Bálint Tóth: Percolation 1.

Introduction

Empirical observations:

① place a piese of porous material (stone, rock, ...) in water and observe whether the interior/centre gets wet or remains dry. Then either it stays all dry except the wet surface, or it gets wet to the very centre – depending on porosity. Nothing in between

② Spraying paint on a coarse vertical surface: it either stays (and dries) or it drops down to the bottom – depending on coarseness of the surface. Nothing in between.

Mathematical model:

S. R. Broadbent, 1954

S. R. Broadbent, J. M. Hammersley, 1957

"Percolation processes I. Crystals and mazes"

and two more papers of Hammersley

in the same year.

Let $\Lambda_n \subset \mathbb{Z}^d$ finite, connected

e.g $\Lambda_n := [-n, n]^d \cap \mathbb{Z}^d$

$$\Lambda_n \nearrow \mathbb{Z}^d, \quad \text{as } n \longrightarrow \infty$$

Let the lattice edges in Λ_n be

"open" with probab p

"closed" with probab $1-p$

independently of one-another

Q. P(there is an <u>open path</u> $0 \longleftrightarrow \partial \Lambda_n$) = ?

in particular in the limit $\Lambda_n \nearrow \mathbb{Z}^d$.

Do the same (edges open/closed) on the whole $\mathbb{Z}^d$.

[Mind: Need ∞-ly many coin tosses!]

$x \in \mathbb{Z}^d$ $\quad C_x := \left\{ \begin{array}{l} \text{connected component} \\ \text{of "open" edges, containing } x \end{array} \right\}$

Q Distribution of $|C_x|$? (asymptotics)

In particular:

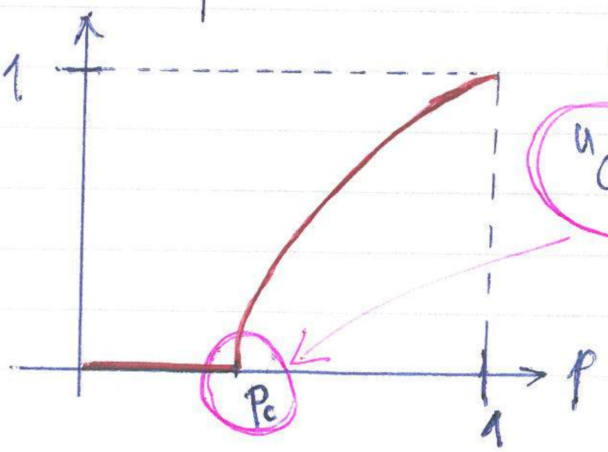
$$\theta_x(p) := \mathbb{P}(|C_x| = \infty)$$

How does the function $p \mapsto \theta_x(p)$ behave?

