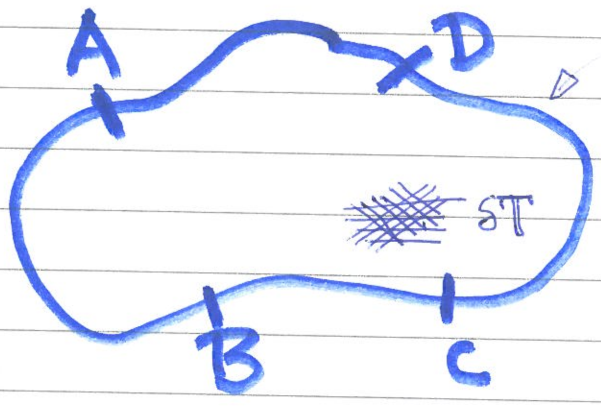
Bálint Tóth: Percolation 7.

Conformal invariance of 2d critical percolation

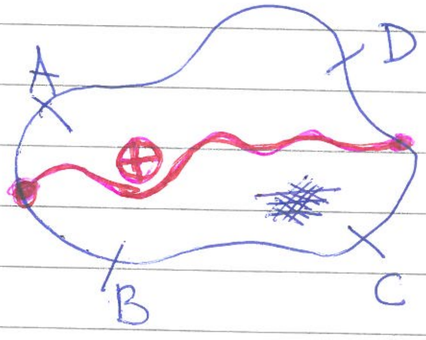
$\mathbb{R}^2 = \mathbb{C}$; $\mathcal{D} \subset \mathbb{C}$ simply connected domain, open, bdd,



(or $\delta\mathbb{Z}^2$: ▦)

$\delta \ll 1$

$$\lim_{\delta \downarrow 0} \mathbb{P}_p \Bigg(\quad \Bigg) = \begin{cases} 0 & p < p_c \\ ? & p = p_c \\ 1 & p > p_c \end{cases}$$



$\boxed{p < p_c}$ follows from Menshikov / Aizenman-Barsky / Duminil-Copin – Tassion

$\boxed{p > p_c}$ follows from Lemma 1 / LN6, p16.

We stay at p_c.

RSW arguments used:

Mosaics with symmetries of $\mathbb{Z}^2$

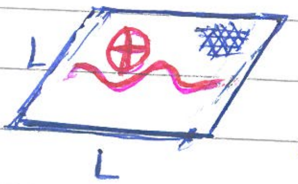
$$0 < \inf_L \underbrace{R_{L,1}(p_c)}_{=1-R^*_{L1}(p_c^*)} \le \sup_L R_{L1}(p_c) < 1$$

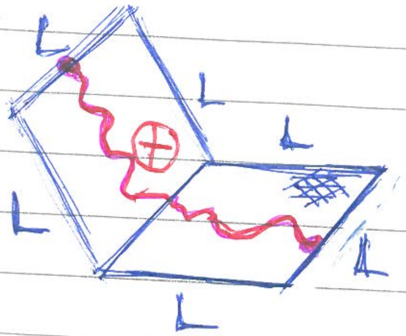
follows from
Lemma 2 / LN6, p16.
and $\Theta(p_c) = 0$

Hence (by RSW): $\forall k$:

$$0 < \inf R_{Lk}(p_c) \le \sup R_{Lk}(p_c) < 1$$

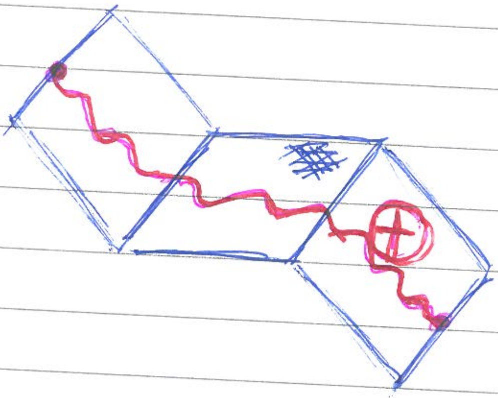
Triangular lattice, $p_c = \frac{1}{2}$:

$\mathbb{P}_{\frac{1}{2}}\Big($  $\Big) = \frac{1}{2}$ by self-duality

$$\mathbb{P}_{1/2}\Bigg($$ 

$$\Bigg) \geq \left(\frac{1}{2}\cdot\frac{1}{2}\right)^2 = \frac{1}{16}$$

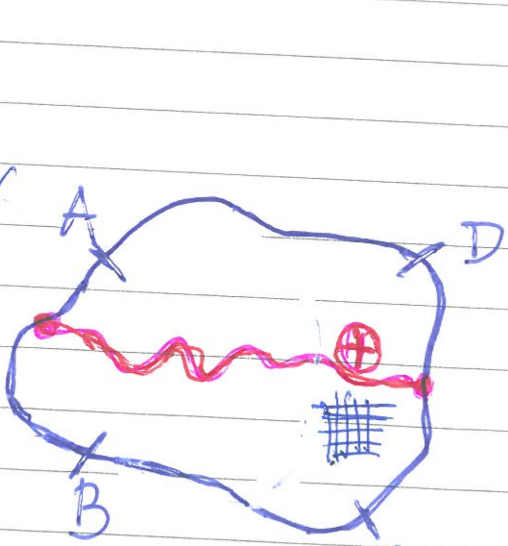
by Smirnov's RSW argument

$$\mathbb{P}_{1/2}\Bigg($$ 
$$\Bigg) \geq \frac{1}{16}\cdot\frac{1}{16}\cdot\frac{1}{2} = \frac{1}{512}$$

by Harris.

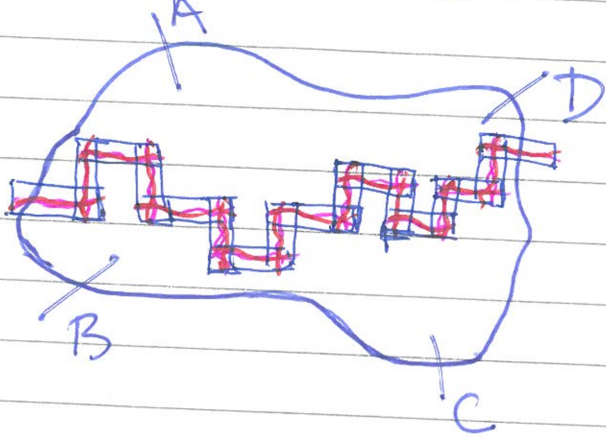
etc.

<u>Proposition:</u>

$$0 < \overline{\lim_{\varepsilon \downarrow 0}}\, \mathbb{P}_{p_c}\Bigg($$ 

$$\Bigg) < 1$$

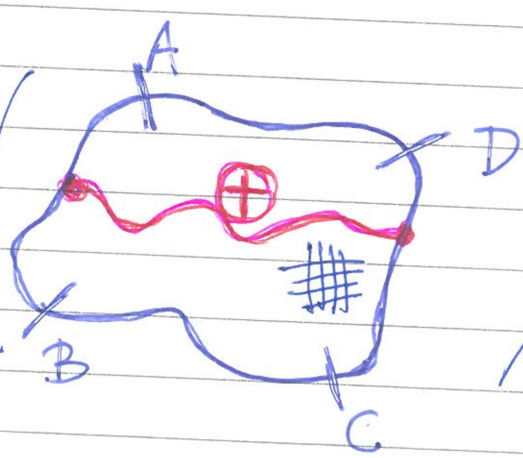
Proof. Lower bound



Apply RSW + Harris ineq.

Uper bound: by duality. □

Conjecture 1:

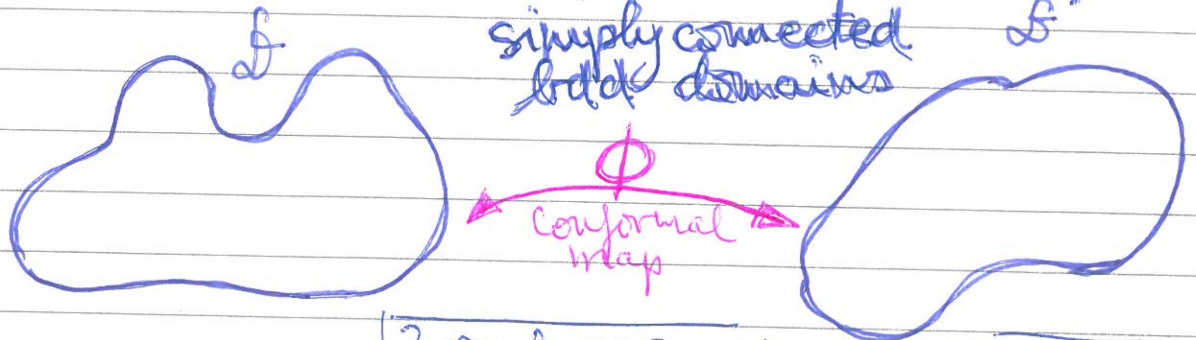
$\lim_{s \downarrow 0} \mathbb{P}_{p_c}$ () exists,

and similar macroscopic limits also exist.

