}_t(z) = \frac{2}{\tilde{g}_t(z) - \tilde{U}(t)} \; ; \; \tilde{U}(t) = \alpha^{-1} U(\alpha^2 t)}$$

The way backwards: from $U(t)$ to (K_t, γ_t):

$U: [0,\infty]$ continuous, $U(0)=0$ ✓

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U(t)} \qquad g_0(z) = z, \quad z \in \overline{\mathbb{H}}.$$

$$T_z := \sup\{t : s \in [0,t) \quad g_s(z) \neq U(s)\}^{*\%}$$

$K_t := \{z \in \overline{\mathbb{H}} : T_z \leq t\}$ compact, ↑

$H_t := \{z \in \mathbb{H} : T_z > t\}$ open, simply connected
$= \mathbb{H} \setminus K_t$

Prop: $g_t : \mathbb{H} \setminus K_t \longrightarrow \mathbb{H}$ is conformal, onto

$$g_t = g_{K_t}$$

Proof:

- $z \in \mathbb{H} \setminus K_t$: $\quad g_t'(z) = \exp - \int_0^t \frac{2}{(g_s(z) - U_s)^2}\, ds \in \mathbb{C} \setminus \{0\}$
- $\operatorname{Im} g_t(z) \geq 0 \Rightarrow \operatorname{Ran} g_t \subseteq \mathbb{H}.$
- backwards flow: $\partial_s g_s^{*(t)}(z) = -\frac{2}{g_s^*(z) - U(t-s)}$ $\quad s \in [0,t) \quad g_0^*(z) = z$, $z \in \mathbb{H}$

$$\tilde{T}_z = \sup\{t : s \in [0,t) : \text{Im}\, g_s(z) > 0\}$$

a priori $\tilde{T}_z \leq T_z$

in fact $\tilde{T}_z = T_z$. ✓

$$g_s^{*(t)}\left(g_t(z)\right) = g_{t-s}(z) \qquad 0 \leq s \leq t$$

$$g_t^{*(t)} = g_t^{-1}$$

hydrodynamic normalization

Question : Is K_t a hull generated by a path γ_t for general driving U_t ?

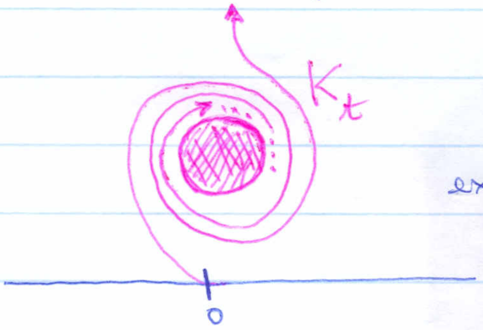
Deterministic results (Marshall & Rhode 2001)

(i) If $t \mapsto U(t)$ is Hölder-1/2 with

$$\sup_t \overline{\lim_{h\to 0}} \frac{|U_{t_0+h} - U_t|}{\sqrt{h}}$$ sufficiently small : YES

(ii) In general NO ! Counterexample : $\exists\ t \mapsto U(t)$ smooth except of one $t_0 \in (0,\infty)$ and

$$\overline{\lim_{h\to 0}} \frac{|U_{t_0+h} - U_{t_0}|}{\sqrt{h}}$$ finite but large

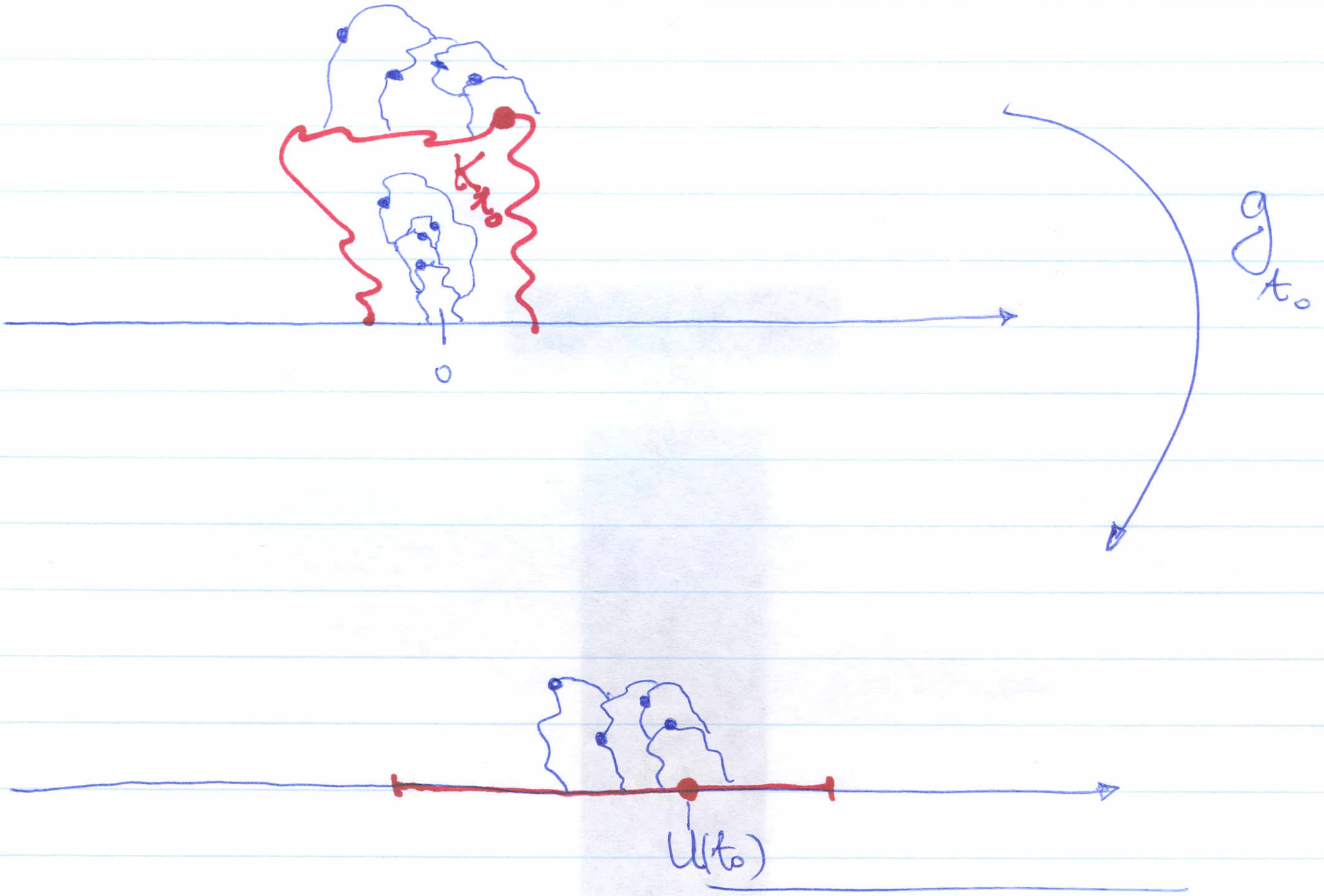


explain...

Mind that for $U_t = \kappa$ Brownian motion

a.s. $\forall t$ $$\overline{\lim_{h\to 0}} \frac{|U_{t+h} - U_t|}{\sqrt{h \log\log 1/h}} = K$$

Stochastic Löwner evolution



$$K_{t_0,\,t_0+t} := g_{t_0}\left(K_{t_0+t} \setminus K_{t_0}\right)$$

Requirements:

① stationary and independent "increments"

$$\{t \longmapsto K_{t_0,\,t_0+t} - U(t_0) : t \geq 0\}$$

|| in law

$$\{t \longmapsto K_t : t \geq 0\}$$

and independent of $\{s \mapsto K_s : s \in [0, t_0]\}$

② $R : \mathbb{H} \to \mathbb{H}$ $\quad R(x+iy) = -x+iy$

$$\{t \to RK_t : t \geq 0\} \overset{law}{=} \{t \to K_t : t \geq 0\}$$

In terms of Löwner :

① $\{t \mapsto U(t) : t \geq 0\}$ is a stoch. process with independent + stationary increments (i.e. a Levy process)

② $\{t \mapsto -U(t) : t \geq 0\} \overset{law}{=} \{t \mapsto U(t) : t \geq 0\}$

③ $t \mapsto U(t)$ is a. s. continuous
$U(0) = 0$

$$\boxed{⓪ \ \& \ ① \ \& \ ② \iff U(t) = \sqrt{K}\, B_t}$$

SLE$_K$:
chordal

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \sqrt{K} B_t}$$

$$g_0(z) = z \in \mathbb{H}$$

Q: Does it define indeed a K_t, $\underline{\underline{\gamma_t}}$ process?

A: YES (Rhode, Schramm 2005: $\kappa \geq 0, \kappa \neq 8$; LSW 2004: $\kappa = 8$)

Scaling of chordal SLE_κ:

$$\{t \longmapsto \alpha^{-1/2} K_{\alpha t} : t \geq 0\} \overset{law}{=} \{t \mapsto K_t : t \geq 0\}$$

$$\{t \longmapsto \alpha^{-1/2} \gamma_{\alpha t} : t \geq 0\} \overset{law}{=} \{t \mapsto \gamma_t : t \geq 0\}$$

$$\{t \longmapsto \alpha^{-1/2} g_{\alpha t}(\alpha^{1/2}\cdot) : t \geq 0\} \overset{law}{=} \{t \longmapsto g_t(\cdot) : t \geq 0\}$$

This is actually conformal invariance of SLE:

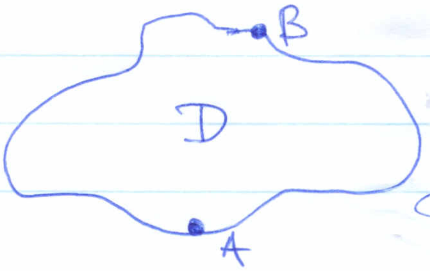
$D_\alpha : \mathbb{H} \to \mathbb{H}$ $\qquad D_\alpha z = \alpha^{-1/2} z$ $\qquad \alpha > 0$

are the only conformal maps $\mathbb{H} \to \mathbb{H}$ which fix 0 and ∞.

$$\tilde{K}_t := \alpha^{-1/2} K_{\underset{\uparrow}{\alpha t}} = D_\alpha(K_{\alpha t})$$

time rescaled so that $a_1(\tilde{K}_t) = 2t$.

