Balint Tóth: Percolation 2.

$p > p_c$: continuity of $p \mapsto \theta(p)$, and uniqueness of the ∞ cluster on $\mathbb{Z}^d$

Recall: $p \mapsto \theta(p)$ is non-decreasing & continuous from right.

Q. Is it continuous?

Theorem (J. van den Berg - M. Keane (1982))

$$\theta_x(p) - \theta_x(p^-) = \mathbb{P}_p\left(|C_x| = \infty \text{ and } p_H(C_x) = 1\right)$$

or, equivalently

$$\theta_x(p^-) = \mathbb{P}_p\left(|C_x| = \infty \text{ and } p_H(C_x) < 1\right)$$

(valid on any connected, locally finite graph)

Corollary: Let $\mathcal{G}$ be connected, and locally finite, and $p > p_H(\mathcal{G})$ (strictly!). Then

$$\boxed{\mathbb{P}_p(\exists \text{ unique } \infty \text{ cluster})} \Longrightarrow \boxed{p \mapsto \theta(p) \text{ is continuous at } p.}$$

Proof of the Theorem: for site percolation

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ connected, locally finite

Lemma 1 $\forall x \in \mathcal{V}$:

$$\theta_x(1^-) := \lim_{p \nearrow 1} \theta_x(p) = \begin{cases} 0 & \text{if } p_H = 1 \\ 1 & \text{if } p_H < 1 \end{cases}$$

Proof (of Lemma 1):

$\boxed{p_H = 1}$ trivial: $\forall p < p_H = 1$: $\theta_x(p) = 0$ ✓

$\boxed{p_H < 1}$ Fix $\varepsilon > 0$

① Choose n so large that

$$\mathbb{P}_{\frac{1+p_H}{2}}\left(\partial B_{x,n} \longleftrightarrow \infty\right) > 1 - \frac{\varepsilon}{2}$$

$$\left(\text{Note: } p > p_H \Rightarrow \lim_{n\to\infty} \mathbb{P}_p\left(\partial B_{x,n} \leftrightarrow \infty\right) = 1\right)$$

② Choose $\delta < \frac{1-p_H}{2}$ so small that

$$\mathbb{P}_{1-\delta}\left(\omega_y = 1, y \in B_{x,n}\right) = (1-\delta)^{|B_{x,n}|} > 1 - \frac{\varepsilon}{2}$$

Then: If $p \geq 1-\delta$:

$$\mathbb{P}_p(x \longleftrightarrow \infty) \geq \mathbb{P}_p\left(\{\omega_y = 1, y \in B_{x,n}\} \wedge \{\partial B_{x,n} \longleftrightarrow \infty\}\right) \geq$$

$$\underset{\text{Harris' ineq}}{\geq} \mathbb{P}_p\left(\omega_y = 1, y \in B_{x,n}\right) \cdot \mathbb{P}_p\left(\partial B_{x,n} \longleftrightarrow y\right)$$

$$\underset{\text{domination}}{\geq} \mathbb{P}_{1-\delta}\left(\omega_y = 1, y \in B_{x,n}\right) \cdot \mathbb{P}_{\frac{1+p_H}{2}}\left(\partial B_{x,n} \longleftrightarrow y\right) \geq \left(1 - \frac{\varepsilon}{2}\right)^2$$

<u>Remark</u> Lemma 1 is the statement of the Theorem for $p = 1$.

<u>Proof of the Theorem</u> – continued:

Let

$\underline{\omega}^1$ Bernoulli site perc with density p

$\underline{\omega}^2$ Bernoulli site perc with density q

independent of one-another

$\underline{\omega} := \underline{\omega}^1 \wedge \underline{\omega}^2 \quad (\omega_x = \min\{\omega^1_x, \omega^2_x\})$