The limit is (of course) invariant

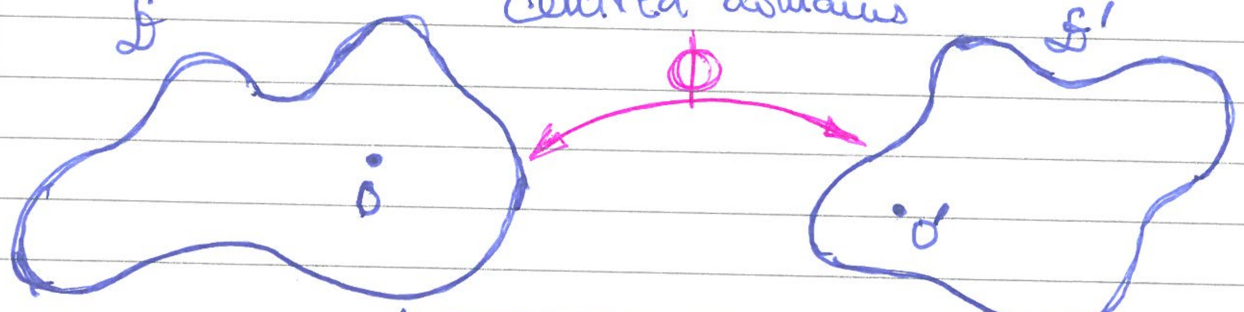
1. to translations
2. to the symmetries of the lattice
3. to (global) dilations

Terminology & basic facts about conformal mappings (all follow from Riemann's conformal mapping thm.



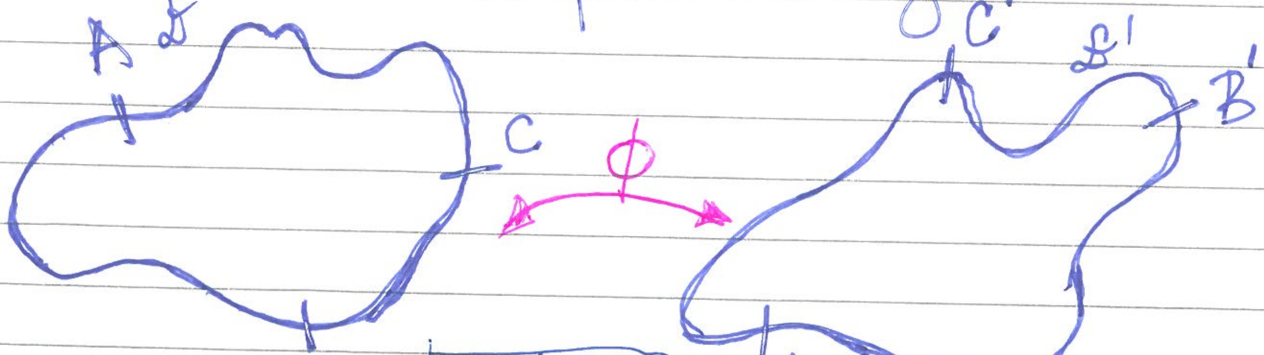
$\boxed{\text{3 real parameters left free}}$

centred domains



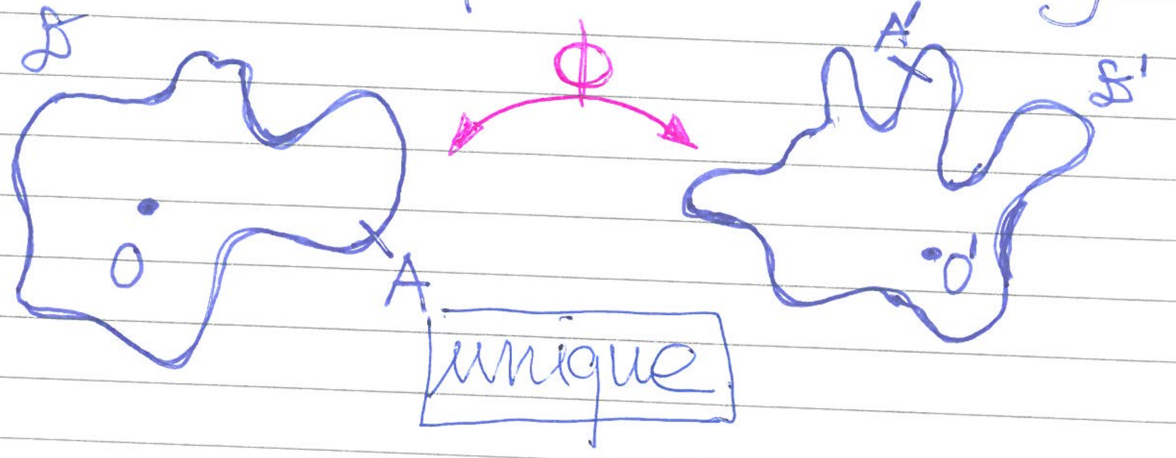
$\boxed{\begin{array}{l}\text{1 real parameter left free} \\ \text{"rotation" around } 0/0'\end{array}}$