Chordal SLE $A \to B$



$\varphi(K_t)$ ← chordal SLE in $\mathbb{H}$

$\varphi : \mathbb{H} \to D$ $\qquad \varphi(0) = A, \varphi(\infty) = B$

φ is not unique, but the $\underline{\text{law}}$ is determined

Main facts about SLE:

① $\forall\ \kappa \geq 0$: K_t is generated by a curve γ_t

$\kappa \neq 8$: $\boxed{\text{Rohde \& Schramm 2005}}$

$\kappa = 8$: $\boxed{\text{LSW 2004.}}$

$t \neq t' \Rightarrow \gamma_t \neq \gamma_{t'}$

② $\kappa \in [0,4]$ $t \mapsto \gamma_t$ is a simple curve

$\boxed{\text{RS2005}}$ $K_t = \gamma_{[0,t]}$

$\kappa \in (4,8)$ $t \mapsto \gamma_t$ is not simple

$\forall z \in \mathbb{H}$ $\underline{\mathbb{P}(z \notin \gamma_{[0,\infty)}) = 1}$

$K_t = \bigcup \{\text{bdd components of } \mathbb{H} \setminus \gamma_{[0,t]}\}$

$\kappa \in [8,\infty)$: $t \to \gamma_t$ is plane-filling (Peano)

$\boxed{\text{RS2005}}$ $K_t = \gamma_{[0,t]}$

$\kappa \in (4,\infty)$: $\bigcup_t K_t \cap \mathbb{R} = \mathbb{R}.$

③ Hausdorff dimensions:

$\boxed{\begin{array}{l} \leq \text{ proved in } \boxed{\text{RS2005}} \\ = \text{ proved in } \boxed{\text{Beffara ??}} \end{array}}$

$$\dim_H(\gamma_{[0,t]}) = \begin{cases} 1 + \frac{\kappa}{8} & 0 \leq \kappa \leq 8 \\ 2 & 8 \leq \kappa \end{cases}$$

$\boxed{\leq \text{ proved in } \boxed{\text{RS2005}}}$

conjectured,

$$\dim_H(\partial K_t) \overset{?}{=} \begin{cases} 1 + \frac{\kappa}{8} & 0 \leq \kappa \leq 4 \\ 1 + \frac{2}{\kappa} & 4 \leq \kappa \end{cases}$$

$\boxed{\begin{array}{c} \kappa = 6 \text{ known} \\ \text{LSW} \ldots \end{array}}$

④ $\boxed{\text{Conjectured}}$

$$\left\{ \partial K_t \,,\ \kappa \geq 4 \right\} = \left\{ \gamma_t ,\ \kappa' = \frac{16}{\kappa} \right\}$$

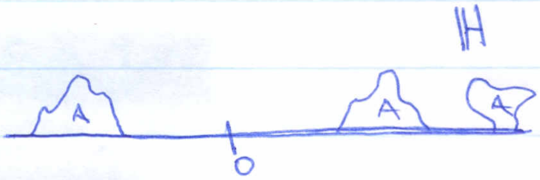
in some sense

$\kappa = 6$ $\kappa' = \frac{8}{3}$ known (LSW)

Locality [LSW 2001]

$A \subseteq \overline{\mathbb{H}}$ compact : $0 \notin A$

$\mathbb{H} \setminus A$ simply connected



$\phi^A : \mathbb{H} \setminus A \to \mathbb{H}$ conformally : $\begin{cases} \phi^A(0) = 0 \\ \phi^A(\infty) = \infty \\ \phi^{A\prime}(\infty) = 1 \end{cases}$

γ_t, K_t, g_t : SLE_κ

$T_A := \inf\{t : K_t \cap A = \phi\}$ stopping time

Remark : ① $0 \leq \kappa \leq 4$: $\mathbb{P}(T_A < \infty) < 1$

② $\infty < \kappa < \infty$: $\mathbb{P}(T_A < \infty) = 1$

for $t \in [0, T_A)$ define

$$\gamma_t^{*A} := \phi^A(\gamma_t), \quad K_t^{*A} := \phi^A(K_t)$$

$$g_t^{*A} := g_{K_t^{*A}} = g_t \circ (\phi^A)^{-1}$$

Remark $$a_1(K^{*A}_t) = a_1(A\cup K_t) - a_1(A)$$

$$0 < a_1(K^{*A}_t) < a_1(K_t)$$

$\uparrow$ "subadditivity" of a_1

Reparametrize so that

$$\alpha(t) := a_1(A\cup K_t) - a_1(A)$$

$$\beta(t) := \alpha^{-1}(2t)$$

$$\begin{cases} (\gamma^A_t, K^A_t, g^A_t) := (\gamma^{*A}_{\beta(t)}, K^{*A}_{\beta(t)}, g^{*A}_{\beta(t)}) \\ 0 \le t \le \frac{1}{2}\alpha(T_A) = \frac{1}{2}\left(a_1(A\cup K_{T_A}) - a_1(A)\right) \end{cases}$$

Then $t \mapsto (\gamma^A_t, K^A_t, g^A_t)$

$0 \le t \le \inf\{s : K^A_s \cap \phi^A(\partial A \cap \mathbb{H}) \neq \emptyset\}$

is a Lowner evolution

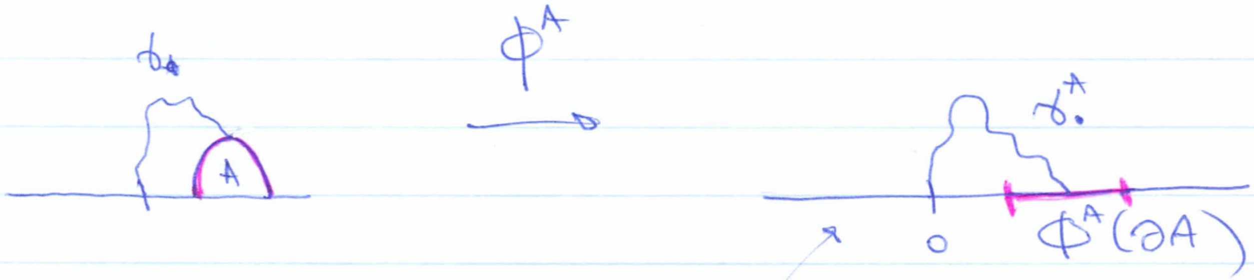
Locality: SLE$_\kappa$ satisfies the $\boxed{\text{locality property}}$

iff for $\forall A$ as above, on the event:

$$t \le \inf\{s : K^A_s \cap \phi^A(\partial A \cap \mathbb{H}) \neq \emptyset\}$$

$[0,t) \ni s \mapsto (\gamma^A_s, K^A_s, g^A_s)$ has the same law as $s \mapsto (\gamma_s, K_s, g_s)$

Remark If $\kappa \leq 4$ SLE_κ can't be local:



but this can't happen for γ

In plain words

locality = γ_t doesn't feel the boundary until hitting it

Remark BM (which is not an SLE) is local.

Remark $t \mapsto (K_t, \gamma_t)$ is Markov

"locality means that what happens in $[t, t+\varepsilon)$ depends only on $K_t \cap B(\gamma_t, \varepsilon)$"

Theorem SLE_κ is local iff $\kappa = 6$.

The restriction property : [LSW 2003]

$A \subseteq \mathbb{H}$, ϕ^A as before.

$\kappa \leq 4$, SLE_κ simple curve

$$\mathbb{P}(\gamma[0,\infty) \cap A = \emptyset) > 0 \quad !$$

Restriction : $\forall A$ as above

$$\left(t \mapsto \gamma^A_t \;\middle|\; \gamma_{[0,\infty)} \cap A = \emptyset \right) \overset{law}{=} \left(t \mapsto \gamma_t \right)$$

$$\left(t \mapsto \gamma^A_t \;\middle|\; \gamma^A_{[0,\infty)} \cap \phi^A(\partial A) = \emptyset \right) \overset{law}{=} \left(t \mapsto \gamma_t \right)$$

Thm (SLE_κ satisfies RP) iff $\kappa = \frac{8}{3}$ and in this case

$$\mathbb{P}(\gamma_{[0,\infty)} \cap A = \emptyset) - \left(\phi^{A\prime}(0) \right)^{5/8}$$

Remark : $\underset{\text{"Brownian excursion"}}{Z_t} = \underset{\text{BM}}{X_t} + i \underset{\text{BES}^3}{Y_t} = \left(B_t \mid \operatorname{Im} B_t > 0 \; \forall t \right)$

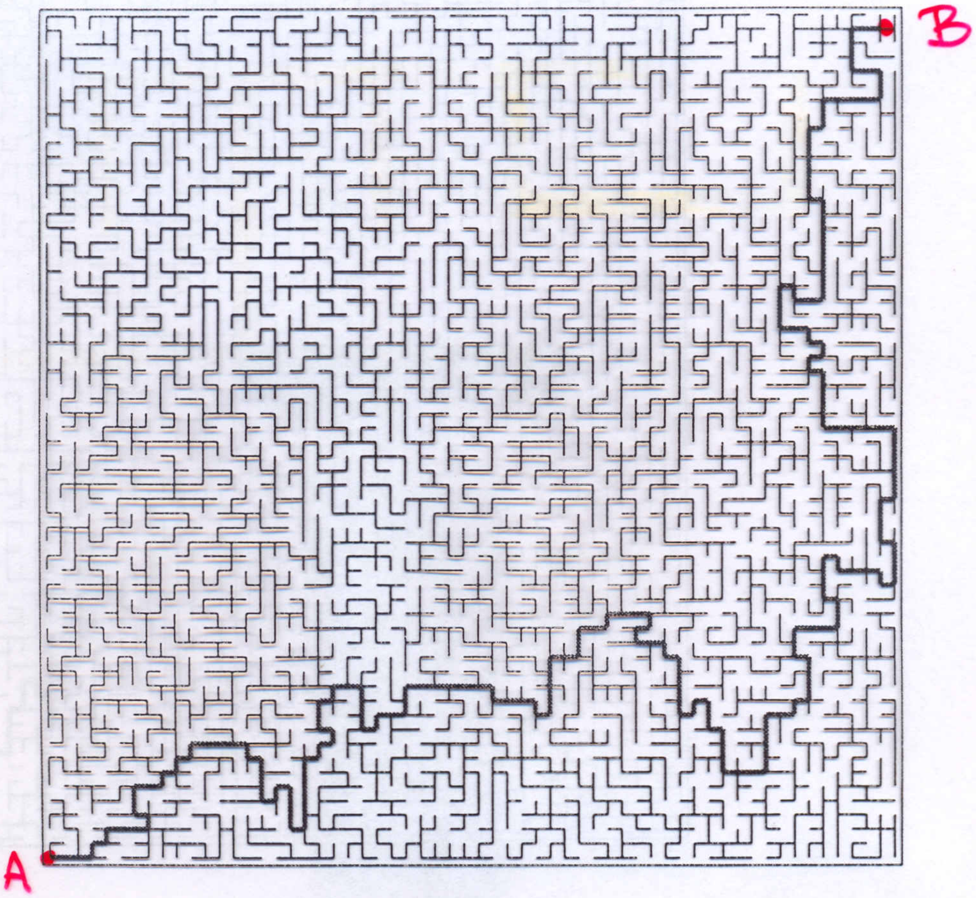
fact $\mathbb{P}(Z_{[0,\infty)} \cap A \qquad) = \phi^{A\prime}(0)$

