Bálint Tóth: Percolation 4.

Sharpness of the phase transition on $\mathbb{Z}^d$

Recall:

$$p_H := \sup\{p : \theta(p) = 0\}$$
$$= \inf\{p : \theta(p) > 0\}$$

$$p_T := \sup\{p : \chi(p) < \infty\}$$
$$= \inf\{p : \chi(p) = \infty\}$$

and one could easily define other natural threshold values of p.

A priori:

$$p_T \leq p_H \quad \checkmark$$

Q: Are they actually equal?

Expect: yes, of course.

Theorem (M.V. Menshikov 1986
M. Aizenman – D. Barsky 1987)

For $p \in [0, p_H)$ $\exists\, \psi(p) > 0$:

$$\mathbb{P}_p\left(0 \longleftrightarrow \partial B_{0,n}\right) \leq e^{-\psi(p) n}.$$

Corollary If $p \in [0, p_H)$:

$(\forall k < \infty)$: $\mathbb{E}_p\left(|\mathcal{C}_0|^k\right) < \infty$.

In particular:

$$p_T = p_H =: p_c$$

and all "reasonably" defined threshold values of p coincide.

I.e. there is a $\boxed{\text{sharp phase transition}}$ at p_c.

- Aizenman-Barsky use methods/techniques of statistical physics and derive differential inequalities for $\theta(p)$, $\chi(p)$
- Menshikov's proof is more geometric-probabilistic
- Both proofs rely on B-K inequality and Russo's formula and are very convoluted
- They also prove (both AB, and M), as a corollary

for: $p_c \leq q \leq p \leq 1$: $\theta(p) - \theta(q) \geq (1-\theta(q))(q-p)$

In particular: $p \geq p_c$: $\theta(p) \geq p - p_c$

Note: $\theta(p) \sim (p-p_c)^{\beta}$ expected with $\beta = 1$ for $d > 6$.

<u>Recently</u>: New "elementary" proof

H. Duminil-Copin – V. Tassion '15

Details for $\boxed{\text{bond}}$ percolation on $\mathbb{Z}^d$

<u>Notation</u>:

$\Lambda_n := [-n, n]^d \cap \mathbb{Z}^d$

$\partial \Lambda_n := \Lambda_n \setminus \Lambda_{n-1}$ (internal boundary of Λ_n)

$S \subseteq \mathbb{Z}^d$: $|S| < \infty$, $0 \in S$

$\Delta S := \{ \langle x, y \rangle : x \in S, y \notin S \}$

edge-boundary of S.

$\{ x \overset{S}{\longleftrightarrow} y \} :=$ x and y are connected <u>within S</u>

A new – very handy – threshold value

$$\varphi_p(S) := \mathbb{E}_p\left(\#\{\langle xy\rangle \in \Delta S : 0 \overset{S}{\longleftrightarrow} x,\ \omega_{\langle xy\rangle} = 1\}\right)$$

$$= p \sum_{\langle xy\rangle \in \Delta S} \mathbb{P}_p\left(0 \overset{S}{\longleftrightarrow} x\right)$$

$$\tilde{p}_c := \sup\{p : (\exists S \subset \mathbb{Z}^d) : |S| < \infty,\ 0 \in S \text{ and } \varphi_p(S) < 1\}$$

$$= \inf\{p : \forall S \subset \mathbb{Z}^d,\ |S| < \infty,\ 0 \in S\ \ \varphi_p(S) \geq 1\}$$

<u>Note</u> $\forall S \subseteq \mathbb{Z}^d,\ |S| < \infty,\ 0 \in S$

$$\varphi_{\tilde{p}_c}(S) \geq 1.$$

Theorem (H. Duminil-Copin, V. Tassion, 2015)

① For $p < \tilde{p}_c$: $\exists\, \psi(p) > 0$ s.t.

$$\mathbb{P}_p(0 \longleftrightarrow \partial\Lambda_n) \leq e^{-\psi(p) n}$$

② For $p > \tilde{p}_c$:

$$\theta(p) \geq \frac{p - p_c}{p(1-p_c)}$$

it follows that $\tilde{p}_c = p_H = p_T$ "

Proof :

① $\boxed{p < \tilde{p}_c}$ Let S be such that, $0 \in S$

$$\varphi_p(S) < 1$$

and $L < \infty$ such that $S \subseteq \Lambda_{L-1}$

$$\mathcal{C} := \{ z \in S : 0 \overset{S}{\longleftrightarrow} z \}$$

$$\{0 \leftrightarrow \partial\Lambda_{kL}\} =$$

$$\bigcup_{C \subseteq S} \ \bigcup_{\substack{\langle xy\rangle \in \Delta S: \\ x \in C}} \{\mathcal{C} = C\} \cap \{\omega_{\langle xy\rangle} = 1\} \cap \{y \xleftrightarrow{\Lambda_{kL}\setminus C} \partial\Lambda_{kL}\}$$

these events are independent !!!