conformal triangles

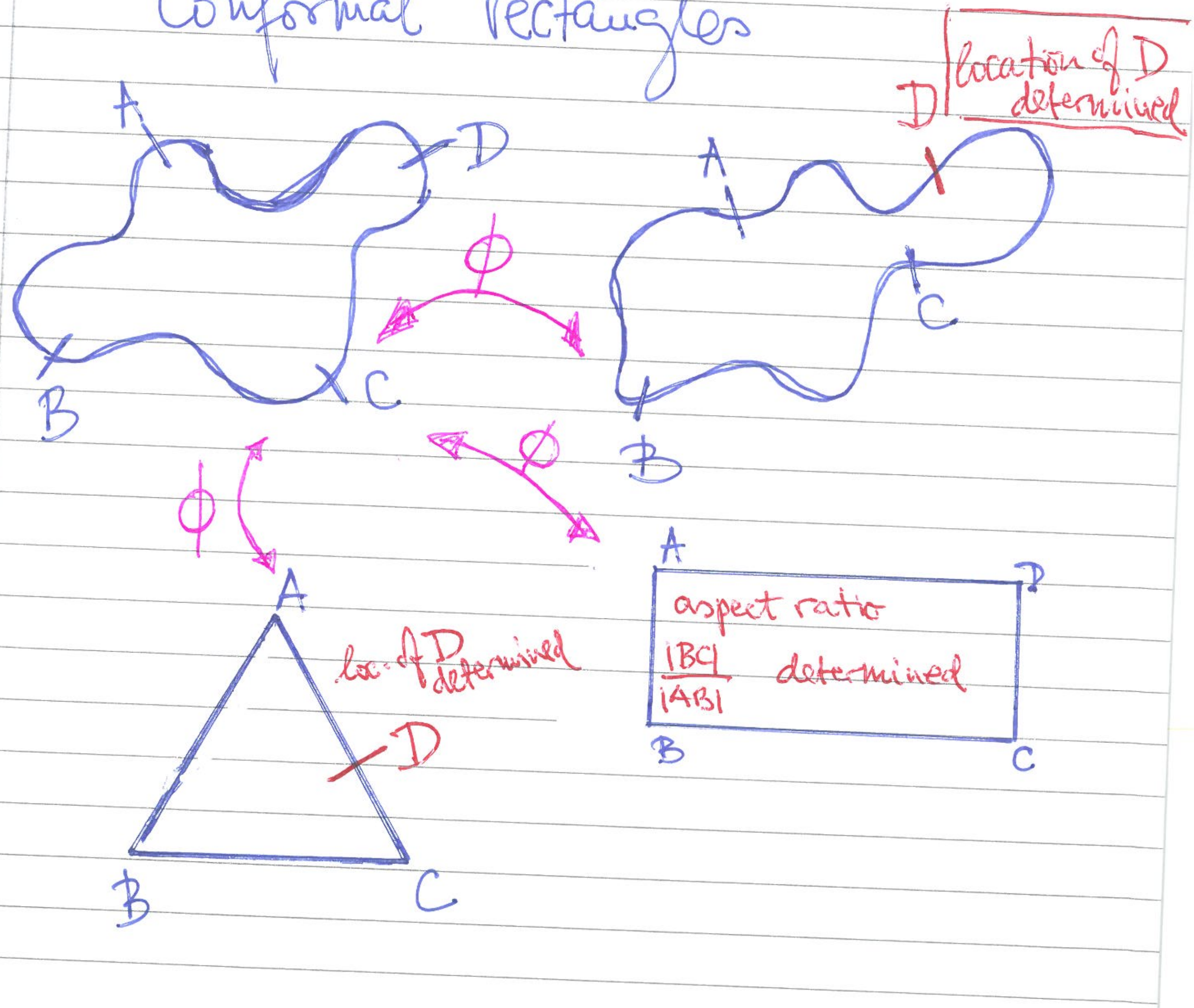


$\boxed{\text{unique}}$

centred Conformal domains with one marked point on the boundary



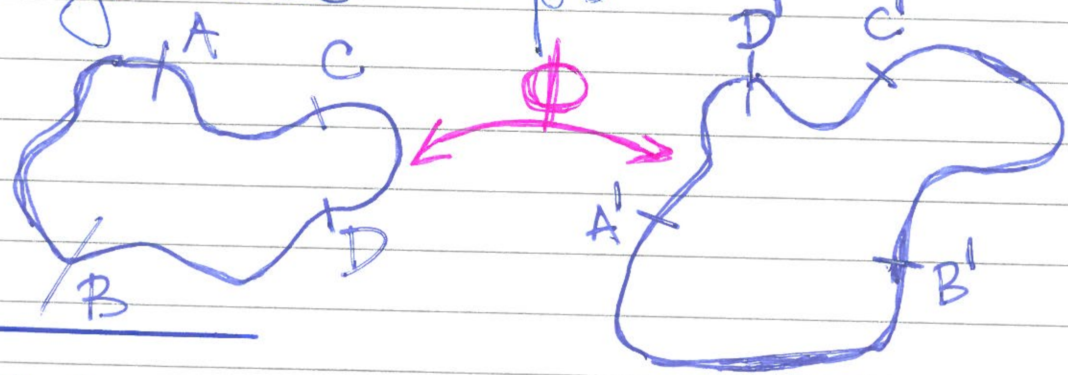
Conformal rectangles



John Cardy (1992)

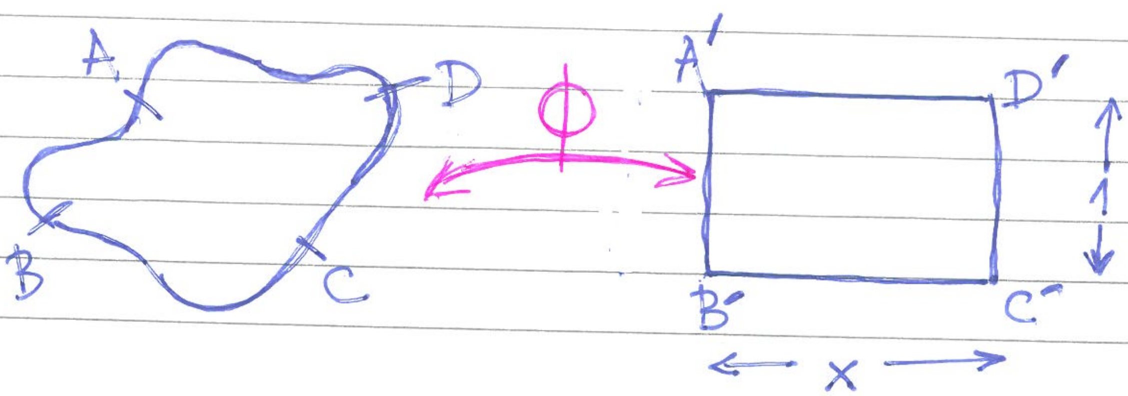
Conjecture 2 : Assume that the underlying mosaic $\mathcal{G}$ is invariant to at least one non-trivial rotation (e.g. $\mathcal{S}$, $\mathcal{T}$, $\mathcal{H}$, ... drawn properly)

Then the limit is invariant under conformal maps



Recall basics of complex analysis and conformal mappings ...

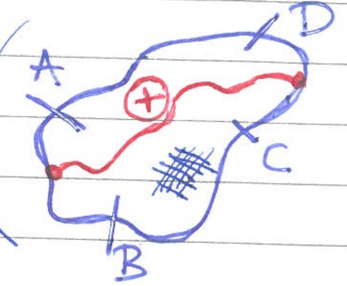
There is a unique (up to similarity) rectangle, such that



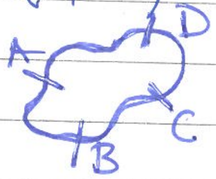
Cardy's Formula (John Cardy 1992)

Under the symmetry assumptions of Conjecture 2

$$\lim_{\delta \downarrow 0} \mathbb{P}_{p_c}\left(\right) = \pi(x)$$



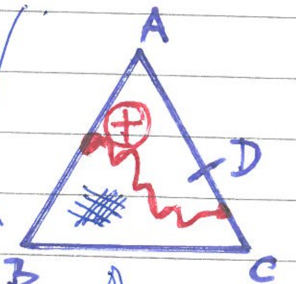
where $x \in (0,\infty)$ is the aspect ratio of the domain



and $x \mapsto \pi(x)$ is $\boxed{\text{explicit}}$ in terms of a hypergeometric function

Carleson's reformulation (Lennart Carleson 199?)

$$\lim_{\delta \downarrow 0} \mathbb{P}_{p_c}\left(\right) = \frac{|DC|}{|AC|}$$



(regular: $|AB| = |BC| = |CA|$).

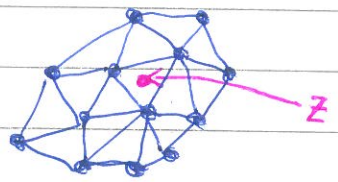
Suggests that the triangular structure / triangular lattice will have some special rôle in the story

Smirnov's Theorem — preparations.

We will consider only fully triangulated planar graphs and eventually the triangular lattice

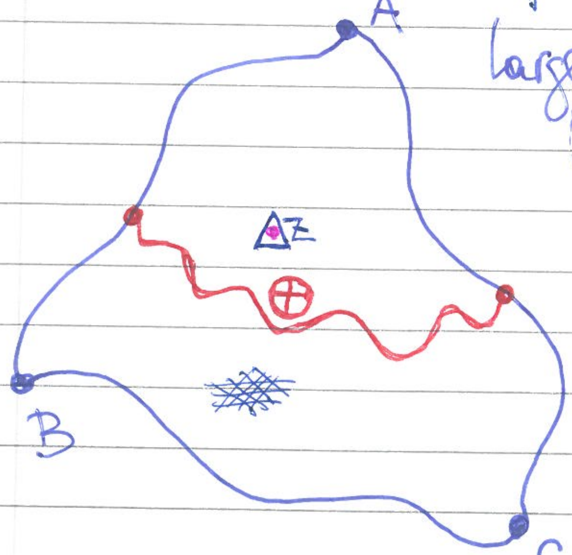
Microscopically (on the graphs):

planar points = (centre of) elementary triangular faces of the graph.



a point z on the discretized plane.