interpret : "(5 independent BE)-hull" = "(8 indep $SLE_{8/3}$)-hull"

An LERW path

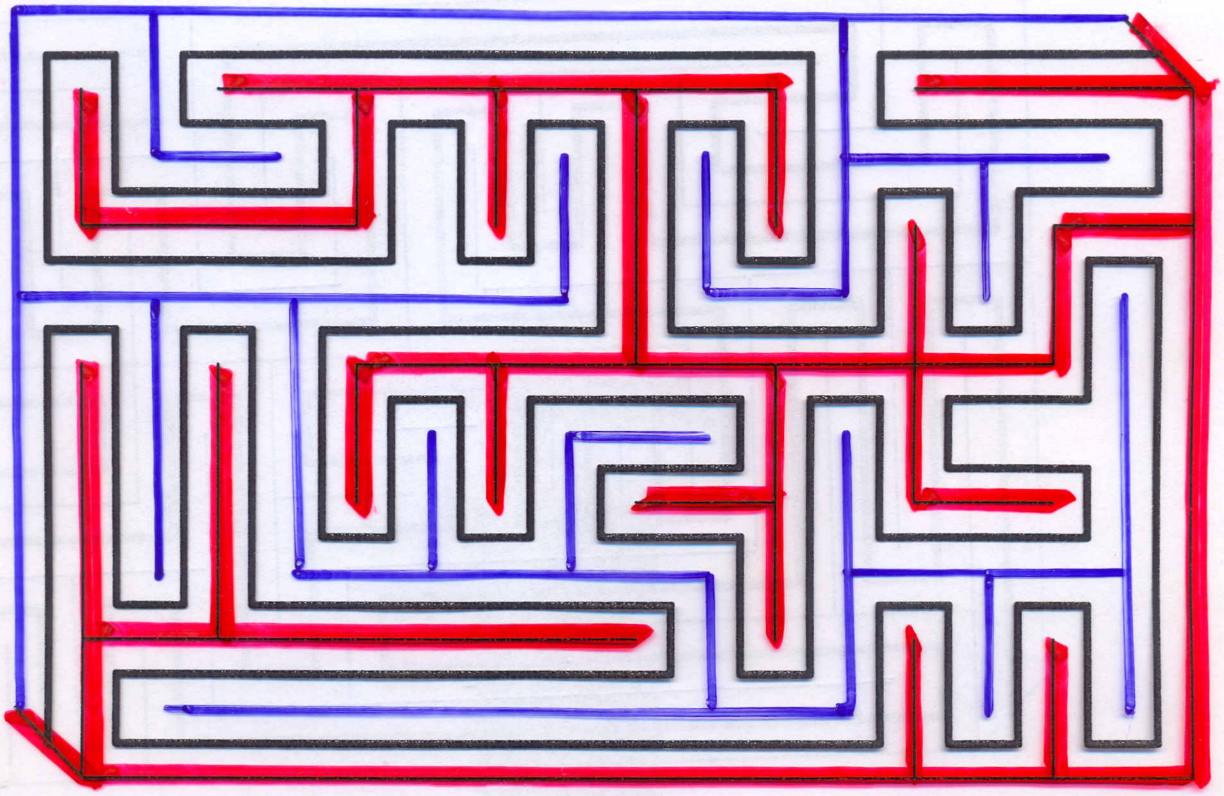
$\sim SLE_2$

(from Schramm 2006)



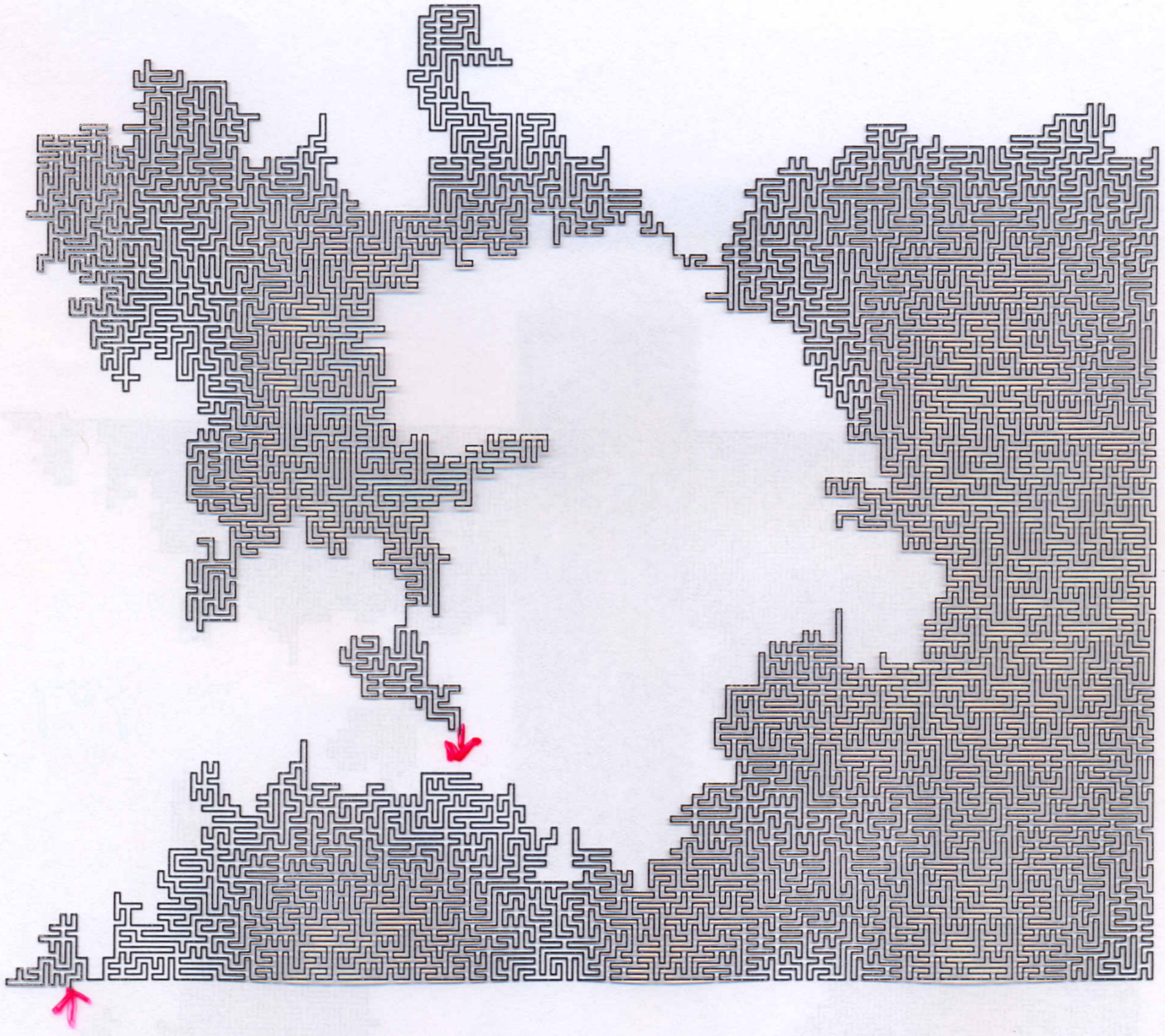
Loop Erased Random Walk
from A to B as a subpath
of the Uniform Spanning Tree

(from Werner 2004)



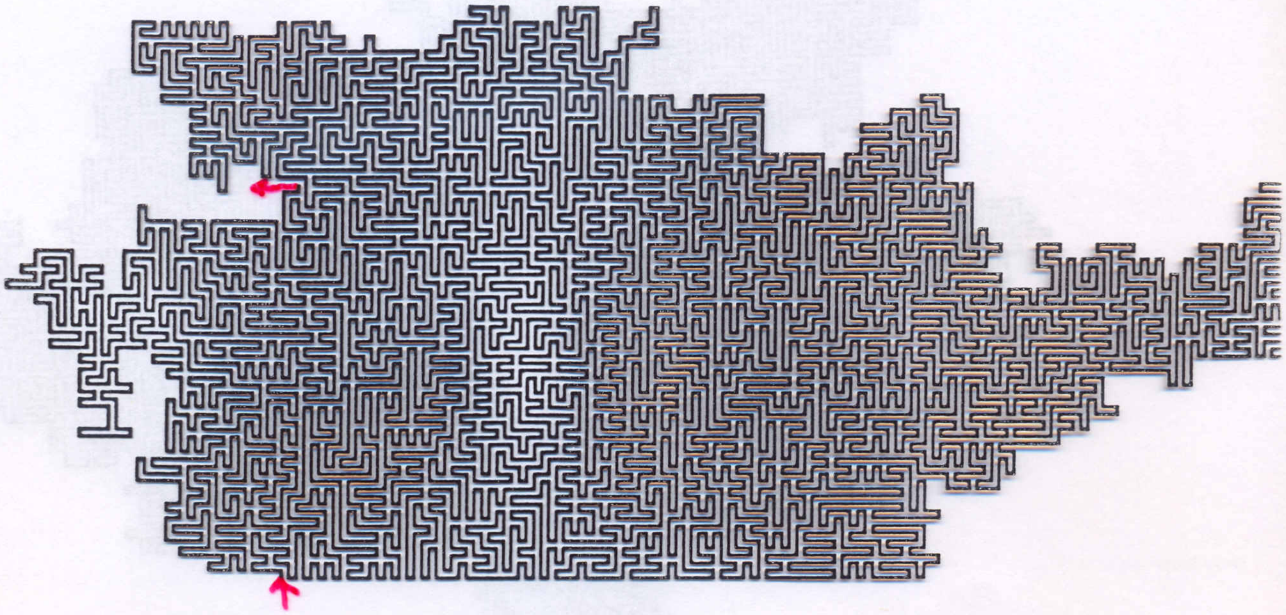
UST (with half wired boundary), its dual (also a UST on the Whitney dual graph) and the lattice filling (discrete Peano) path between them