Bernoulli site perc. with density $p \cdot q$

$$\theta_x(p \cdot q) = \mathbb{E}_p\Big(\mathbb{E}\big(\mathbb{1}(x \leftrightarrow \infty) \,\big|\, \underline{\omega}^1\big)\Big) =$$

$$= \mathbb{E}_p\Big(\mathbb{1}(|C^1_x| = \infty \text{ on } \mathcal{G})\, \mathbb{E}_q\big(\mathbb{1}(|C^2_x| = \infty \text{ on } C^1_x)\big)\Big)$$

$$= \mathbb{E}_p\Big(\mathbb{1}(|C^1_x| = \infty \text{ on } \mathcal{G})\, \theta_x(q, \text{ on } C^1_x)\Big)$$

let now $q \nearrow 1$ and quote Monotone Convergence:

$$\theta_x(p^-) = \mathbb{E}_p\left(\mathbb{1}\left(|C_x^1| = \infty \text{ on } \mathcal{G}\right)\theta_x(1^-, \text{on } C_x^1)\right)$$

by Lemma 1 $= \mathbb{P}_p\left(|C_x^1| = \infty \text{ and } p_H(C_x) < 1\right)$

$\square$ Thm.

<u>Proof of the Corollary:</u>

<u>Lemma 2</u> Let $p \geq p_H = p_H(\mathcal{G})$ be such that:

$$\mathbb{P}_p(\exists \text{ a unique } \infty \text{ connected cluster}) = 1$$

Denote <u>the</u> ∞ cluster $\hat{\mathcal{G}} = \hat{\mathcal{G}}(\underline{\omega})$

[it is a <u>random</u> connected, locally finite graph]

Then $\mathbb{P}_p\left(p_H(\hat{\mathcal{G}}) = \frac{p_H(\mathcal{G})}{p}\right) = 1.$

Proof of Lemma 2:

① $p_H(\tilde{\mathcal{G}}(\underline{\omega}))$ is Tail - measurable.

By Kolmogorov 0-1: $p_H(\tilde{\mathcal{G}}(\underline{\omega})) = \text{const}$, $\mathbb{P}_p$ - a.s.

② apply the same coupling as before:

$$\mathbb{P}_{pq}(\exists \infty \text{ cluster on } \mathcal{G}) = \mathbb{E}_p\left(\mathbb{P}_q\left(\exists \infty \text{ cluster on } \tilde{\mathcal{G}}\right)\right)$$

Hence $\quad pq > p_H(\mathcal{G}) \Longrightarrow q \geq p_H(\tilde{\mathcal{G}})$

$$q > p_H(\tilde{\mathcal{G}}) \Longrightarrow pq \geq p_H(\mathcal{G})$$

$\Box$ L2

The Corollary follows, since assuming uniqueness of the ∞ cluster at $p > p_H(\mathcal{G})$,

$$p_H(\tilde{\mathcal{G}}) = \frac{p_H(\mathcal{G})}{p} < 1.$$

$\Box$ Coroll.

Actually: if $\mathcal{G}$ is vertex transitive then uniqueness of the ∞ cluster is not necessary for continuity of $p \mapsto \theta(p)$, at $p > p_H$.

<u>Proposition</u> (R. Lyons (?)):
If $\mathcal{G}$ is transitive then,

$$\mathbb{1}(|C_x| = \infty)\, p_H(C_x) = \mathbb{1}(|C_x| = \infty) \cdot \frac{p_H(\mathcal{G})}{p}$$

<u>Conclusion</u> $p \mapsto \theta(p)$ is continuous on $p \in (p_c, 1]$

| Continuity <u>at</u> p_c remains open

The number of ∞ clusters on $\mathbb{Z}^d$:

(more generally: on any periodic graph embedded in $\mathbb{R}^d$)

$\mathcal{G} = \mathbb{Z}^d$, $\Omega = \{0,1\}^{\mathbb{Z}^d}$ (site-perc)

$\tau_z : \Omega \to \Omega$, $(\tau_z \underline{\omega})_x = \omega_{z+x}$

shift by $z \in \mathbb{Z}^d$

$(\Omega, \mathcal{F}, \mathbb{P}_p, \tau_z : z \in \mathbb{Z}^d)$ is <u>ergodic</u>

$$\nu(\underline{\omega}) := \#\{\infty \text{ clusters in } \underline{\omega}\}$$

$$(\forall z \in \mathbb{Z}^d) \quad \nu(\tau_z \underline{\omega}) = \nu(\underline{\omega})$$

So, by the Ergodic Theorem

$$\mathbb{P}_p(\nu(\omega) = \text{constant}) = 1$$

<u>**Proposition**</u> (simple, C. Newman – L. Schulman, 1981)

$\mathbb{P}_p$ – a.s. either $\nu = 0$ or $\nu = 1$ or $\nu = \infty$.