The microscopic object:



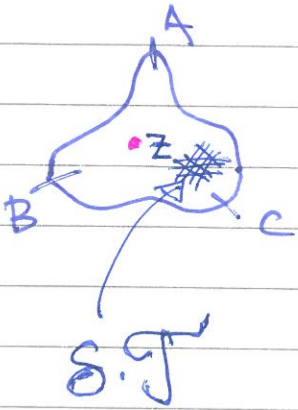
large triangle on a fully triangulated planar graph

← event denoted $Q_A(z)$

similarly define $Q_B(z)$

$Q_C(z)$

$Q_A^\delta(z)$ same with

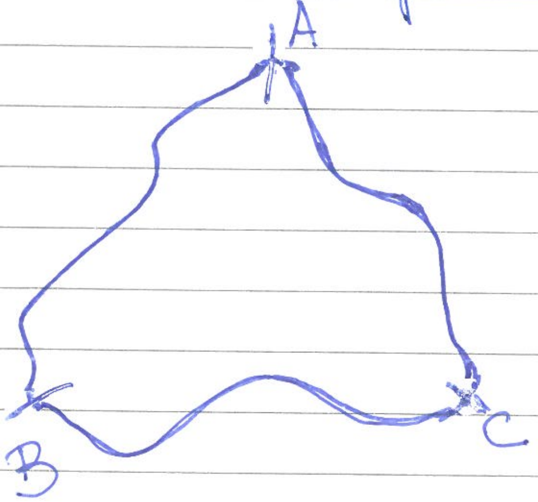


$\subset \mathbb{R}^2 = \mathbb{C}$ fixed

$$H_A(z) := \mathbb{P}_{1/2}\big(Q_A(z)\big)$$ and similarly $H_B(z), H_C(z)$

$$H_A^\delta(z) := \mathbb{P}_{1/2}\big(Q_A^\delta(z)\big)$$ $H_B^\delta(z), H_C^\delta(z)$

The macroscopic object:



$$h_A : \bar{\mathcal{D}} \longrightarrow [0,1]$$

$$\Delta h_A = 0 \text{ in } \mathcal{D}^\circ$$

$$h_A(A) = 1$$

$$h_A\big|_{\widehat{BC}} = 0$$

$$\tau = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$\tau^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

$$\frac{\partial h_A}{\partial(\tau\nu)}\bigg|_{\widehat{AB}} = 0 = \frac{\partial h_A}{\partial(\tau^2\nu)}\bigg|_{\widehat{AC}}$$

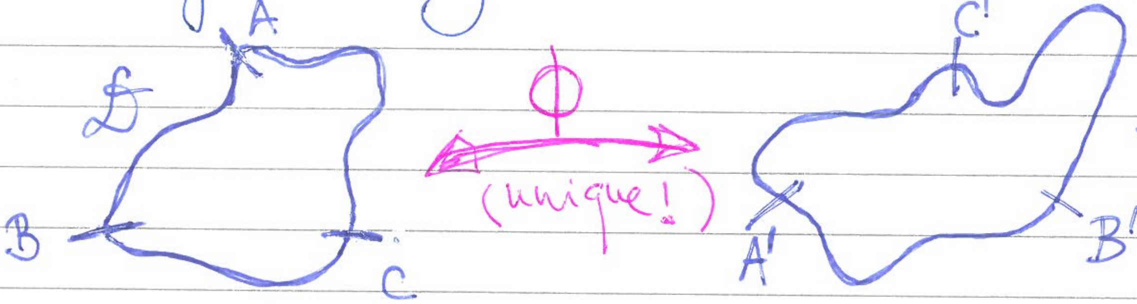
ν = counterclockwise unit tangent

Define similarly h_B and h_C.

Facts

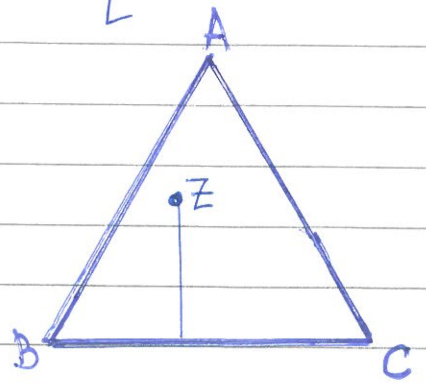
① h_A, h_B, h_C are uniquely determined by the PDE + BC.

② The definition of h_A, h_B, h_C is conformally invariant:



$$h'_{A'} \circ \phi = h_A \quad ; \text{etc.}$$

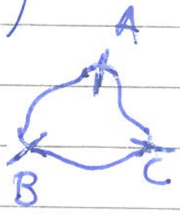
③ Particularly simple form on the equilateral triangle



$$h_A(z) = \frac{\text{dist}(z, \overline{BC})}{\text{dist}(A, \overline{BC})}$$

linear.

Theorem (S. Smirnov, 2002)

Let the conformal triangle 

$\subset \mathbb{R}^2$ be fixed. Then

$$H^{\delta}_{A,BC} \longrightarrow h_{A,BC}$$

uniformly for $z \in \overline{\mathcal{D}}$.

Corollary: Cardy's Formula holds.

Remark: Proof valid only for the triangular lattice $\delta\mathcal{T}$, $\delta \downarrow 0$.
We'll see why.

Proof of Smirnov's Theorem

Main Steps

I: Uniform equicontinuity of

$$H^{\delta}_{A,B,C} : \mathcal{D} \to \mathbb{R} \ , \quad \delta > 0$$

Hence – by Arzela-Ascoli – uniform pre-compactness of $\{H^{\delta}_{A,B,C} : \delta > 0\}$ in $C(\mathcal{D})$.

II: Discrete complex analysis on δT. Discrete Cauchy integrals. Analyticity of certain linear combinations of $\lim_{\delta \downarrow 0} H^{\delta}_{A,B,C}$. Hence: harmonicity of

$$\lim_{\delta \downarrow 0} H^{\delta}_{A,B,C}$$

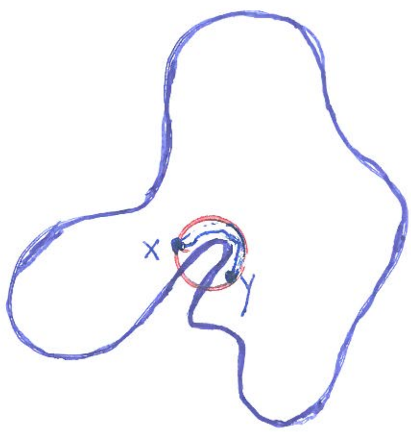
III. Identification of $\lim_{\delta \downarrow 0} H^{\delta}_{A,B,C}$ – by checking the boundary conditions

I. Uniform equicontinuity

Define $w: \mathbb{R}_+ \to \mathbb{R}_+$ (depending on $\mathcal{D}$)

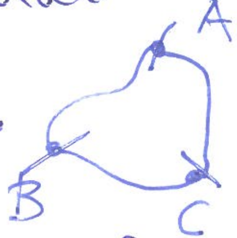
$w(r) := \inf \{ \delta : x,y \in \mathcal{D},\ |x-y| \leq r \Rightarrow$ an arc connecting x & y in $\mathcal{D}$ is within a disc of diameter $\delta \}$

- $w(r) \geq r$, and $w(r) \equiv r$ if $\mathcal{D}$ is convex
- $r \mapsto w(r)$ is nondecreasing, $\lim_{r \downarrow 0} w(r) = 0$



turn page

Proposition 1 (uniform equicontinuity)

There exists $\alpha > 0$, such that for any conformal triangle 

$\subseteq \mathbb{R}^2$ there is a $C < \infty$ (also depends on $\mathcal{D}$). such that

$(\forall \delta > 0)$ $\forall z, z' \in \mathcal{D}$ with $|z - z'| > \delta$

$$\left| H_A^\delta(z) - H_A^\delta(z') \right| \leq C \cdot w(|z - z'|)^\alpha ,$$

and similarly for H_B^δ, H_C^δ.

