Bálint Tóth : Percolation 5

Two dimensions 1:
Planar duality and its consequences

2d percolation is very special for two reasons:

(a) topology:
Connections & cutsets are both 1-d. This leads to a natural notion of duality

(b) analysis: The plane is endowed in natural way with complex structure: $\mathbb{R}^2 = \mathbb{C}$

Both have enormously important consequences.

Topological duality:

Whitney duality of plane graphs

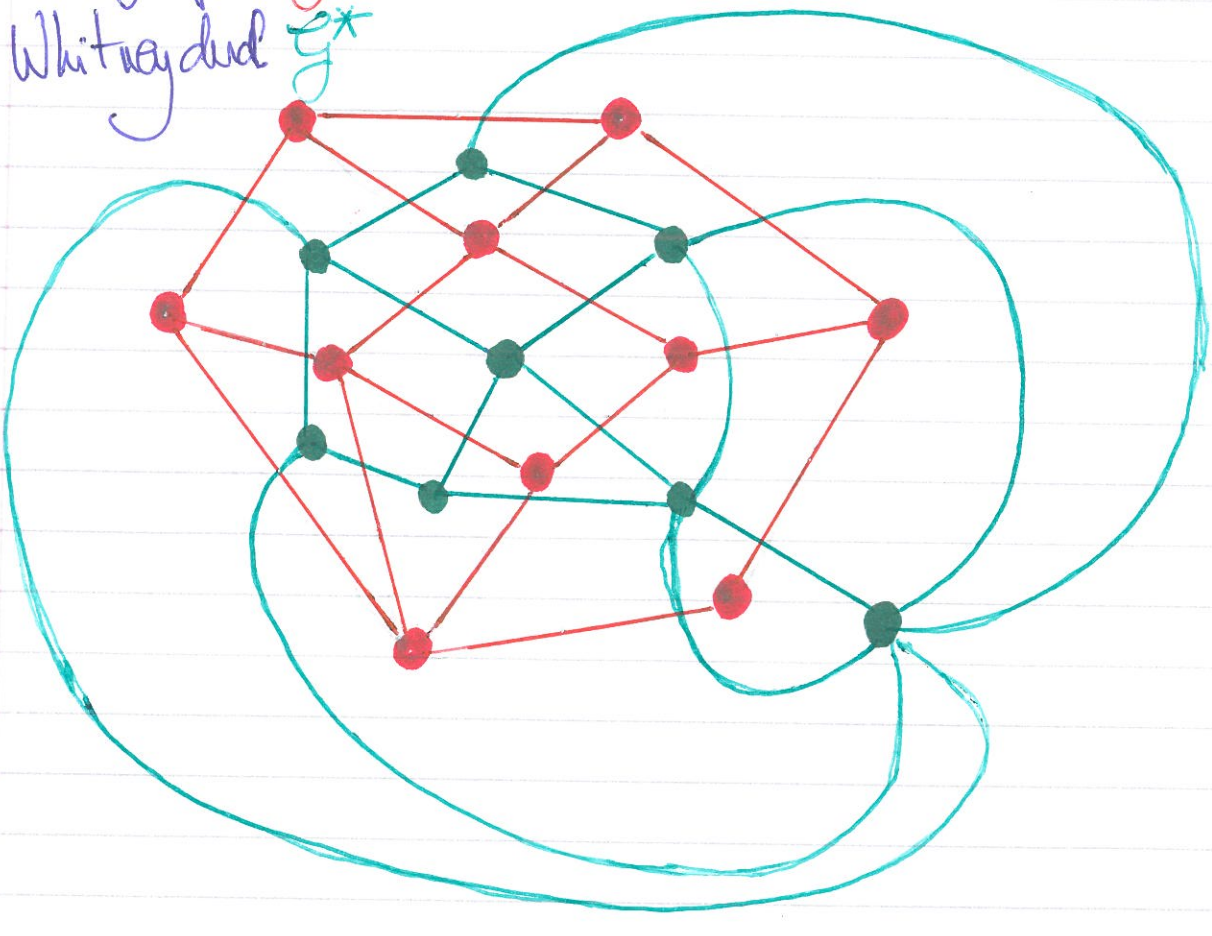
plane graph = • planar graph <u>embedded</u> in $\mathbb{R}^2$