[More precisely:

$p < p_H$: $\mathbb{P}_p$ – a.s. $\nu = 0$

$p > p_H$: $\mathbb{P}_p$ – a.s. either $\nu = 1$ or $\nu = \infty$

$p = p_H$: as in the statement of Prop.]

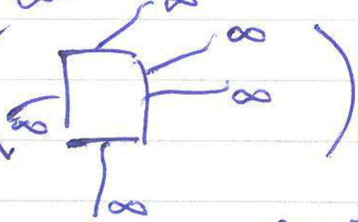
<u>**Proof**</u>: Straightforward. Let $1 < k < \infty$.

Assume $\mathbb{P}_p(\nu = k) = 1$

Let n be so large that

$$\mathbb{P}_p(\text{all } k\ \infty \text{ clusters intersect } \partial B_{0,n}) > 0.$$

Then $\mathbb{P}_p(\nu = 1) \geq \mathbb{P}_p\Big($  $\Big) \geq$

$\mathbb{P}_p\Big($  $\Big) \cdot \mathbb{P}_p\Big($  $\Big) > 0$

contradiction $\square$ P.

<u>Remark</u>: The arguments remain valid on any transient (not necessarily amenable) graph $\mathcal{G}$.

Q Can it be $\nu = \infty$, on $\mathbb{Z}^d$?

HW: Prove that on $\mathbb{T}_d$ ($= \infty$ tree of degree $d \geq 3$), at density $p > p_H = \frac{1}{d-1}$ there are ∞-ly many ∞ clusters.

However, on $\mathbb{Z}^d$ (or any $\mathbb{R}^d$-embedded periodic graph) there seems to be not enough room for $\nu = \infty$.

<u>Theorem</u> On $\mathbb{Z}^d$ (or...)

$\forall p \in [0,1]$, $\mathbb{P}_p$-a.s. either $\nu = 0$ or $\nu = 1$.

Historical notes:

$\boxed{d=2}$ Bond percolation on $\mathbb{Z}^2$:

T.E. Harris (1960) proves uniqueness of the ∞ cluster, if $\Theta(p)>0$.

Idea: If $\Theta(p)>0$ then the origin is surrounded by ∞-ly many disjoint occupied circuits, not letting room for more than one ∞ cluster — we'll see later these 2-d arguments

M. Fisher (1962), L. Russo (1978)... extend the 2-d arguments to a wide range of 2-d-embedded graphs

The arguments are 2-d topological. Don't extend to $d\geq 3$.

$\boxed{d\geq 3}$ Conjecture H. Kesten (1982)

first proof M. Aizenman - H. Kesten - C. Newman (1987)

The Proof from The Book

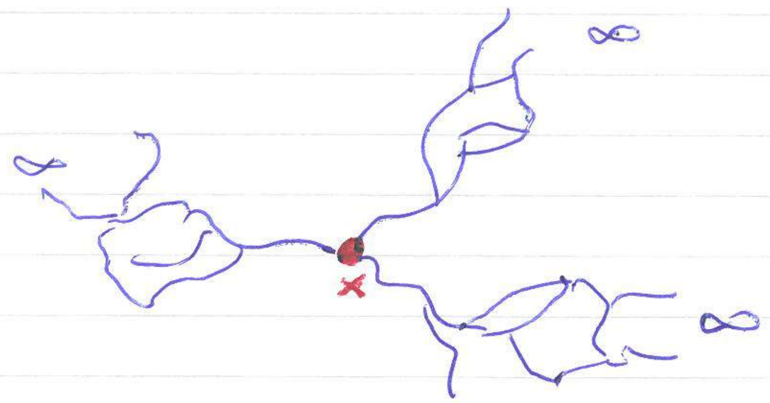
R. Burton – M. Keane (1988)

"encounter points":

$x \in \mathbb{Z}^d$ is encounter point for $\underline{\omega} \in \Omega$ if:

1. $|C_x(\underline{\omega})| = \infty$, and

2. $C_x(\underline{\omega}) \setminus \{x\}$ falls apart into exactly 3 infinite (and no finite) connected clusters