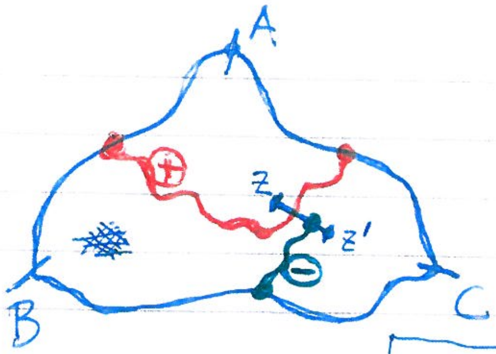
Proof of Proposition 1:

$$\left| H_A^\delta(z) - H_A^\delta(z') \right| \leq$$

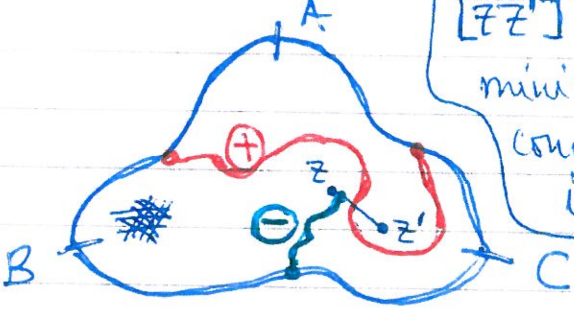
$$\mathbb{P}\left(Q_A^\delta(z) \setminus Q_A^\delta(z') \right) +$$

$$\mathbb{P}\left(Q_A^\delta(z') \setminus Q_A^\delta(z) \right)$$

$$Q_A^\delta(z) \setminus Q_A^\delta(z') \subseteq$$



$$Q_A^\delta(z') \setminus Q_A^\delta(z) \subseteq$$



[zz'] denotes a minimal arc connecting z & z' in D

$\exists\, c > 0$ (depending on  only) such that

$$\max\left\{ \operatorname{dist}([z,z'], \widehat{AB}), \operatorname{dist}([z,z'], \widehat{BC}), \operatorname{dist}([zz'], \widehat{CA}) \right\} > c$$

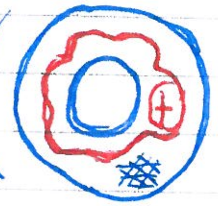
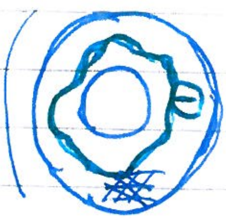
(assuming: $|z-z'|$ sufficiently small)

$\Rightarrow$ $[z,z]$ is separated from the distant arc by

$$\left|\log_2 w(|z-z'|)\right| - c$$

concentric, disjoint annuli 

By RSW:

$\mathbb{P}_{1/2}\Big($  $\Big) = \mathbb{P}_{1/2}\Big($  $\Big) \geq \varepsilon > 0$

uniformly for $\delta > 0$.

Hence...

$$\left| H_A^\delta(z) - H_A^\delta(z') \right| \leq C\,(1-\varepsilon)^{\left|\log_2 w(|z-z'|)\right|}$$

$$\boxed{\alpha = \left|\log_2 (1-\varepsilon)\right|}$$

$$= C\, w\big(|z-z'|\big)^{\alpha}, \quad \square.$$

By Arzela-Ascoli:

$\{ H_A^\delta : \delta > 0 \}$ is (pre)compact in $C(\overline{\Omega})$. $\square$

<u>Remark</u> Only RSW was used!!

II. Discrete Cauchy integrals

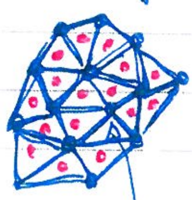
Let $\mathcal{D} \subset \mathbb{C}$ be open & simply connected

$f : \mathcal{D} \to \mathbb{C}$ is holomorphic if and only if

$$\oint_{\Gamma} f(z)\,dz = 0$$

on all simple loops $\Gamma \subset \mathcal{D}$

Cauchy loop integrals on triangulations:

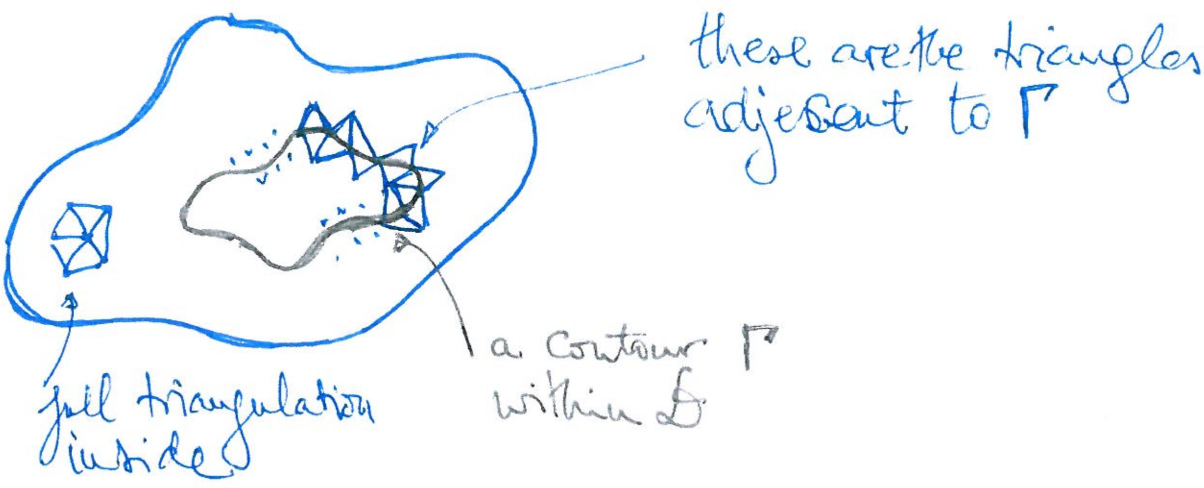
$F :$  $\longrightarrow \mathbb{C}$

z

(defined on the $\boxed{\text{triangular faces}}$, not on vertices!)

Notation: $F\,\nabla\!\!\nearrow = \underbrace{\pm F(\nabla)}_{\text{value of } F \text{ on } \nabla} \cdot \underbrace{\nearrow}_{\text{this vector as complex number}}$

(value of F on ∇) $\times$ (this vector as complex number)



The discrete Cauchy contour integral:

$$\text{"}\oint_\Gamma F\,dz\text{"} := \sum_{\in \Gamma} F$$

An identity:

$$\boxed{\text{is inside } \Gamma}$$

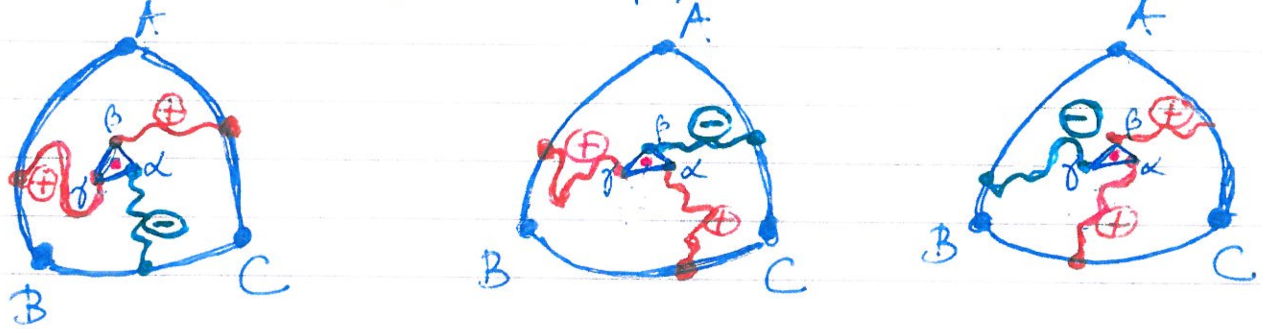
$$\sum_{\in \Gamma} F = \sum_{\in \operatorname{int}\Gamma} F$$

$$= \sum_{\in \operatorname{int}\Gamma} F \left\{ \quad + \quad + \quad \right\}$$

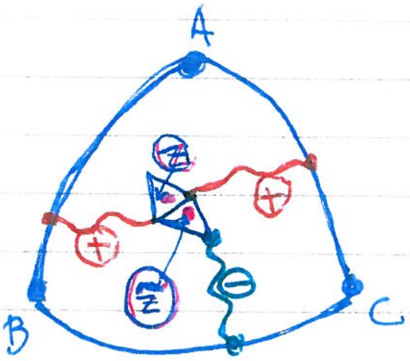
We compute "$\oint_\Gamma (\alpha H_A + \beta H_B + \gamma H_C)\,dz$"

on contours Γ within the fixed conformal triangle

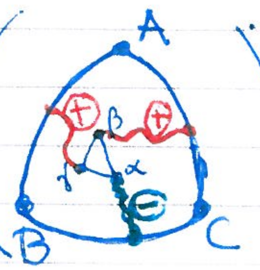
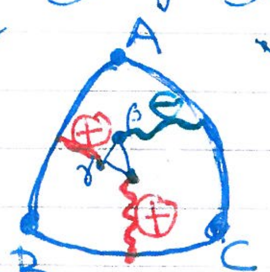
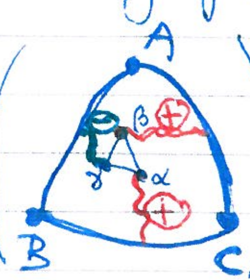
The main (microscopic) events considered