(from Werner 2004)



Initial segment of the lattice filling (discrete Peano) path surrounding UST in a quadrant of $\mathbb{Z}^2$

(from Schramm 2006)



Lattice filling (discrete Peano) path: interface of UST on $\mathbb{Z}^2$

$\sim \mathrm{SLE}_8$

(from Werner 2004)

Note that the two families L^+ and L^- create a random maze, with one single connected component (this is a simple consequence of the coalescing property). One single possible path starting at the middle of the 'entrance gate' $[(-1,1),(0,1)]$, i.e. from the point of coordinates $(-1/2,1)$, explores this maze. (See Fig. 2.) The random walk $(S_i)_{i\geq 0}$ can then easily be deterministically constructed from this random maze. More precisely, we define a continuous function $(\tilde{S}_t, H_t)$ on $\mathbb{R}_+$ with $\tilde{S}_0 = -1/2$ and $H_0 = 1$, that explores the maze, at constant speed (the speed is $\sqrt{2}$; in other words, for almost all $t \geq 0$, $|\tilde{S}'_t| = |H'_t| = 1$), as shown on the picture. Note that at all integer times i, $\tilde{S}_i + 1/2 \in \mathbb{Z}$ and $H_i \in \mathbb{N} \setminus \{0\}$ (these points are dotted on the picture).

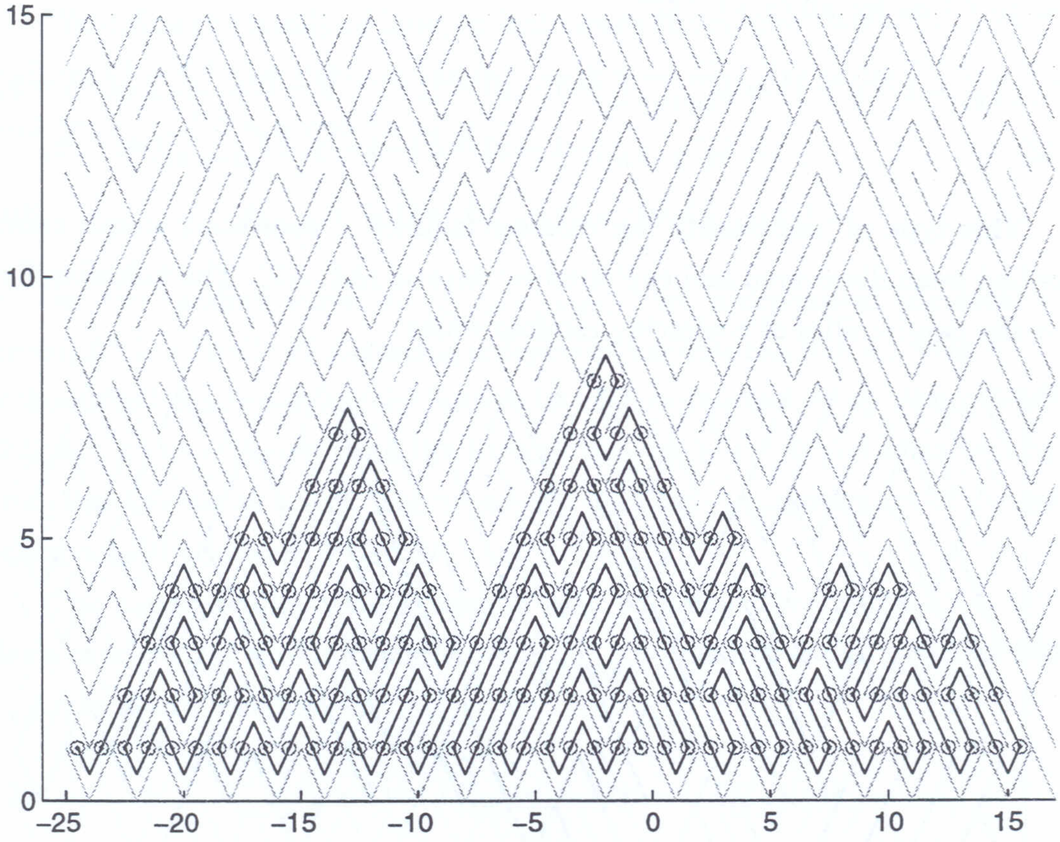
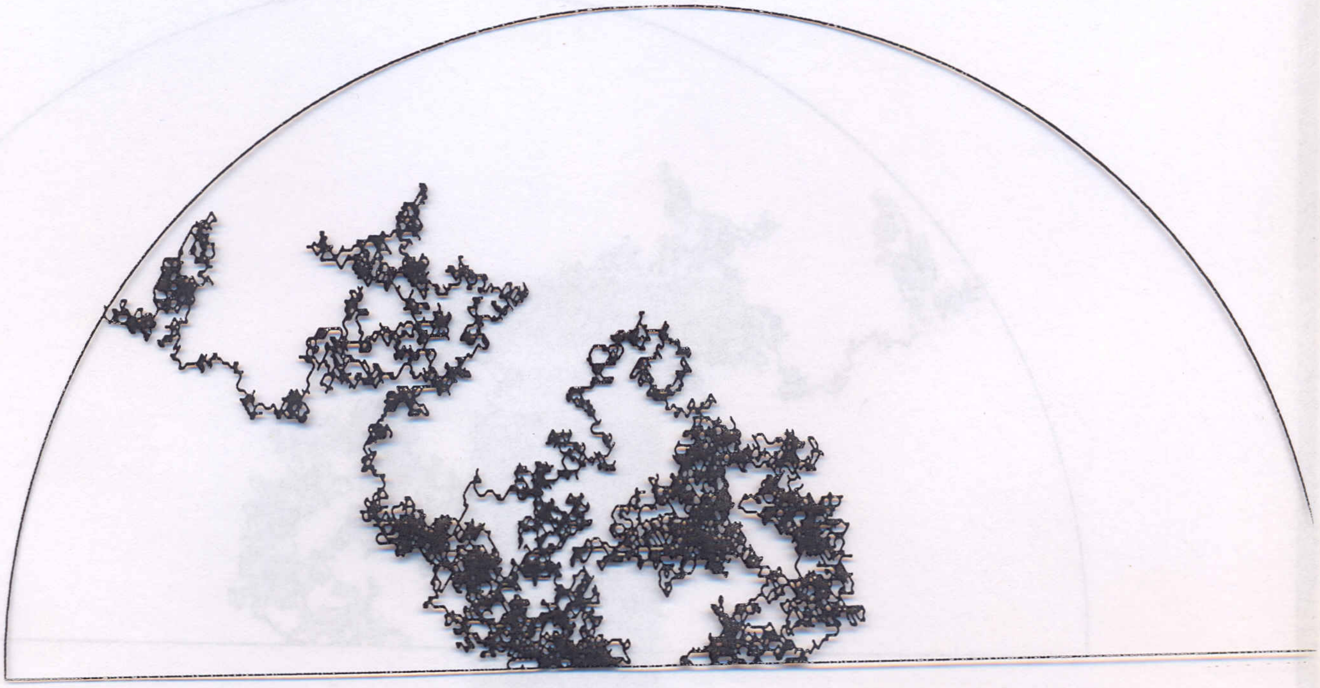


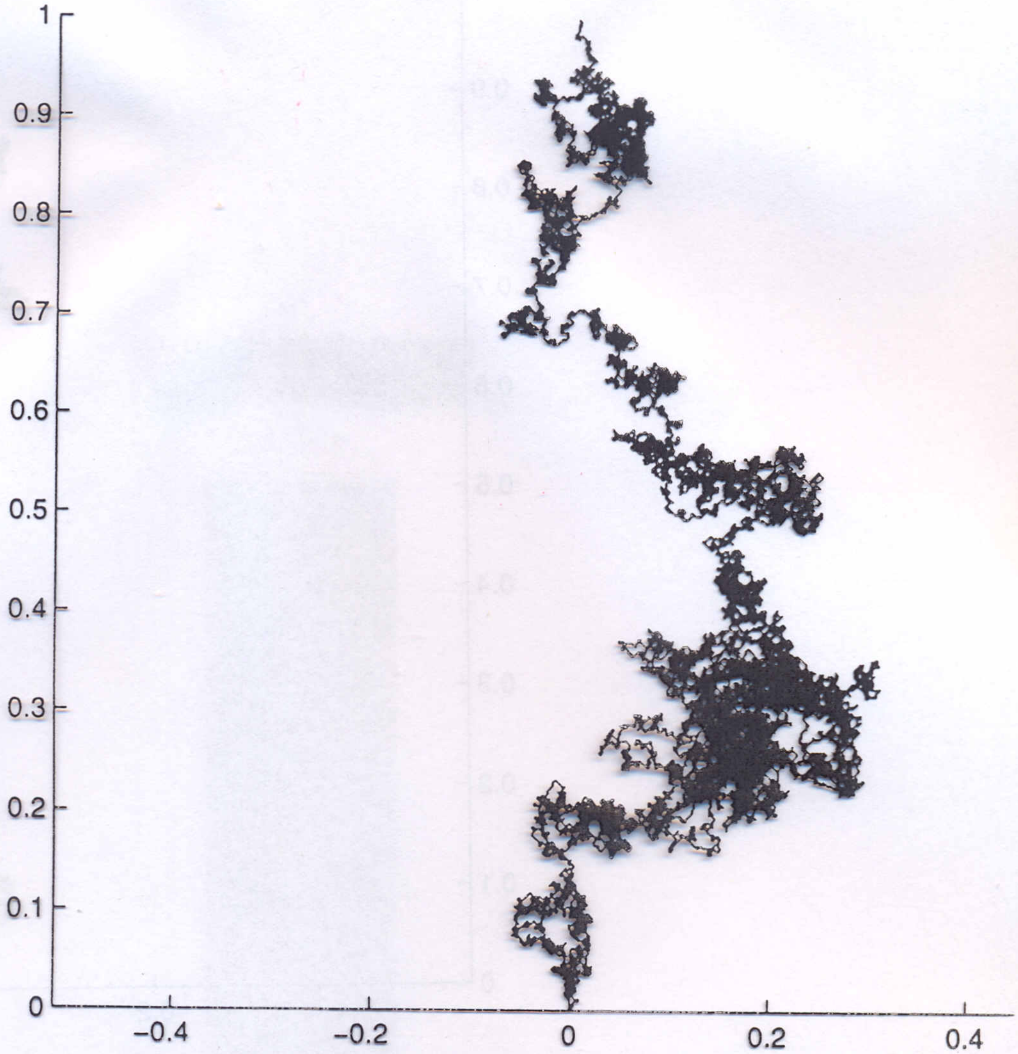
Fig.2: The maze and the path of $(\tilde{S}_t, H_t)$. *The points* $(\tilde{S}_i, H_i)$ *are circled*