Qualitative picture

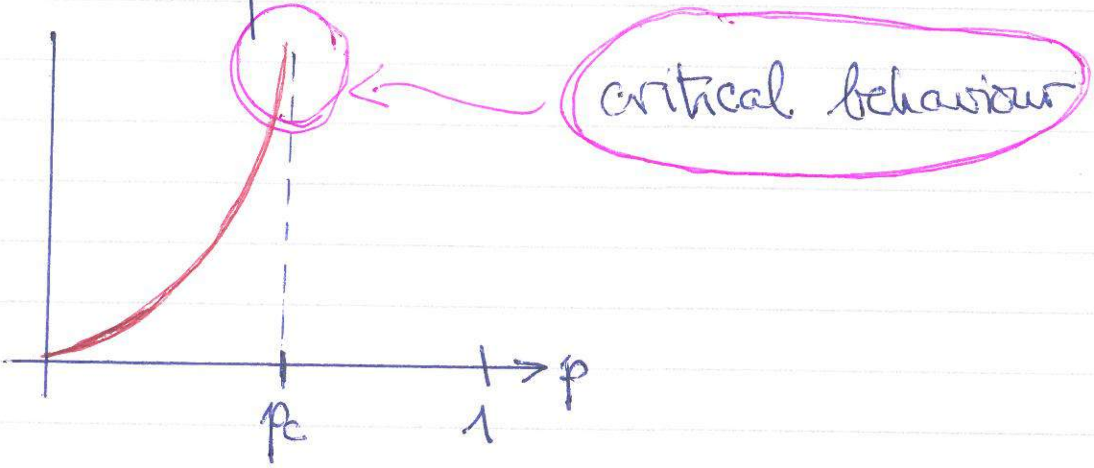
dramatic change at $p_c \in (0,1)$

"critical behaviour"



Q $\chi_x(p) := \mathbb{E}(|C_x|)$?

Qualitative picture:



Further motivation:

- conductivity of disordered electric networks
- spread of infection in spatially disordered population
- design of micro-structures
- phase transitions & critical phenomena in statistical physics
- gives rise to interesting and beautiful mathematics

main!

Variations on the same theme

- $\boxed{\text{bond = edge}}$ $\boxed{\text{site = vertex}}$
- $\boxed{\text{unoriented}}$ $\boxed{\text{oriented}}$
- based on PPP (rather than $\mathbb{Z}^d$)

  (Poisson Point Process on $\mathbb{R}^d$)

  $\boxed{\text{Boolean}}$

  $\left.\begin{array}{l}\boxed{\text{Voronoi tessellation}} \\ \boxed{\text{Delaunay triangulation}}\end{array}\right\} d=2$

  $\boxed{\text{first / last passage}}$

  $\boxed{\text{invasion}}$ $\boxed{\text{bootstrap}}$ etc.

More remarks:

- Erdős, Rényi: On random graphs 1959
actually: a mean-field version of the percolation model.

- MathSciNet: number of publications with the word "percolation" in the title

1957 - 1960 : 6

1961 - 1970 : 16

1971 - 1980 : 72

1981 - 1990 : 439

1991 - 2000 : 419

2001 - 2010 : 609

2011 - 2014 : 316

Definitions, Notation, Basics

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ infinite, unoriented, connected graph with bounded degree

E.g. • $\mathcal{V} = \mathbb{Z}^d$, $\mathcal{E} = \{(x,y) : |x-y| = 1\}$

• $\mathcal{T}_d = \infty$ tree of degree d

• triangular lattice on $\mathbb{R}^2$

• any periodic "decoration" of $\mathbb{Z}^d$

• can be not-vertex-transitive

The <u>bond percolation</u> problem:

$\Omega = \{0,1\}^{\mathcal{E}}$, $\mathcal{F} =$ σ-alg. generated by finite cylinders

$\mathbb{P}_p = \bigotimes_{e \in \mathcal{E}} \mu_e$, $\mu_e(0) = 1-p$, $\mu_e(1) = p$

The <u>site percolation</u> problem:

$\Omega = \{0,1\}^{\mathcal{V}}$, $\mathcal{F} =$ σ-alg. generated by finite cylinders

$\mathbb{P}_p = \bigotimes_{v \in V} \mu_v$; $\mu_v(0) = 1-p$, $\mu_v(1) = p$.

$\boxed{\text{* remark at bottom of page}}$

<u>Notation</u>: $\underline{\omega} = (\omega_e)_{e \in \mathcal{E}} \in \Omega^{\mathcal{E}}$

$\underline{\omega} = (\omega_v)_{v \in \mathcal{V}} \in \Omega^{\mathcal{V}}$

Simple paths: $x, y \in \mathcal{V}$ $x \neq y$

$x_0, e_1, x_1, e_2, \dots, e_n, x_n$

where: $x = x_0$, $y = x_n$

$e_j = (x_{j-1}, x_j)$

$x_i \neq x_j$ if $i \neq j$

$n =$ the length of the path.