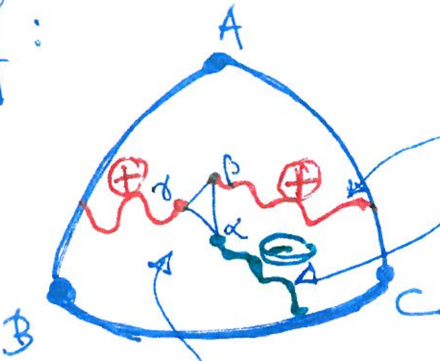
Relation to $Q_A(z)$:

 $= Q_A(z) \setminus Q_A(\tilde{z})$

Lemma: (valid on any fully triangulated graph)

$\mathbb{P}_{1/2}\Big($  $\Big) = \mathbb{P}_{1/2}\Big($  $\Big) = \mathbb{P}_{1/2}\Big($  $\Big)$

Proof:

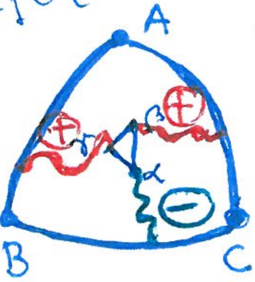
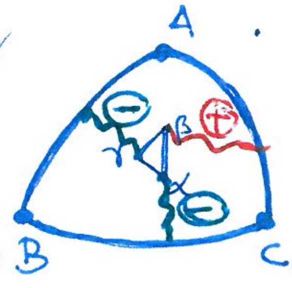


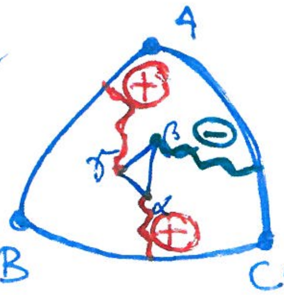
condition on these being the most-clockwise / most-counterclockwise ⊕/⊖ crossings

$\beta \leftrightarrow \oplus \ \widehat{AC}, \quad \alpha \leftrightarrow \ominus \ \widehat{BC}$

and flip colours here (using the geometric Markov property!)

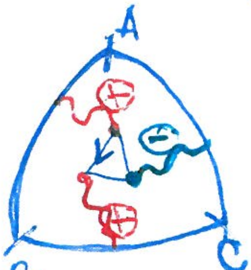
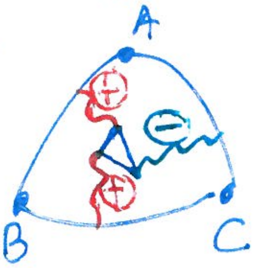
Get:

$\mathbb{P}_{1/2}\Big($  $\Big) = \mathbb{P}_{1/2}\Big($  $\Big)$

$= \mathbb{P}_{1/2}\Big($  $\Big)$

$\square$

Notation (from now on)

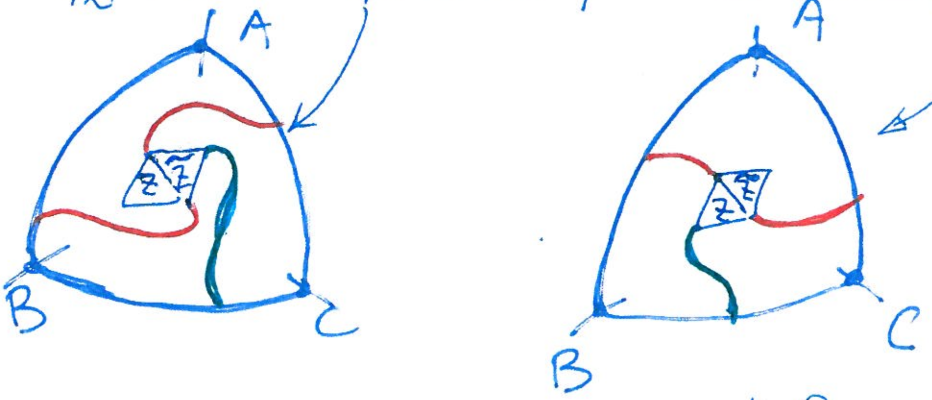
 $:= \mathbb{P}_{1/2}\Big($  $\Big) \times$ 

this little vector, as a complex number

somewhere in $\mathcal{D}$:

$$H_A(z) - H_A(\tilde z) = \mathbb{P}_{1/2}\left(Q_A(z)\right) - \mathbb{P}_{1/2}\left(Q_A(\tilde z)\right)$$

$$= \mathbb{P}_{1/2}\left(\underbrace{Q_A(z) \setminus Q_A(\tilde z)}\right) - \mathbb{P}_{1/2}\left(\underbrace{Q_A(\tilde z) \setminus Q_A(z)}\right)$$

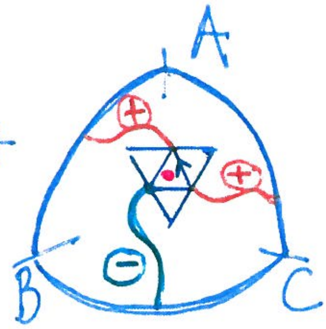


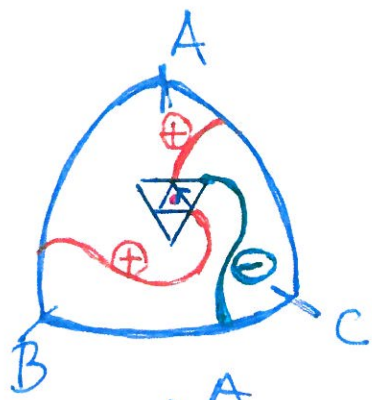
Inserting in the expression of "$\oint_\Gamma H_A \, dz$" =

$$\sum_{\Delta \in \mathrm{int}\,\Gamma} H_A \Big\{$$  $+$  $+$  $\Big\} =$

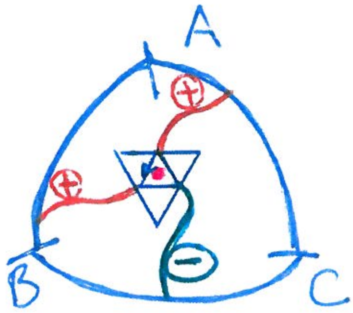
(turn page)

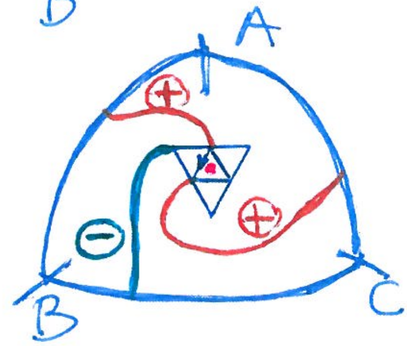
$$\sum_{\triangle \in \operatorname{int}\Gamma}$$

\+ 

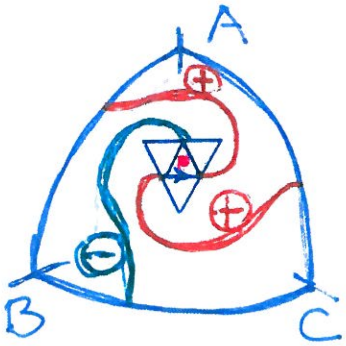
− 

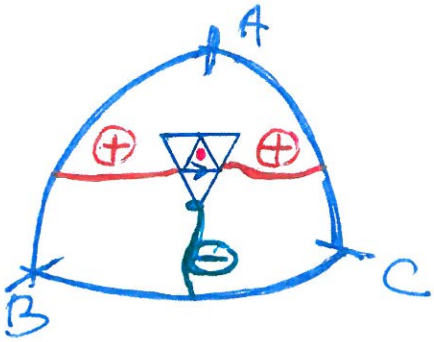
mind the little arrow!