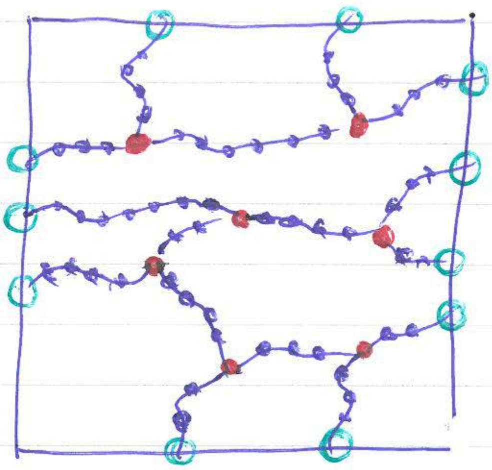
an encounter point

Let: $\Lambda \subset \mathbb{Z}^d$ finite box

$$g_\Lambda = g_\Lambda(\underline{\omega}) := \#\{\text{encounter points in } \Lambda\}$$

Then $$g_\Lambda \leq |\partial \Lambda|$$

for simple graph-enumeration reasons



Let $\mathcal{T}$ be a tree/forest of open sites which contain the encounter point

g_Λ = # encounter points in Λ

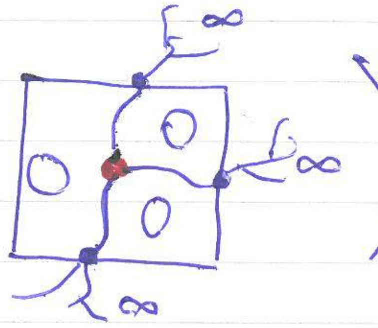
b_Λ = # $\mathcal{T} \cap \partial\Lambda$

$g_\Lambda = b_\Lambda - 2 \cdot$ # of disjoint components of $\mathcal{T}$

Assume $P_p(N=\infty)=1$.

Then $$P_p(0 \text{ is an encounter point}) > 0$$
$$\parallel$$
$$P_p(x \in \mathbb{Z}^d \text{ is an encounter point})$$
$$\parallel$$
$$\alpha$$

$$\mathbb{P}_p(0 \text{ is an encounter point}) \geq$$

$\mathbb{P}_p\Big($ 

$\Big) > 0 \quad .$

$$(\forall \Lambda) \quad \begin{cases} g_\Lambda \leq |\partial \Lambda| & \text{a.s.} \\ \mathbb{E}(g_\Lambda) = \alpha \cdot |\Lambda| \end{cases}$$

Can hold only if $\alpha = 0$

Since $\inf \dfrac{|\partial \Lambda|}{|\Lambda|} = 0$ $\square$

[Not valid for non-amenable graphs.]

<u>Note that the proof is very robust.</u>

Extends easily to transitive (or "periodic") graphs with $\inf \frac{|\partial\Lambda|}{|\Lambda|} = 0$

and (transitive) measures $\mathbb{P}$ with the finite energy property:

$\forall \Lambda \subset V$ finite, $\underline{\omega}^\Lambda \in \Omega^\Lambda$

$$T_{\underline{\omega}^\Lambda} : \Omega \to \Omega \ ; \ \left(T_{\underline{\omega}^\Lambda}\, \underline{\omega}\right)_x = \begin{cases} \omega_x & \text{if } x \notin \Lambda \\ \omega^\Lambda_x & \text{if } x \in \Lambda \end{cases}$$

[inside Λ overwrite $\underline{\omega}$ by $\underline{\omega}^\Lambda$]

$$A \in \mathcal{F} : \ T_{\underline{\omega}^\Lambda} A = \{ T_{\underline{\omega}^\Lambda}\, \underline{\omega} : \underline{\omega} \in \Lambda \}$$

<u>Finite energy property</u>:

If $A \in \mathcal{F}$, $\mathbb{P}(A) > 0$, then for $\forall \Lambda \subset V$ finite $\forall\, \underline{\omega}^\Lambda \in \Omega^\Lambda$:

$$\mathbb{P}(T_{\underline{\omega}^\Lambda} A) > 0.$$

i.e. no local restrictions.