TSRM path
(from Tóth – Werner '97)



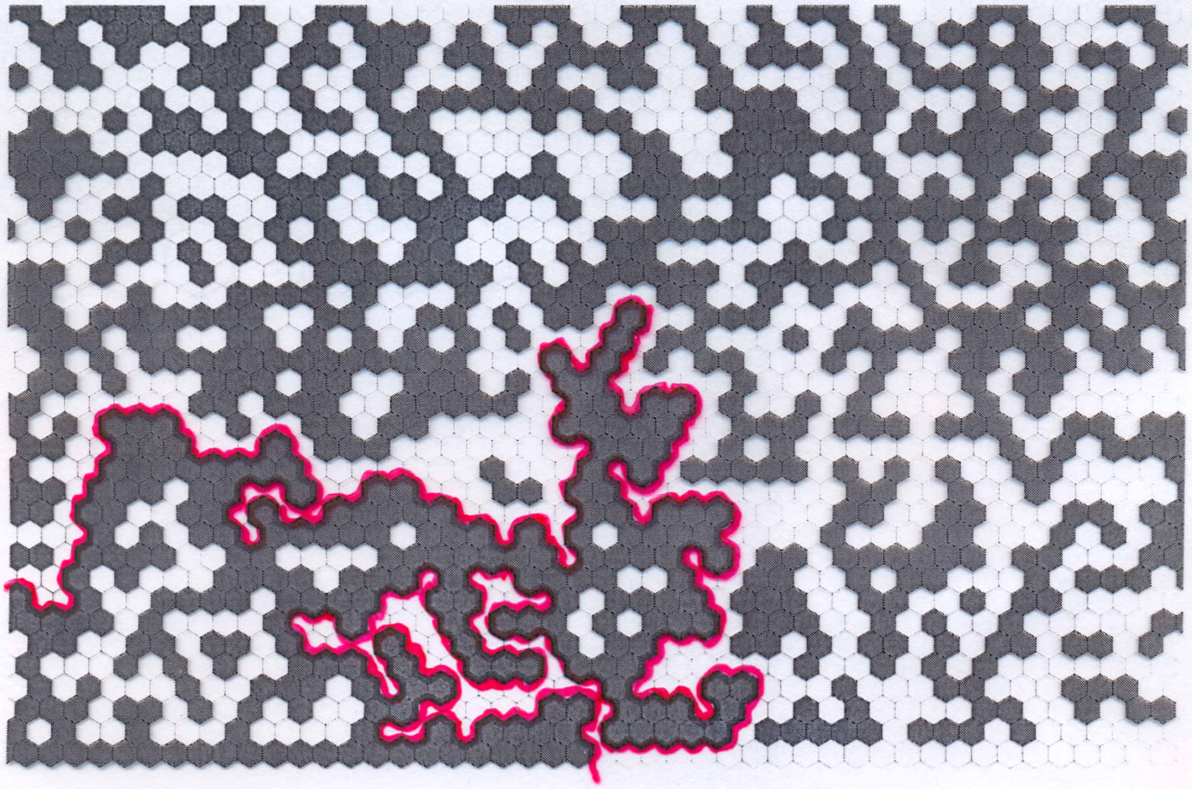
Brownian motion path reflected on $\operatorname{Im} z = 0$.

(from Werner 2004)



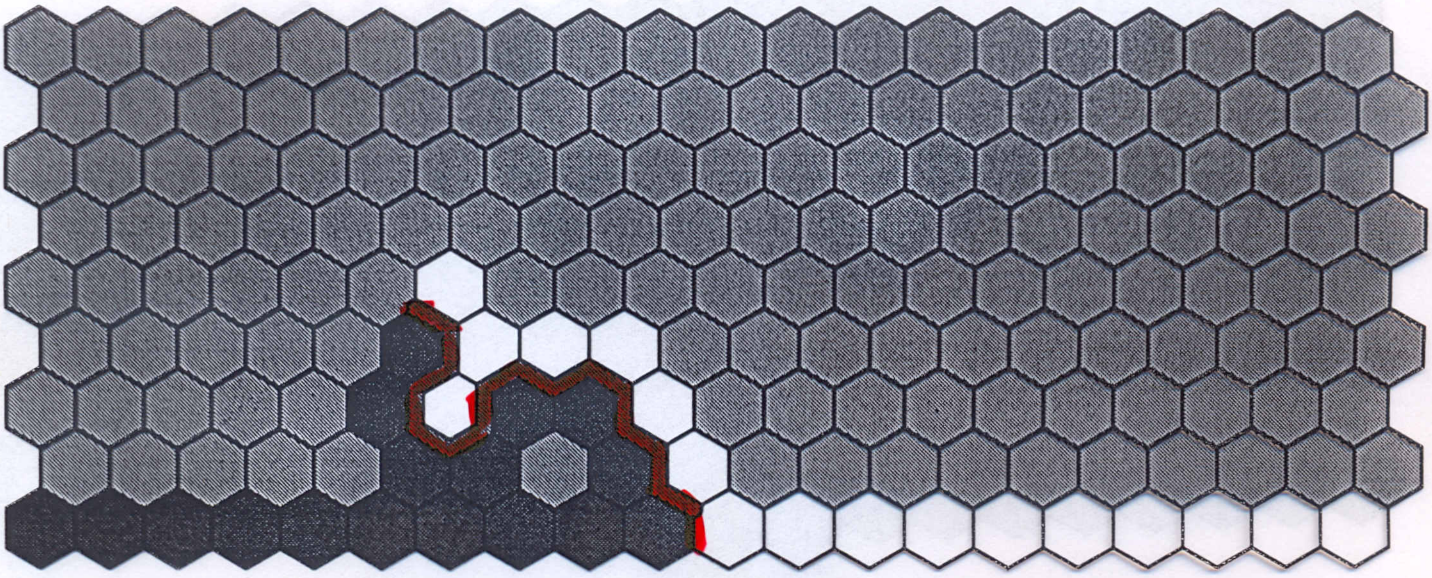
Brownian excursion path from 0 to i in the strip $\{z : \mathrm{Im}\, z \in [0,1]\}$

(from Werner 2004)



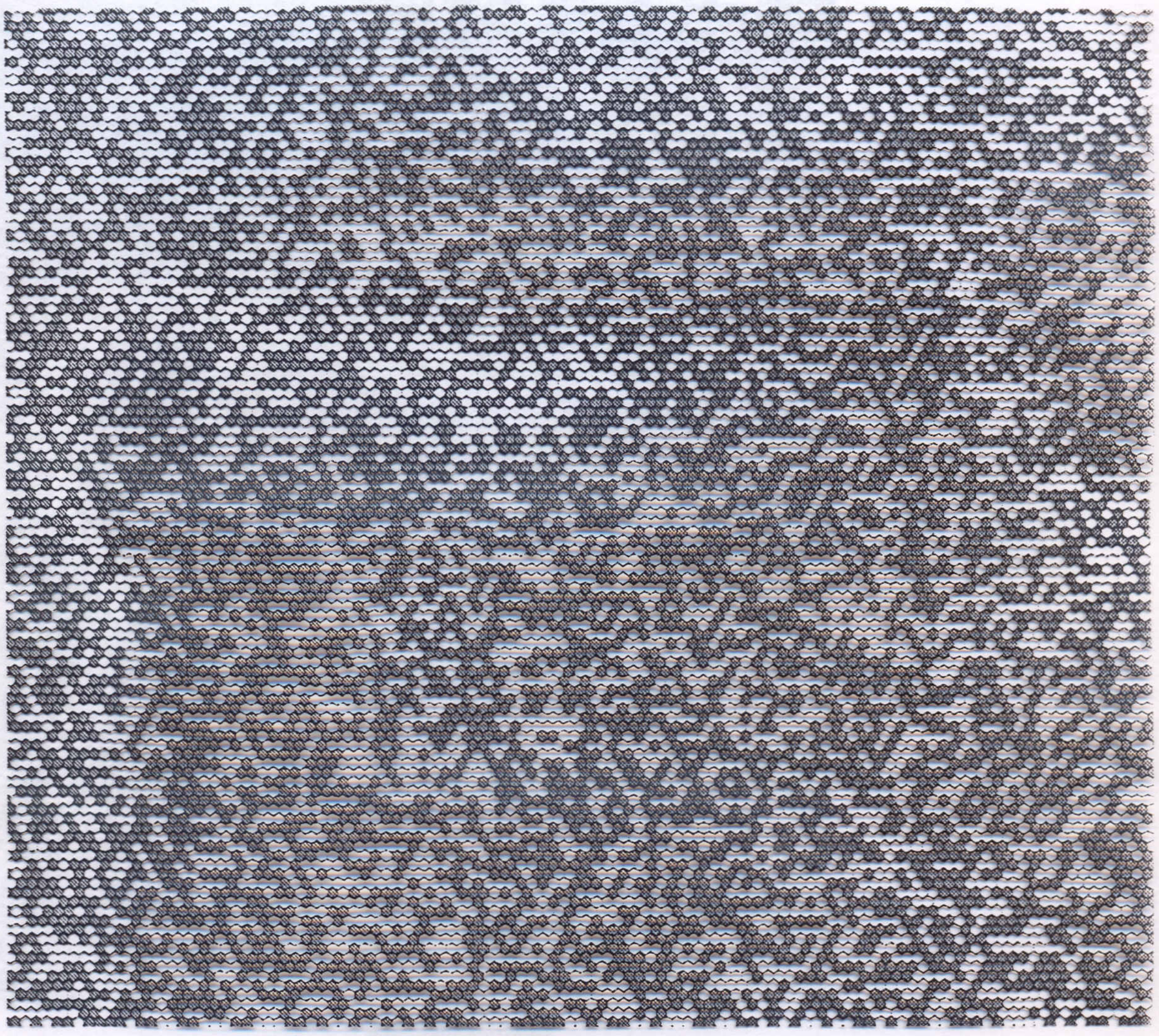
Initial segment of the critical percolation interface (or exploration path) in the upper half plane, ~ SLE_6