\+ 

− 

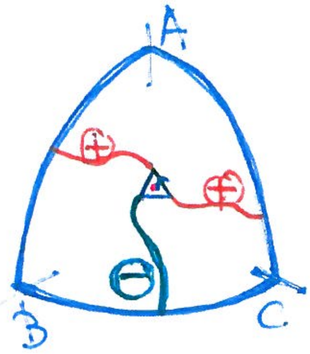
$\boxed{\text{mind the pink dots •}}$

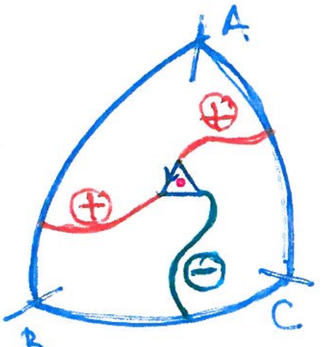
\+ 

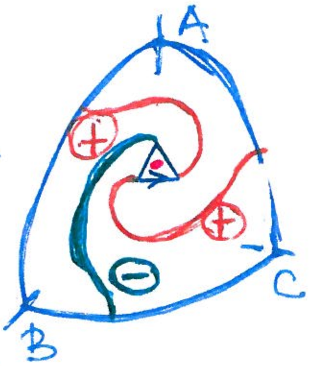
− 

=

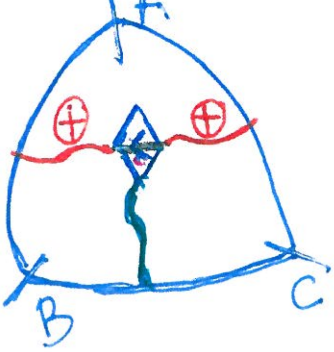
$$2\sum_{\triangle \in \operatorname{int}\Gamma}$$

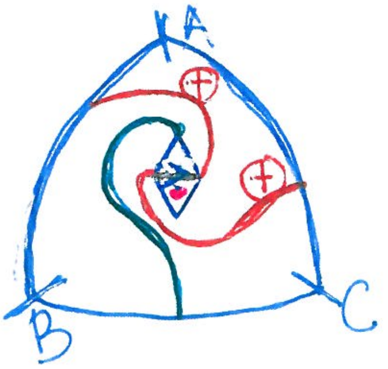


\+ 

\+ 

$$\sum_{\diamond \in \Gamma}$$

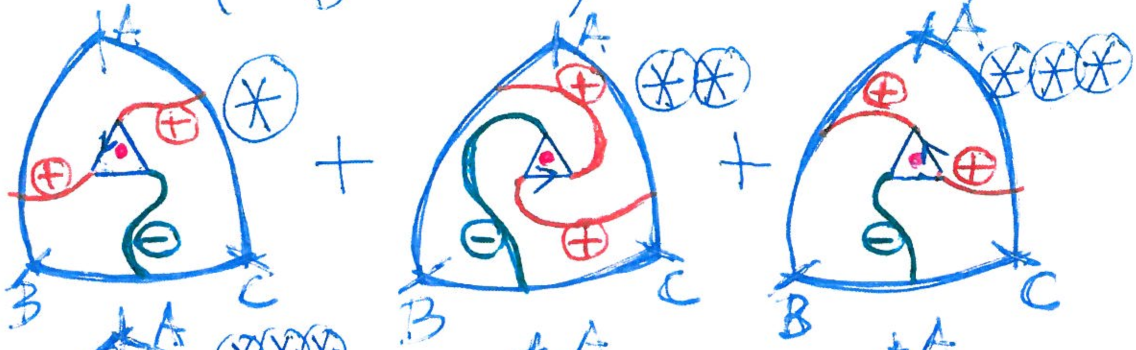
− 

− 

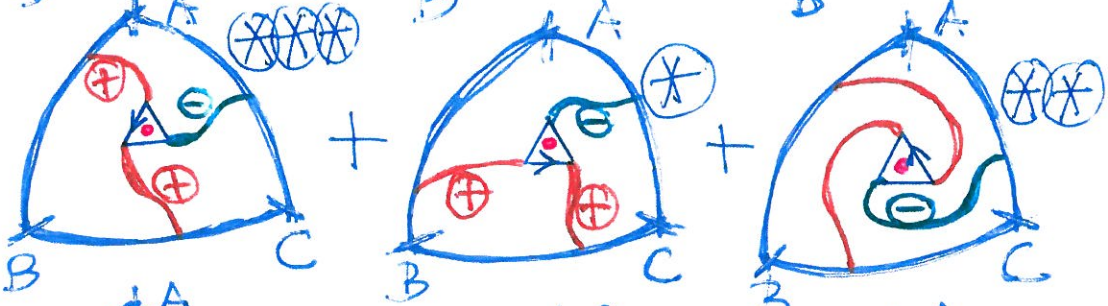
The last sum (boundary terms) is $o(1)$ as $\delta \downarrow 0$.

$$\text{"}\oint_\Gamma (\alpha H_A + \beta H_B + \gamma H_C)\, dz\text{"} = \qquad \alpha, \beta, \gamma \in \mathbb{C}$$

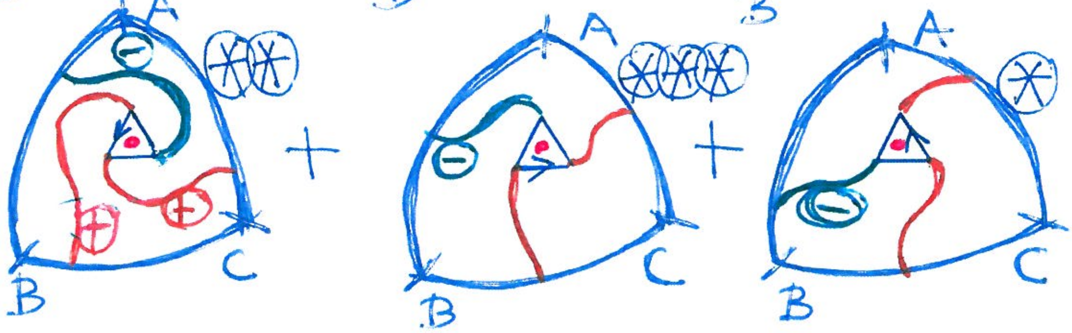
$\alpha \sum_{\Delta \in \mathrm{int}\,\Gamma}$



$\beta \sum_{\Delta \in \mathrm{int}\,\Gamma}$



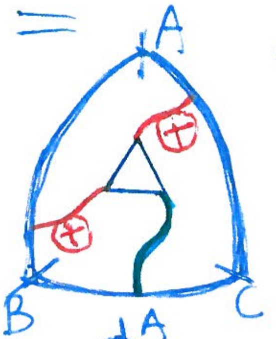
$\gamma \sum_{\Delta \in \mathrm{int}\,\Gamma}$

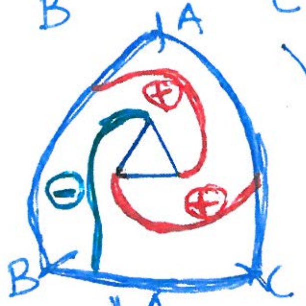


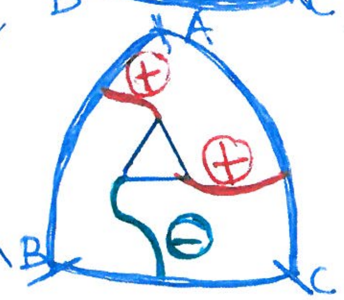
+ the boundary sum.

by the Lemma the terms marked (*) / (*)(*) / (*)(*)(*) are equal

$$\text{“}\oint_\Gamma \left(\alpha H_A + \beta H_B + \gamma H_C\right) dz\text{”} =$$

$$\sum_{\Delta \in \operatorname{int}\Gamma} \left(\alpha \Delta + \beta \Delta + \gamma \Delta\right) \mathbb{P}_{1/2}\Bigg($$  $$\Bigg) +$$

$$\sum_{\Delta \in \operatorname{int}\Gamma} \left(\alpha \Delta + \beta \Delta + \gamma \Delta\right) \mathbb{P}_{1/2}\Bigg($$  $$\Bigg) +$$

$$\sum_{\Delta \in \operatorname{int}\Gamma} \left(\alpha \Delta + \beta \Delta + \gamma \Delta\right) \mathbb{P}_{1/2}\Bigg($$  $$\Bigg)$$

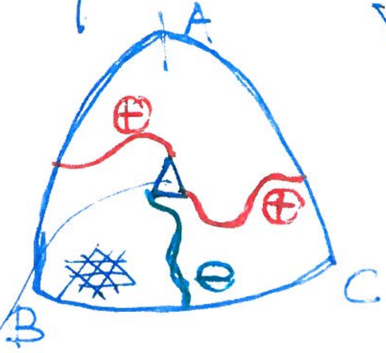
+ the boundary sum.