(finite or denumerable)

• (locally) bdd degrees

• (locally) bdd – from below & from above – edge length

a plane graph G and its Whitney dual G^*



Remarks and more examples:

① embedding matters

② $(\mathcal{G}^*)^* = \mathcal{G}$

③ (external) boundary of simply connected spanned subgraph of $\mathcal{G}$ ($\mathcal{G}^*$) is a simple circuit on $\mathcal{G}^*$ ($\mathcal{G}$)

④ $\mathcal{G}$ = square lattice $\mathcal{G}^*$ = square lattice (self dual)

$\mathcal{G}$ = triangular lattice $\mathcal{G}^*$ = hexagonal lattice

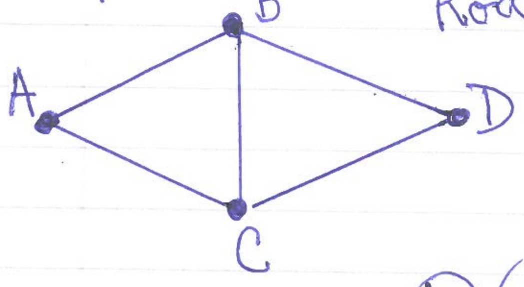
Whitney duality is relevant for $\boxed{\text{bond}}$ percolation:
we consider $\boxed{\text{jointly}}$ bond percolation problems on $\mathcal{G}$ and $\mathcal{G}^*$:

⊕ open on $\mathcal{G}$ / ⊖ closed on $\mathcal{G}^*$ with probab p
⊖ closed on $\mathcal{G}$ / ⊕ open on $\mathcal{G}^*$ with probab $1-p$

Connected path on $\mathcal{G}$ = blocking boundary on $\mathcal{G}^*$

For aesthetic / symmetry reasons we change notation: open = ⊕, closed = ⊖

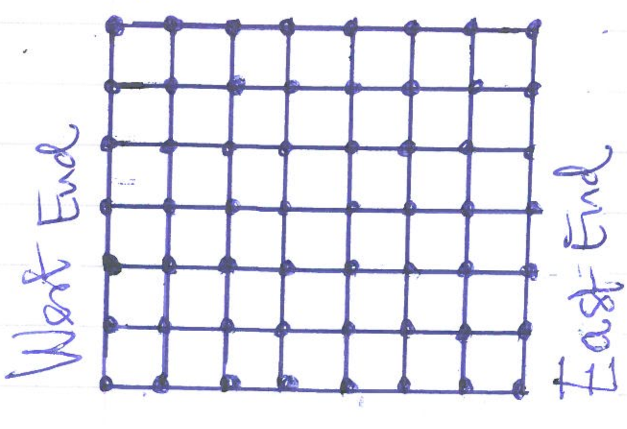
Example 1.



Road map between four cities.
Roads blocked by snowstorm with probab $\frac{1}{2}$, independently

$$\mathbb{P}(\text{one can drive from } A \text{ to } D) = ?$$

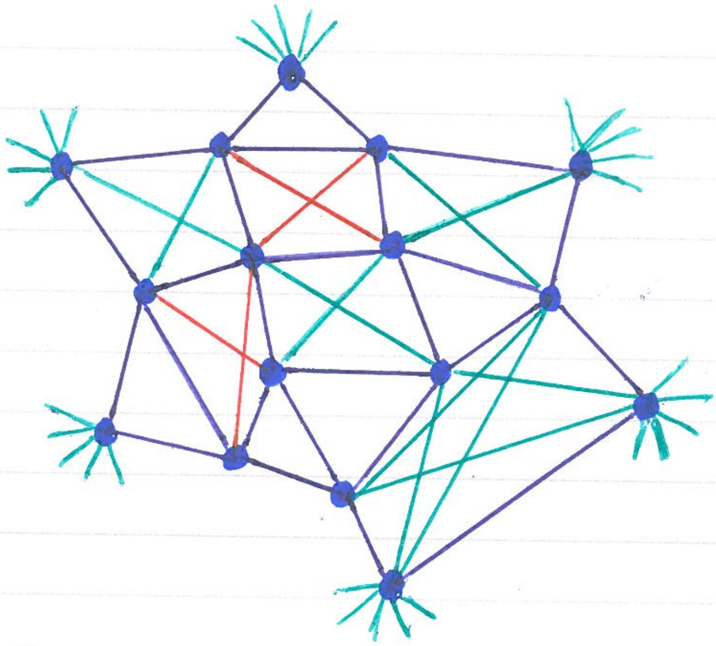
Example 2.:



street map of Xanadu
streets blocked by accident / traffic jam / robbery / ... with probab 1/2 — indep.

$$\mathbb{P}\left(\begin{matrix}\text{one can drive from East End}\\ \text{to West End}\end{matrix}\right) = ?$$

Towards a more general notion of topological duality: draw the line-graphs of $\mathcal{G}$ and $\mathcal{G}^*$:



Consists of
- Skeleton: plane graph (vertices + blue edges)
- faces fully decorated with all diagonals either red or green

Blue + Red = line graph of $\mathcal{G}$

Blue + Green = line graph of $\mathcal{G}^*$

Matching pairs of mosaic graphs:

$\mathcal{M}$ = finite or countable/locally finite planar graph embedded in $\mathbb{R}^2$

$\mathcal{S}$ = subset of faces of $\mathcal{M}$

$\mathcal{S}^*$ = complementary subset of faces of $\mathcal{M}$

$\mathcal{G} = \mathcal{M}$ + all diagonals of all faces in $\mathcal{S}$

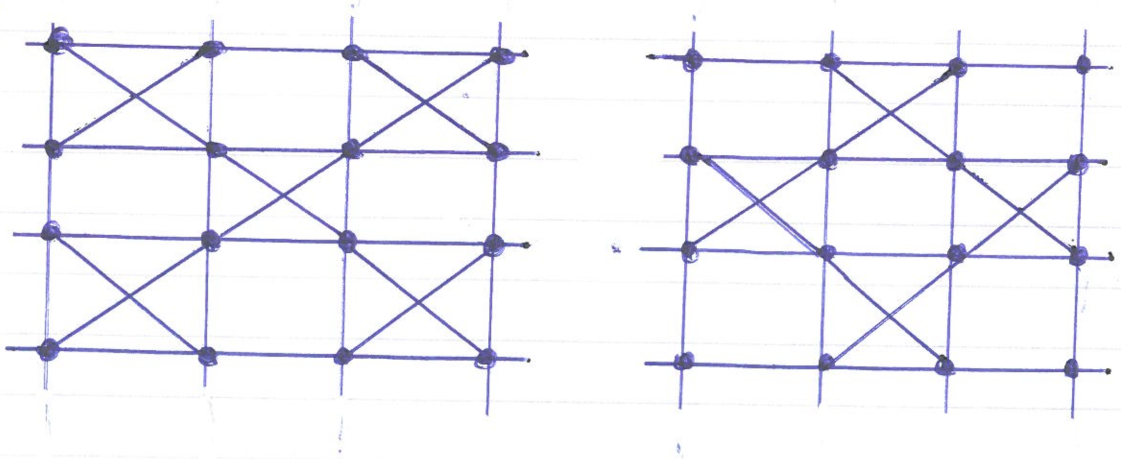
$\mathcal{G}^* = \mathcal{M}$ + all diagonals of all faces in $\mathcal{S}^*$

[Remark: $(\mathcal{G}^*)^* = \mathcal{G}$]

Examples:

① (pairs of) line-graphs of Whitney dual pairs of plane graphs:

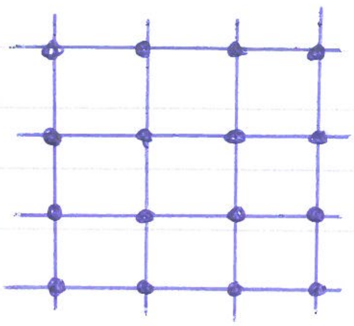
e.g.