(from Schramm 2006)



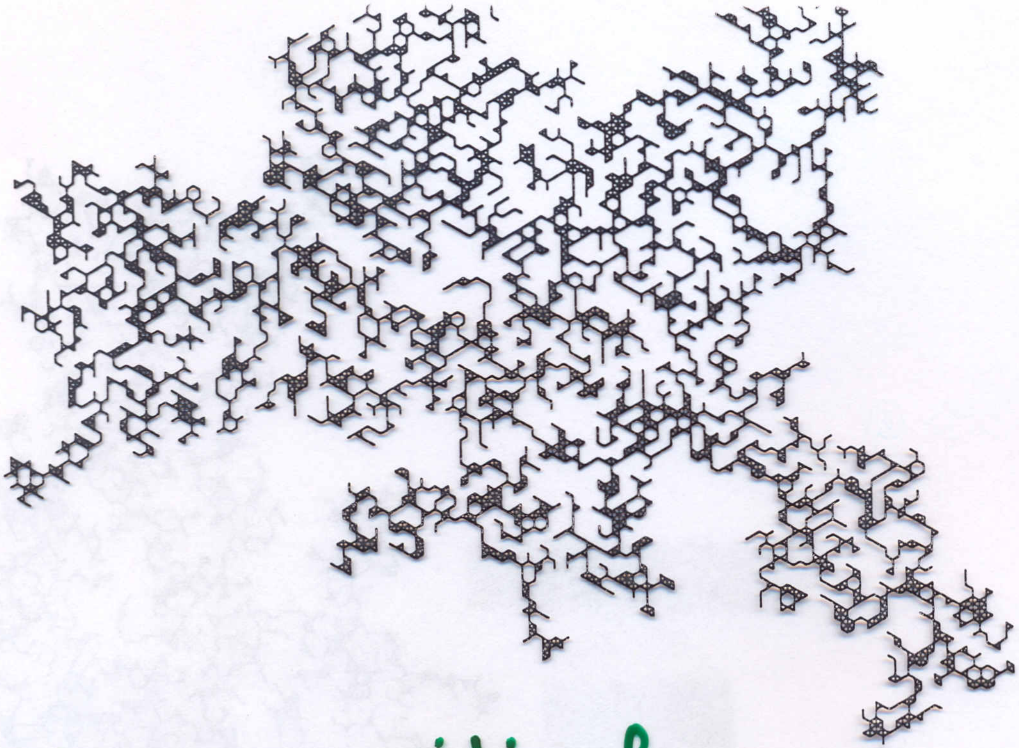
Beginning of the critical percolation exploration path in the upper half plane

(from Schramm 2006)



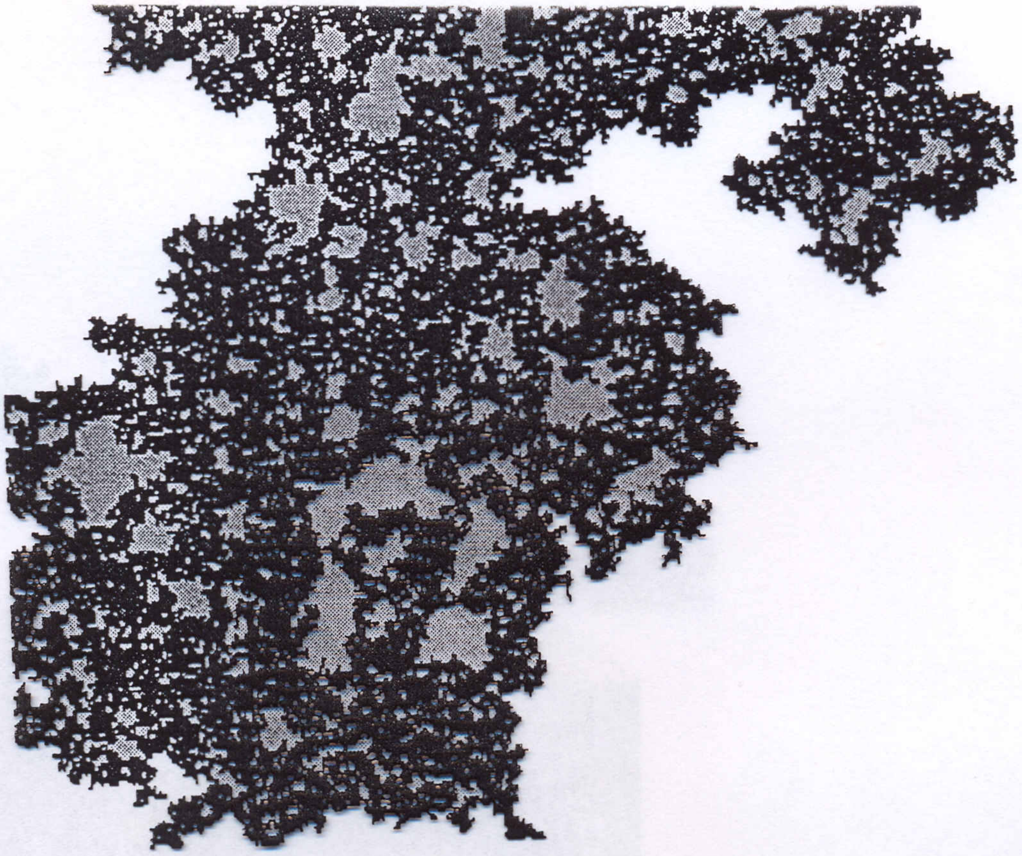
Critical percolation on the triangular grid

(from Werner 2004)



A large critical percolation cluster on the triangular grid $p_c = \frac{1}{2}$

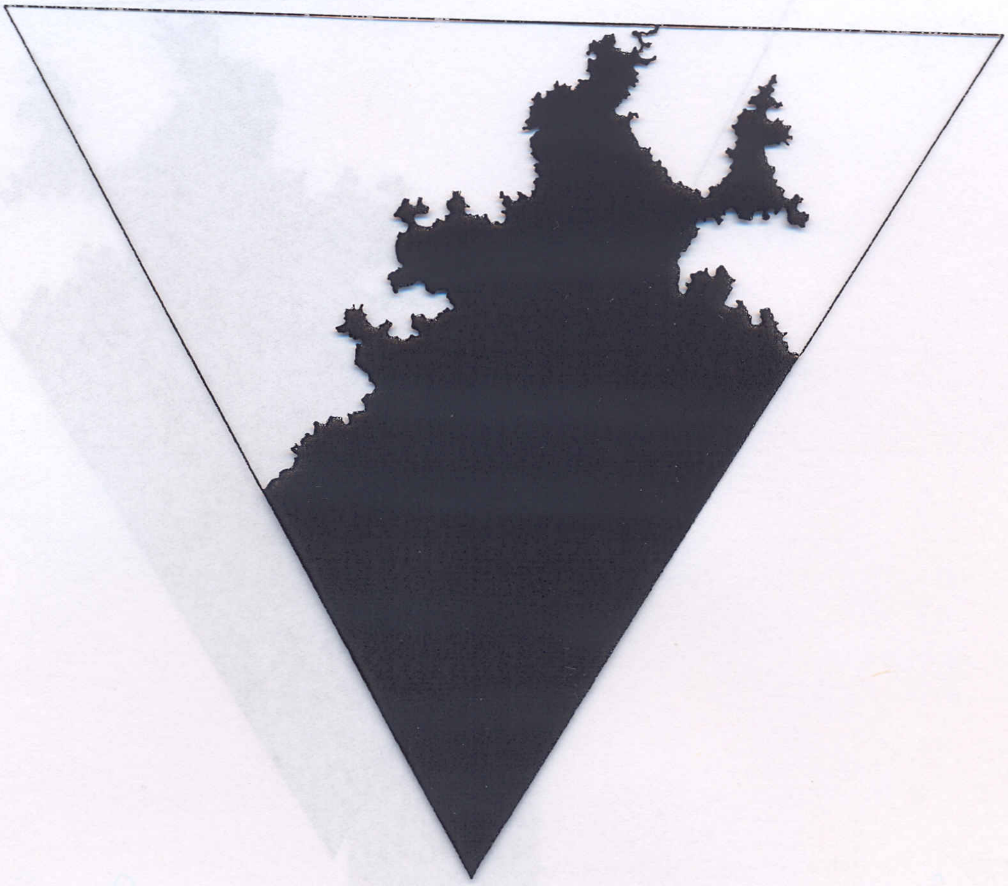
(from Werner 2004)