Hence:

$$\mathbb{P}_p(0 \leftrightarrow \partial\Lambda_{kL}) \leq \ldots$$

$$\sum_{C \subseteq S} \ \sum_{\substack{\langle xy\rangle \in \Delta S: \\ x \in C}} \mathbb{P}_p(\mathcal{C} = C)\, p\, \mathbb{P}_p(y \xleftrightarrow{\Lambda_{kL}\setminus C} \partial\Lambda_{kL})$$

<u>Note:</u> $C \subset \Lambda_{L-1}$, $d(y, C) = 1$

Then $\mathbb{P}_p(y \xleftrightarrow{\Lambda_{kL}\setminus C} \partial\Lambda_{kL}) \leq$

$$\mathbb{P}_p(0 \leftrightarrow \partial\Lambda_{(k-1)L})$$

(since $y + \Lambda_{(k-1)L} \subseteq \Lambda_{kL}$ ....)

$$\mathbb{P}_p(0 \leftrightarrow \partial\Lambda_{kL}) \leqslant$$

$$\mathbb{P}_p(0 \leftrightarrow \Lambda_{(k-1)L})\, \underbrace{p \sum_{C \subseteq S} \sum_{\substack{\langle xy\rangle \in \Delta S \\ x \in C}} \mathbb{P}_p(\mathcal{C} = C)}_{= \varphi_p(S)}$$

Conclusion:

$$\mathbb{P}_p(0 \leftrightarrow \partial\Lambda_{kL}) \leqslant \varphi_p(S) \cdot \mathbb{P}_p(0 \leftrightarrow \partial\Lambda_{(k-1)L})$$

$$\mathbb{P}_p(0 \leftrightarrow \partial\Lambda_{kL}) \leqslant (\varphi_p(S))^k$$

Hence 1 $\square$

Remark We haven't used B-K ineq.!
For more complicated geometries
- e.g. long range bonds -
we need it

(2) We rely on Russo's formula

Lemma Let $p \in (0,1)$ and $n \geq 1$.

$$\frac{d}{dp} \mathbb{P}_p(0 \leftrightarrow \partial\Lambda_n) \geq \frac{1}{p(1-p)} \inf_{\substack{S \subseteq \Lambda_n \\ 0 \in S}} \varphi_p(S) \cdot \left(1 - \mathbb{P}_p(0 \leftrightarrow \partial\Lambda_n)\right)$$

(2) follows by integration: $p \geq \tilde{p}_c$

$$\frac{d}{dp} \ln \frac{1}{1 - \mathbb{P}_p(0 \leftrightarrow \partial\Lambda_n)} \geq \frac{d}{dp} \ln \frac{p}{1-p}$$

integrate $\int_{\tilde{p}_c}^{p} \dots$ to get

$$\frac{1 - \mathbb{P}_{\tilde{p}_c}(0 \leftrightarrow \partial\Lambda_n)}{1 - \mathbb{P}_p(0 \leftrightarrow \partial\Lambda_n)} \geq \frac{p(1-\tilde{p}_c)}{\tilde{p}_c(1-p)}$$

Hence ... (2)

Proof of the Lemma:

Let $\mathcal{S} := \{ x \in \Lambda_n : x \not\leftrightarrow \partial\Lambda_n \}$

Russo's formula:

$$\frac{d}{dp} \mathbb{P}_p(0 \leftrightarrow \partial\Lambda_n) = \sum_{\langle xy\rangle \text{ in } \Lambda_n} \mathbb{P}_p(\langle xy\rangle \text{ is pivotal for } 0 \leftrightarrow \partial\Lambda_n)$$

$$= \frac{1}{1-p} \sum_{\langle xy\rangle \text{ in } \Lambda_n} \mathbb{P}_p(0 \overset{\Lambda_n}{\longleftrightarrow} x,\ y \leftrightarrow \partial\Lambda_n,\ 0 \not\leftrightarrow \partial\Lambda_n)$$

$$= \frac{1}{1-p} \sum_{\substack{S \subseteq \Lambda_n \\ 0 \in S}} \sum_{\langle xy\rangle \in \Delta S} \mathbb{P}_p(0 \overset{S}{\longleftrightarrow} x,\ \mathcal{S} = S)$$

independant

$$= \frac{1}{p(1-p)} \sum_{\substack{S \subseteq \Lambda_n \\ 0 \in S}} \varphi_p(S)\, \mathbb{P}_p(\mathcal{S} = S) \geq$$

$$\frac{1}{p(1-p)} \min_{\substack{S \subseteq \Lambda_n \\ 0 \in S}} \varphi_p(S) \cdot \sum_{\substack{S \subseteq \Lambda_n \\ 0 \in S}} \mathbb{P}_p(\mathcal{S} = S) =$$

$$\frac{1}{p(1-p)} \min_{\substack{S \subseteq \Lambda_n \\ 0 \in S}} \varphi_p(S)\, \mathbb{P}_p(0 \not\leftrightarrow \Lambda_n) \quad \square \; (2)$$