So far: arguments valid for arbitrary triangulations.

Now we turn to periodic triangulations with symmetries, so that RSW at $p = \frac{1}{2}$ holds.

By the equi-continuity argument

$$\mathbb{P}_{1/2}\Bigg($$



$$\Bigg) \leq C \cdot \delta^{\alpha}$$

with some $\alpha \in (0,1)$

this is an elementary triangular face

Hence:

1. Given a piecewise smooth macroscopic contour $\Gamma \subset \mathcal{D}$

$$\sum_{\diamond \in \Gamma} \Big| \text{boundary terms} \Big| \leq C \, \delta^{\alpha}$$

The boundary sum is negligible as $\delta \to 0$.

2. The bulk sum vanishes (only) if

$$\left.\begin{aligned} \alpha \triangle + \beta \triangle + \gamma \triangle &= 0 \\ \alpha \triangle + \beta \triangle + \gamma \triangle &= 0 \\ \alpha \triangle + \beta \triangle + \gamma \triangle &= 0 \end{aligned}\right\} \text{ on all triangular faces}$$

$\alpha = \beta = \gamma = 1$ is certanly OK ✓

If all triangular faces are equilateral then

$\alpha = \tau^0 = 1$, $\beta = \tau = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$; $\tau^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

is another solution.

And there are no more (linearly independent)

<u>Conclusion of step II:</u>

Assume $H^{\delta'}_{A,B,C} \xrightarrow{u} \chi_{A,BC}$ as $\delta' \downarrow 0$ (on some subsequence)

Then for any piecewise smooth contour Γ in $\mathcal{D}$

$$\oint_\Gamma \left(\chi_A(z) + \chi_B(z) + \chi_C(z)\right) dz = 0$$

$$\oint_\Gamma \left(\tau^0 \chi_A(z) + \tau^1 \chi_B(z) + \tau^2 \chi_C(z)\right) dz = 0$$

That is.

$$z \mapsto \chi_A(z) + \chi_B(z) + \chi_C(z)$$

$$z \mapsto \tau^0 \chi_A(z) + \tau^1 \chi_B(z) + \tau^2 \chi_B(z) =: f(z)$$

are holomorphic in $\mathcal{D}$

In particular:

$$\chi_A(z) + \chi_B(z) + \chi_C(z) \equiv 1$$

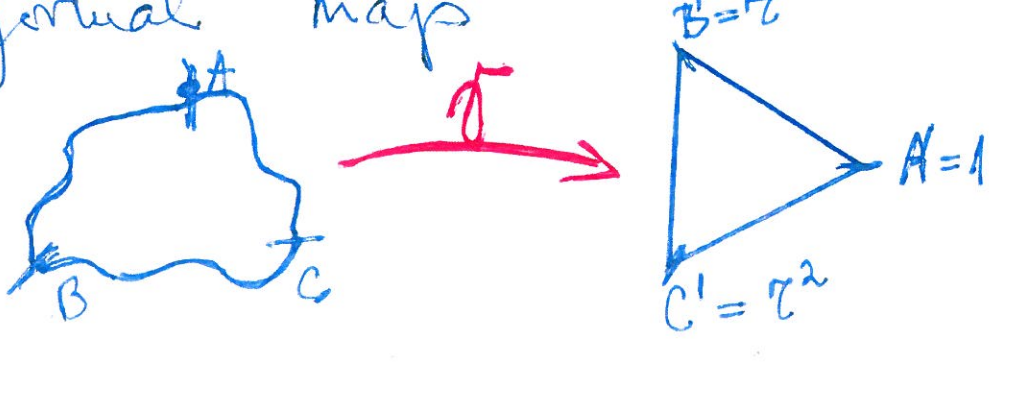
since: it is $\mathbb{R}$-valued & holomorphic

$$\Delta \chi_A(z) \equiv 0, \ \Delta \chi_B(z) \equiv 0, \ \Delta \chi_C(z) \equiv 0 \ \text{ in } \mathcal{D}$$

due to Cauchy - Riemann

III Identification, boundary conditions

Proposition $f : \mathcal{D} \to \mathbb{C}$ is the unique conformal map

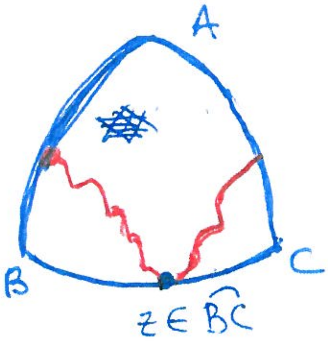


Proof :

① $\chi_A(A) = \chi_B(B) = \chi_C(C) = 1$

② $\chi_A\big|_{\widehat{BC}} \equiv \chi_B\big|_{\widehat{CA}} \equiv \chi_C\big|_{\widehat{AB}} \equiv 0$ ✓

$\mathbb{P}_{1/2}\Bigg($

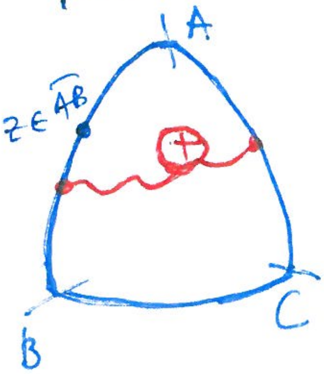


$\Bigg) \longrightarrow 0$

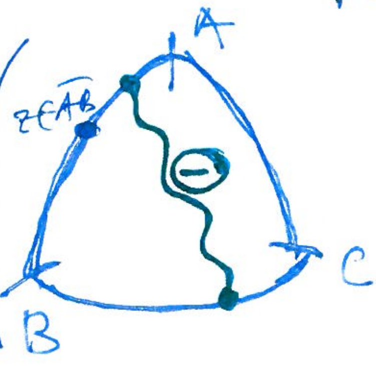
actually $< C \cdot \delta^{\alpha}$

③ $(\chi_A + \chi_B)\big|_{\widehat{AB}} \equiv (\chi_B + \chi_C)\big|_{\widehat{BC}} \equiv (\chi_C + \chi_A)\big|_{\widehat{CA}} \equiv 1$

$\mathbb{P}_{1/2}\Bigg($



$\Bigg) + \mathbb{P}_{1/2}\Bigg($



$\Bigg)$

$= 1$

and ⊕/⊖ flip

④ $\widehat{AB} \ni z \longmapsto \chi_A(z)$ is strictly decreasing

$\widehat{AC} \ni z \longmapsto \chi_A(z)$ — " —

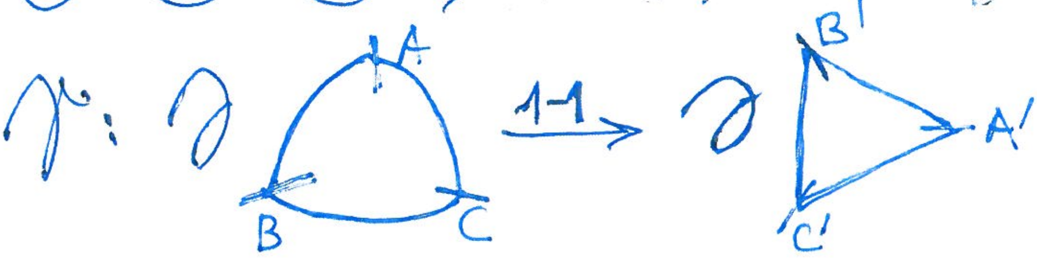
$\widehat{BC} \ni z \longmapsto \chi_B(z)$ — " —

$\widehat{BA} \ni z \longmapsto \chi_B(z)$ — " —

$\widehat{CA} \ni z \longmapsto \chi_C(z)$ — " —

$\widehat{CB} \ni z \longmapsto \chi_C(z)$ — " —

① + ② + ③ + ④ $\Longrightarrow$



+ γ is holomorphic in $\mathcal{D}$

The Proposition follows $\square$

The relations:

$$\chi_A(z) + \chi_B(z) + \chi_C(z) = 1$$

$$\tau^0 \chi_A(z) + \tau \chi_B(z) + \tau^2 \chi_C(z) = \gamma(z)$$

$$\chi_A(z), \chi_B(z), \chi_C(z) \in \mathbb{R}$$

uniquely determine $\chi_A(z), \chi_B(z), \chi_C(z)$