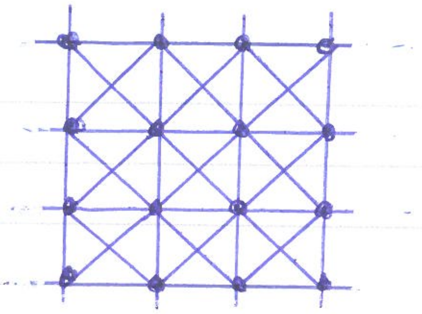
self-dual
(no surprize)

(2) any fully triangulated planar graph

(3)



$\mathcal{G}$



$\mathcal{G}^*$

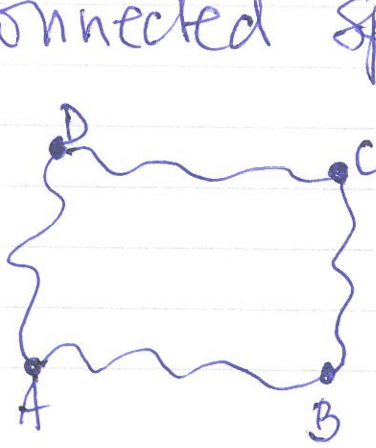
Fundamental Topological Theorem:

Let $\mathcal{G}, \mathcal{G}^*$ be a matching pair of mosaic graphs (based on $\mathcal{M}$)

(1) The external boundary (= vertices at graph distance 1 from) a $\mathcal{G}$-connected spanned subgraph of $\mathcal{G}$ forms a simple circuit on $\mathcal{G}^*$. (and vice versa)

(2) a Simple circuit on $\mathcal{G}^*$ cuts out a Connected spanned subgraph of $\mathcal{G}$.

(3) Let



be a graph-rectangle on $\mathcal{M}$ and colour the vertices within $\oplus$ or $\ominus$ then either

$\overline{DA} \xleftrightarrow[\text{on } \mathcal{G}]{\oplus} \overline{CB}$ or $\overline{DC} \xleftrightarrow[\text{on } \mathcal{G}^*]{\ominus *} \overline{AB}$

We consider $\boxed{\text{site percolation}}$ on matching pairs $\mathcal{G}, \mathcal{G}^*$ of mosaic graphs

The Big Theorem:

(T E Harris 196 , M Fisher 1962,
M Sykes, J Essam 1964, L Russo 1978,
P Seymour, D Welsh 1978, H Kesten 1980,
L Russo 1981)

Let $\mathcal{G}, \mathcal{G}^*$ be a matching pair of mosaic graphs with the following symmetries:

- periodic in $\mathbb{R}^2$
- $\frac{\pi}{2}$-rotation symmetry
-

Then $\quad p_c + p_c^* = 1$

$$\left[p_c = p_c(\mathcal{G}, \text{site}), \; p_c^* = p_c(\mathcal{G}^*, \text{site}) \right]$$

Historical notes:

T.E. Harris 1960:

$$p_H(\text{square lattice, bond}) \geq \frac{1}{2}$$

+ a gold mine of ideas

M Fisher 1961:

$$p_H(\mathcal{G}, \text{bond}) + p_H(\mathcal{G}^*, \text{bond}) \geq 1$$

where $\mathcal{G}$ is a regular (periodic + refl. symmetric + $\frac{\pi}{2}$-rot. invariant) planar lattice. Extends Harris's results.

M Sykes, J. Essam 1964:

$$p_c(\mathcal{G}, \text{site}) + p_c(\mathcal{G}^*, \text{site}) = 1$$

where $\mathcal{G}, \mathcal{G}^*$ matching pair of regular mosaic lattices. Introduce the general duality of matching pairs.

$\boxed{\text{incomplete proof}}$ still open

L. Russo 1978, P. Seymour, D. Welsh 1978:

RSW inequality and consequences

Russo: $p_H + p_T^* = 1$, $\theta(p_H) = 0, \ldots$

H. Kesten 1980:

$$p_H(\text{square lattice, bond}) = p_T(\text{square lattice, bond}) = \tfrac{1}{2}$$

L. Russo 1981:

$$p_c + p_c^* = 1$$

adapts Kesten's proof to general matching pairs $\mathcal{G}, \mathcal{G}^*$.