A large critical percolation cluster on the square grid

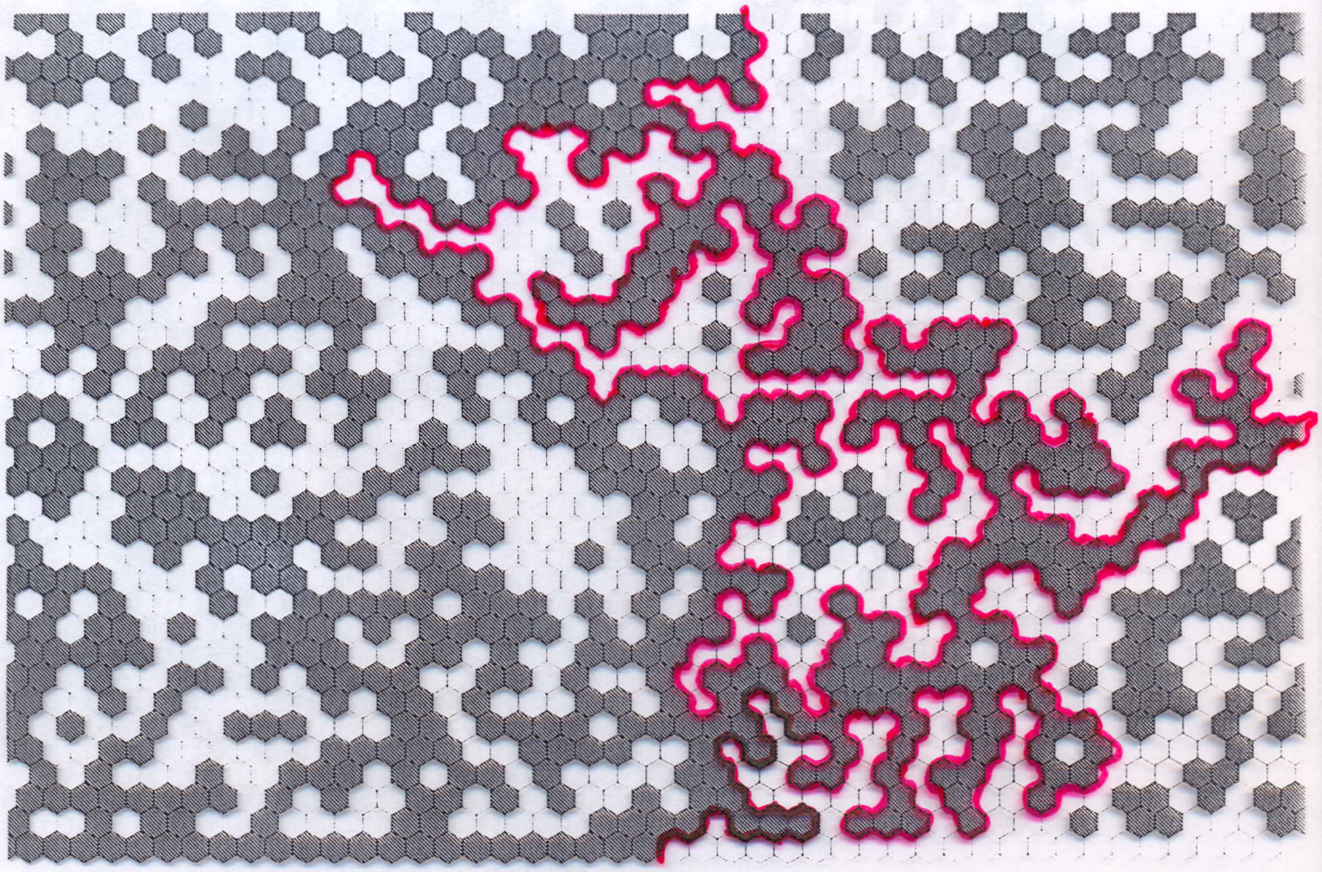
$p_c \approx 0.59$

(from Werner 2004)



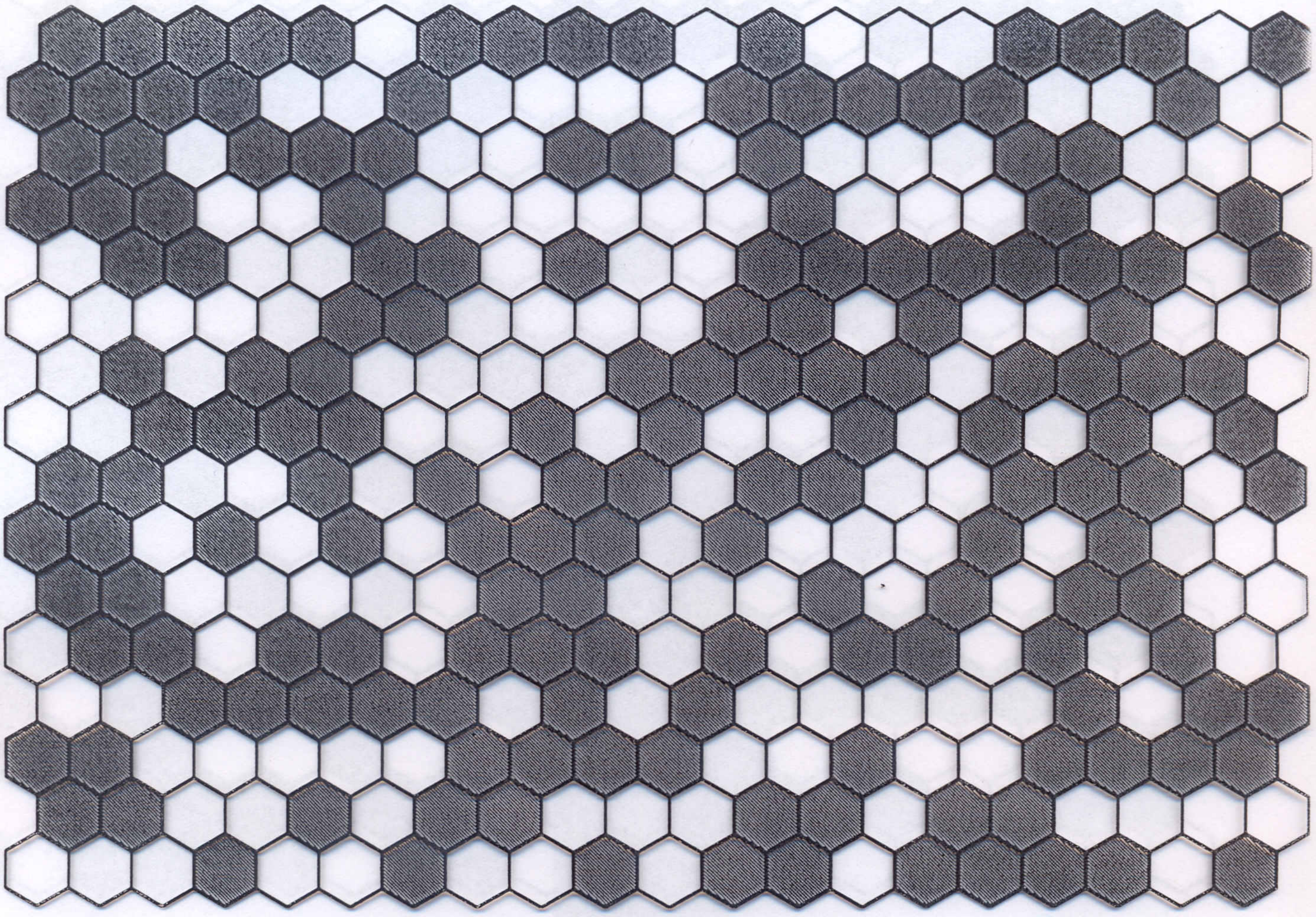
The hull of critical percolation exploration path on the triangular lattice, $\sim SLE_6$

(from Werner 2004)



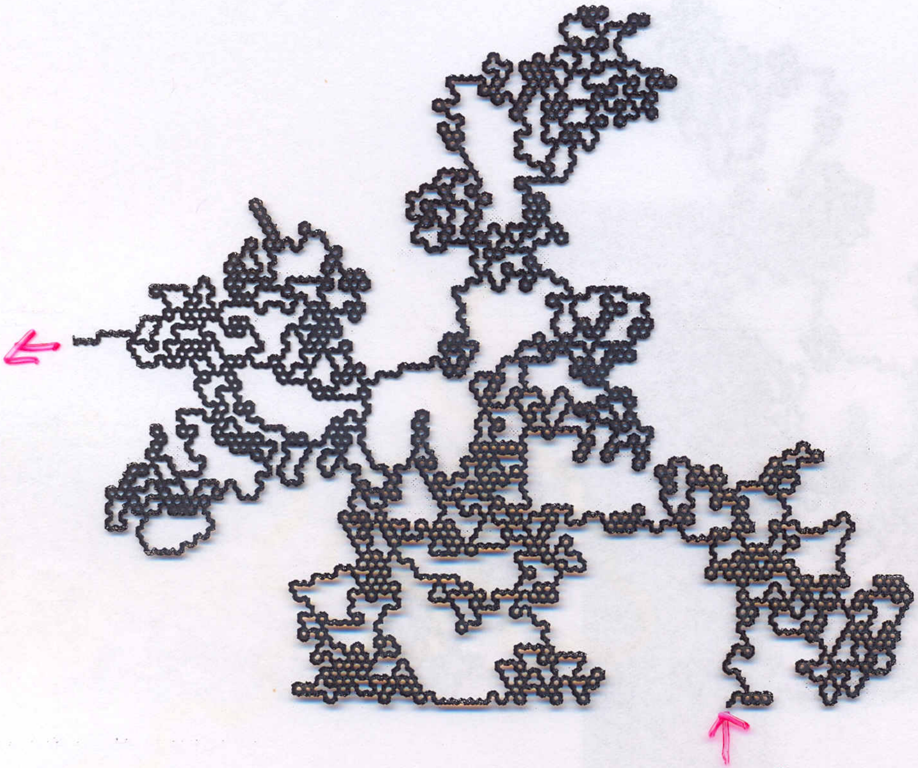
A larger segment of the critical percolation interface (= exploration path) in the upper half plane, $\sim SLE_6$

(from Werner 2004)