[actually: proves for square lattice + its matching pair]

J. Wierman 1981:

$$p_c(\triangle, \text{bond}) = 2 \sin \frac{\pi}{18}$$

$$p_c(\hexagon, \text{bond}) = 1 - 2 \sin \frac{\pi}{18}$$

Kesten–Russo + star-triangle transformation

M. Sykes, J Essam 1964:

Consider square lattice bond percolation

$$K(p) := \mathbb{E}_p\left(|C_0|^{-1}\right) \qquad (\leq 1)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{P}_p(|C_0| = n)$$

$$= \lim_{L\to\infty} \mathbb{E}_p^L\left(|C_0|^{-1}\right)$$ on discrete torus of size $L \times L$

$$= \lim_{L\to\infty} \frac{1}{L^2} \mathbb{E}_p^L(\#\text{ of connected clusters})$$

Theorem: $K(p) + p = K(1-p) + (1-p)$

Remark: More generally: for a matching pair of mosaic graphs $\mathcal{G}, \mathcal{G}^*$

$$K(p) = a\,K^*(1-p) + \varphi(p)$$

constant polynomial

<u>Consequence</u>: $\boxed{\text{Assuming}}$ that

$p \mapsto \kappa(p)$ is analytic in $[0,1] \setminus \{p_c\}$
and singular p_c

it follows that $p_c + p_c^* = 1$.

Status of this assumption

$p \mapsto \kappa(p)$ C^∞ in $[0,1] \setminus \{p_c\}$ ✓
C^2 in $[0,1]$

<u>Conjecture</u>: It is not C^3 at p_c

<u>Proof of the Theorem</u>:

Let $\mathcal{G}$ be a graph drawn on $\mathbb{T}_2$ with no intersections of edges (other than the vertices)

$v = v(\mathcal{G})$ = # of vertices

$e = e(\mathcal{G})$ = # of edges

$f = f(\mathcal{G})$ = # faces (connected components of $\mathbb{T}^2 \setminus \mathcal{G}$)

$c = c(\mathcal{G})$ = # connected components

Euler:

$$c - v + e - f = \begin{cases} -1 & \text{if } \mathcal{G} \sim \cdot \\ 0 & \text{if } \mathcal{G} \sim \\ +1 & \text{if } \mathcal{G} \sim \end{cases}$$

where $\mathcal{G} \sim *$ means that $\mathcal{G}$ (embedded in $\mathbb{T}^2$) can be shrinked by a continuous deformation of $\mathbb{T}^2 \circlearrowleft$, to $*$.

Consider bond percolation of density $p / 1-p$ on the discrete torus $\mathbb{T}_L^2$ and its dual $\mathbb{T}_L^{2*}$ ($\sim \mathbb{T}_L^2$) and let $\mathcal{H}_L, \mathcal{H}_L^*$ be the (random) graphs spanned.

Note: $v(\mathcal{H}_L) = v(\mathcal{H}^*) = L^2$

$$\mathbb{E}(e(\mathcal{H}_L)) = 2L^2 \cdot p$$

$$\mathbb{E}(e(\mathcal{H}_L^*)) = 2L^2 (1-p)$$

$$\left.\begin{array}{l} c(\mathcal{H}_L) = f(\mathcal{H}_L^*) \\ f(\mathcal{H}_L) = c(\mathcal{H}_L^*) \end{array}\right\} \text{by Whitney duality}$$

and

$$\mathbb{E}_p(\chi(\mathcal{H}_L^*)) = \mathbb{E}_{1-p}(\chi(\mathcal{H}_L))$$

<u>Hence</u>:

$$\mathbb{E}_p(c(\mathcal{H}_L)) - \mathbb{E}_{1-p}(c(\mathcal{H}_L)) + L^2(1-2p) = O(1)$$

$$\begin{array}{ccccccc} \dfrac{\mathbb{E}_p(c(\mathcal{H}_L))}{L^2} & - & \dfrac{\mathbb{E}_{1-p}(c(\mathcal{H}_L))}{L^2} & + & 1-2p & \to & 0 \\ \downarrow & & \downarrow & & & & \\ K(p) & - & K(1-p) & + & 1-2p & = & 0 \end{array}$$

Sykes–Essam $\square$

Streamlined proof of the Harris-Kesten-Russo Theorem (the Big Theorem)

Given

(a) Uniqueness of the infinite cluster
(Aizenman-Kesten-Newman 1987
Burton-Keane 1989), and

(b) Sharpness of the phase transition
(Menshikov 1986
Aizenman-Barsky 1987
Duminil-Copin-Tassion 2015)

the arguments simplify
Y. Zhang 1988.

Step 1: $\boxed{p_T + p_H^* \leq 1.}$ (1)

Let $p < p_T$

and $A_k := \{(k,0) \overset{+}{\longleftrightarrow} \{(x,y): x \leq 0, y \in \mathbb{R}\}\}$

Then $A_k \subseteq \{|\mathcal{C}_{(0,k)}| \geq k\}$

$$\mathbb{P}_p(A_k) \leq \mathbb{P}_p(|\mathcal{C}_{(0,k)}| \geq k) = \mathbb{P}_p(|\mathcal{C}_{0,0}| \geq k)$$

use translation invariance or periodicity here

$$\sum_{k=1}^{\infty} \mathbb{P}_p(A_k) \leq \sum_{k=1}^{\infty} \mathbb{P}_p(|\mathcal{C}_{0,0}| \geq k)$$

since $p < p_T$

$$= \mathbb{E}_p(|\mathcal{C}_{0,0}|) < \infty$$

By Borel-Cantelli

$$\mathbb{P}_p\Big(\exists \nu < \infty : (0,\nu) \overset{+}{\longleftrightarrow} \{(x,y): x \leq 0, y \in \mathbb{R}\} \ \&$$
$$\forall k > \nu : (0,k) \overset{+}{\not\longleftrightarrow} \{(x,y): x \leq 0, y \in \mathbb{R}\}\Big)$$
$$= 1$$

But then – for simple topological reasons

$$(0, \gamma + 1) \xleftrightarrow{\Theta^*} \infty$$

(on the dual problem)

We have proved: $\boxed{p < p_T} \Rightarrow \boxed{1-p \geq p_H^*}$

which implies (1)

Step 2: $\boxed{p_H + p_H^* \geq 1}$ This is essentially Harris's Thm (1960)

Actually: $\boxed{\Theta(p) \cdot \Theta^*(1-p) = 0} \Rightarrow$

Lemma ("the square root trick")

Let $A_1, A_2, \dots, A_m$ be increasing events, such that

$$\mathbb{P}_p(A_1) = \mathbb{P}_p(A_2) = \dots = \mathbb{P}_p(A_m).$$

Then, for all $i = 1, 2, \dots, m$

$$\mathbb{P}_p(A_i) \geq 1 - \left\{1 - \mathbb{P}_p\left(\bigcup_{j=1}^{m} A_j\right)\right\}^{1/m}.$$

Proof of the square root trick:

$$1-\mathbb{P}_p\Big(\bigcup_{j=1}^{m} A_j\Big) = \mathbb{P}_p\Big(\big(\bigcup_{j=1}^{m} A_j\big)^c\Big)$$

$$= \mathbb{P}_p\Big(\bigcap_{j=1}^{m} A_j^c\Big) \geq \prod_{j=1}^{m} \mathbb{P}_p(A_j^c)$$

Harris ineq

$$= \Big(1-\mathbb{P}_p(A_i)\Big)^m$$

□ Lemma.

Assume $p_H + p_H^* < 1$ and let $p \in (p_H, 1-p_H^*)$

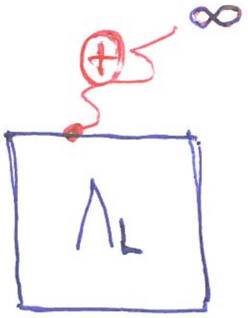
Then $\exists L < \infty$, such that

$$\mathbb{P}_p\big(\partial\Lambda_L \overset{\oplus}{\longleftrightarrow} \infty\big) > 1-\Big(1-\frac{\sqrt{3}}{4}\Big)^4$$