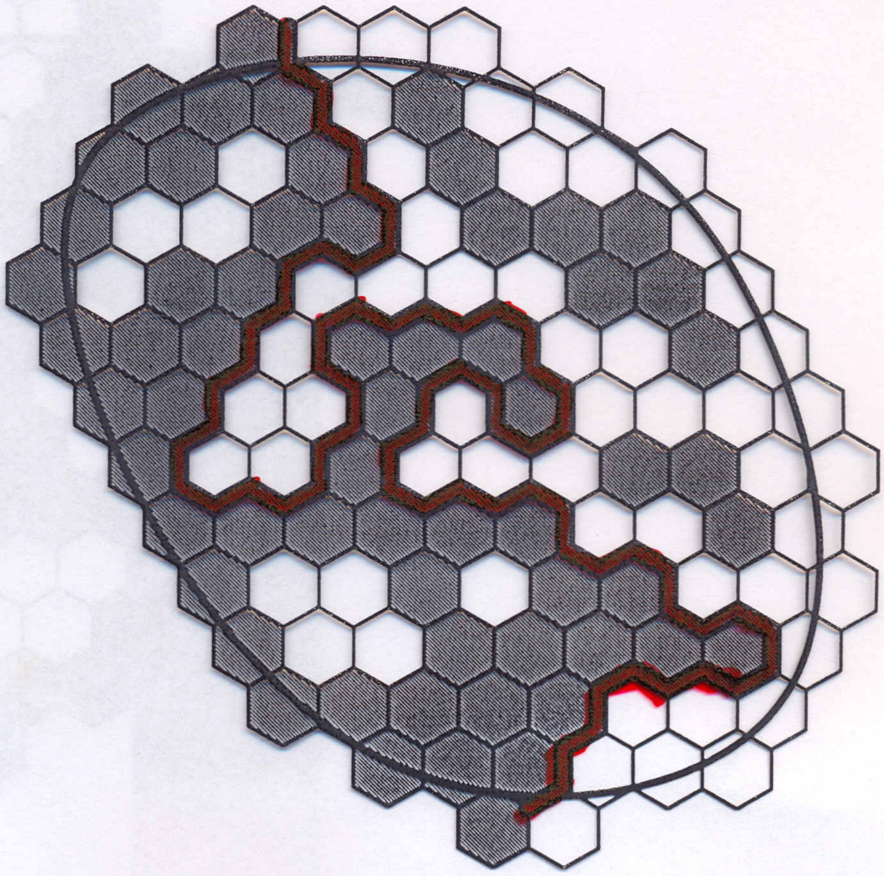
Critical percolation on the triangular grid

(from Schramm 2006)



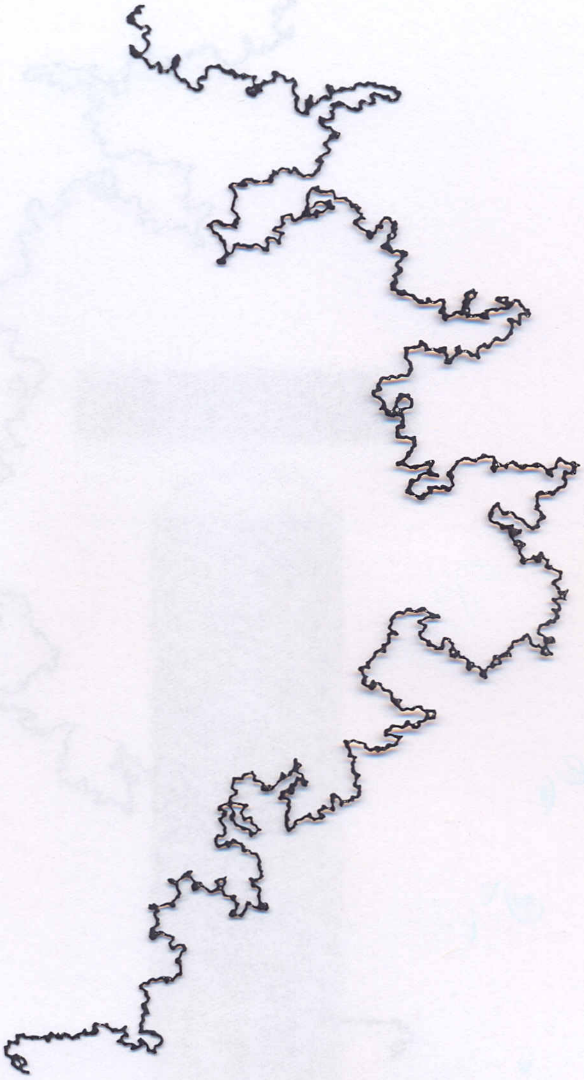
A critical percolation exploration process, $\sim SLE_6$

(from Werner 2004)



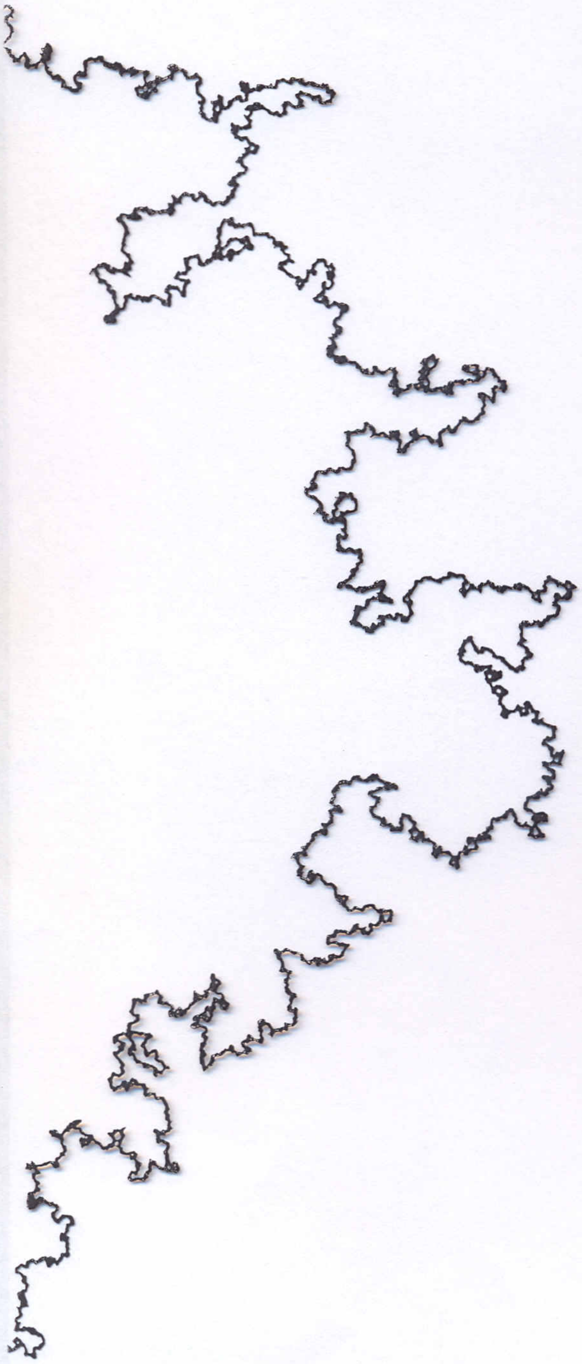
Critical percolation interface
in a compact domain
$\sim SLE_6$

(from Schramm 2006)



Self avoiding walk in the upper half plane, $\sim SLE_{8/3}$?

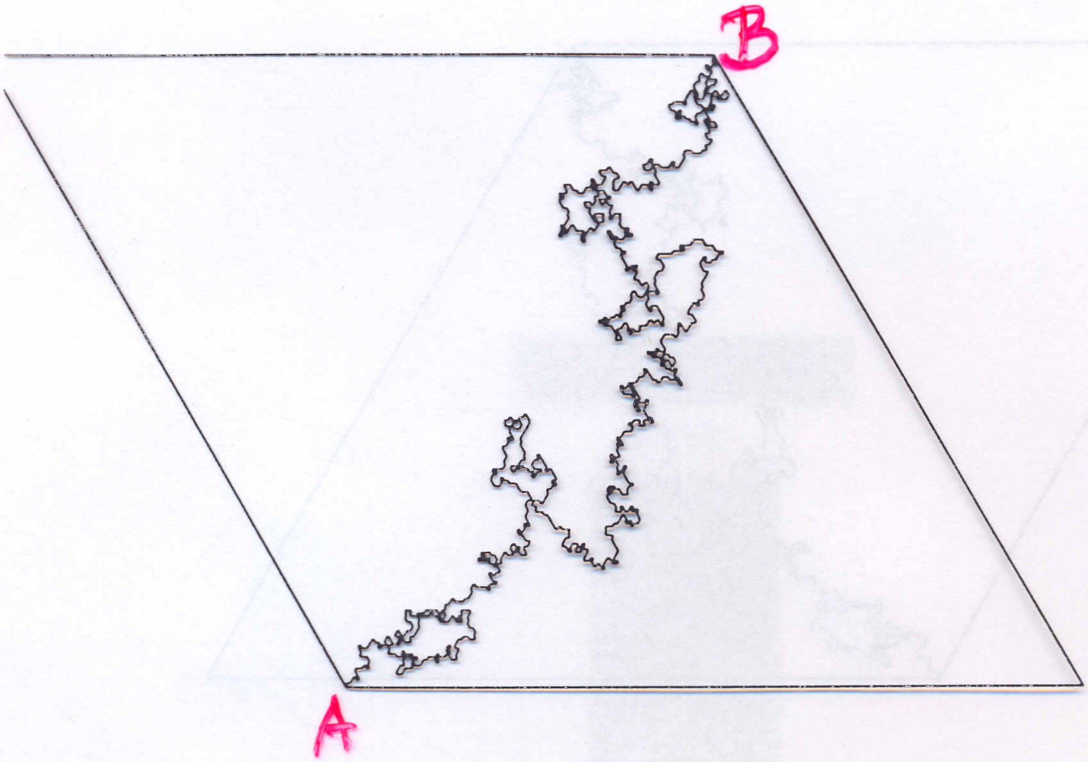
(from Werner 2004)



SARW path in the upper half plane

$\sim SLE_{8/3}$ (?)

(from Werner 2004)



Harmonic explorer path,
$\sim \mathrm{SLE}_4$

(from Schramm 2006)