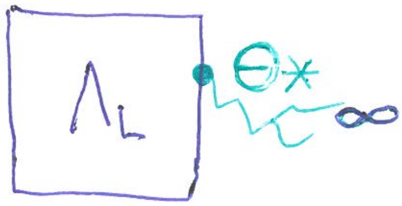
$$\mathbb{P}_p\big(\partial\Lambda_L \overset{\ominus *}{\longleftrightarrow} \infty\big) > 1-\Big(1-\frac{\sqrt{3}}{4}\Big)^4$$

where $\Lambda_L = [-L, L]^2$

Denote

$$\partial\Lambda_L^N \overset{\oplus}{\longleftrightarrow} \infty =$$ 

N = North
S = South

$$\partial\Lambda_L^E \overset{\ominus *}{\longleftrightarrow} \infty =$$ 

E = East
W = West

By the square root trick

$$\mathbb{P}_p\left(\partial\Lambda_L^{N,S} \overset{\oplus}{\longleftrightarrow} \infty\right) > \frac{\sqrt{3}}{2}$$

$$\mathbb{P}_p\left(\partial\Lambda_L^{E,W} \overset{\ominus *}{\longleftrightarrow} \infty\right) > \frac{\sqrt{3}}{2}$$

$\frac{\pi}{2}$ - rot symmetry used here

By Harris's inequality

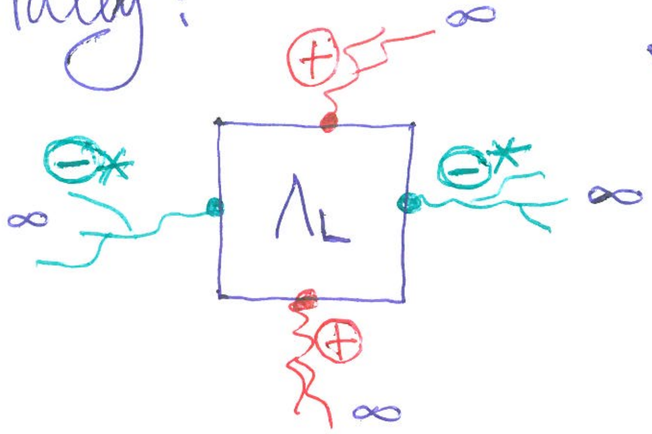
$$\mathbb{P}_p\left(\partial\Lambda_L^N \overset{\oplus}{\longleftrightarrow} \infty \ \& \ \partial\Lambda_L^S \overset{\oplus}{\longleftrightarrow} \infty\right) \geq \frac{3}{4}$$

$$\mathbb{P}_p\left(\partial\Lambda_L^E \overset{\ominus *}{\longleftrightarrow} \infty \ \& \ \partial\Lambda_L^W \overset{\ominus *}{\longleftrightarrow} \infty\right) \geq \frac{3}{4}$$

By union bound

$$\mathbb{P}_p\left(\partial\Lambda_L^N \overset{\oplus}{\longleftrightarrow} \infty; \partial\Lambda_L^S \overset{\oplus}{\longleftrightarrow} \infty; \partial\Lambda_L^E \overset{\ominus *}{\longleftrightarrow} \infty; \partial\Lambda_L^W \overset{\ominus *}{\longleftrightarrow} \infty\right) \geq \frac{1}{2}$$

Pictorially:

$\mathbb{P}_p\Big($  $\Big) \geq \frac{1}{2} > 0$

But this contradicts the uniqueness theorem.

$\boxed{\text{Actually we proved } \theta(p)\cdot\theta^*(1-p)=0}$

3. step Since $p_T = p_H = p_c$ (sharpness)

$$(1) \,\&\, (2) \Longrightarrow p_c + p_c^* = 1$$

□ Harris-Kesten-Russo

Remarks

By-products of step 2:

(1) $\theta(p) > 0 \Rightarrow$ infinitely many "concentric" ⊕-circuits

(2) If $\mathcal{G} = \mathcal{G}^*$ (e.g. $\mathbb{Z}^2$-bond, Δ-site)

$p_c = \frac{1}{2}$ and $\theta(p_c) = 0$

continuity of $p \mapsto \theta(p)$.