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* the site-perc-problem is more general: edge-problem on $\mathcal{G}$ = site problem on the line-graph of $\mathcal{G}$

<u>Notation:</u>

$x \longleftrightarrow y := \{\underline{\omega} \in \Omega : \exists \text{ open path connecting } x \,\&\, y\}$

<u>Note</u>: in bond-perc. $x \leftrightarrow x = \Omega$

in site perc $x \leftrightarrow x = \{\omega_x = 1\}$

$A, B \subset V$

$$A \leftrightarrow B := \bigcup_{x \in A} \bigcup_{y \in B} \{x \leftrightarrow y\}$$

Connected clusters:

$x \in V$: $C_x = C_x(\underline{\omega}) = \{y \in V : x \leftrightarrow y\}$

"the connected cluster of site x"

we may mean the <u>induced</u> connected graph.

Percolation probability.

$$\theta_x(p) := \mathbb{P}_p(|C_x| = \infty) =$$

$$= 1 - \sum_n P_p(|C_x| = n)$$

$$= \lim_{n\to\infty} \mathbb{P}_p(x \longleftrightarrow \partial B_{x,n})$$

boundary of   ball of rad. n around x

= decreasing limit of polynomials

Thus: $p \mapsto \theta_x(p)$ is upper semicontinuous

$$\chi_x(p) := \mathbb{E}(|C_x|)$$

$$= \lim_{n\to\infty} \mathbb{E}(|C_x \cap B_{x,n}|)$$

= increasing limit of polynomials

Thus: $p \mapsto \chi_x(p)$ is lower semicontinuous.

Furthermore:

$p \longmapsto \Theta_x(p)$

$p \longmapsto \chi_x(p)$ are non-decreasing

This is intuitively evident.

Nevertheless, needs a proof.

If so, then $p \mapsto \Theta_x(p)$ is continuous from the right

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Digression: upper/lower semicontinuity

(M, d) metric space

$f : M \to \mathbb{R}$ is u.s.c. iff

$\forall x \in M, \; x_n \to x : \overline{\lim_{n\to\infty}} f(x_n) \leq f(x)$

Thm $f_n : M \to \mathbb{R}$ continuous; $\forall x \in M : f_n(x) \downarrow f(x)$
then f is u.s.c.

Simple tools: stochastic order of Bernoulli measures, Harris's inequality

Let $N<\infty$; $\Omega=\{0,1\}^N$

Ω is a poset:

$\underline{\omega}\leq\underline{\omega}'$ iff $\forall i\in\{1,..,N\}$ $\omega_i\leq\omega_i'$

$f:\Omega\to\mathbb{R}$ is increasing (actually non-decr.)

iff $\underline{\omega}\leq\underline{\omega}'\Rightarrow f(\underline{\omega})\leq f(\underline{\omega}')$

$A\subseteq\Omega$ is called "increasing"

iff $\mathbb{1}_A:\Omega\to\mathbb{R}$ is increasing

Let $\mathbb{P}$ & $\mathbb{P}'$ be probability measures on Ω.

stoch. domination

We say that $\mathbb{P}'$ stochastically dominates $\mathbb{P}$ iff $\forall f:\Omega\to\mathbb{R}$ increasing

$$\mathbb{E}(f(\underline{\omega}))\leq\mathbb{E}(f'(\underline{\omega}))$$

Sufficient: $\forall A \subseteq \Omega$ increasing

$$\mathbb{P}(A) \leq \mathbb{P}'(A)$$

FKG property

We say that $\mathbb{P}$ is an F-K-G measure (for Fortuin-Kasteleyn-Ginibre) iff $\forall f, g : \Omega \to \mathbb{R}$ both increasing $\mathbb{E}(f(\omega)g(\omega)) \geq \mathbb{E}f(\omega)\mathbb{E}g(\omega)$

Sufficient: $\forall A, B \subseteq \Omega$ both increasing

$$\mathbb{P}(A \cap B) \geq \mathbb{P}(A)\mathbb{P}(B).$$

Remark: Both definitions/concepts make perfect sense on general ~~posets~~ or associative lattices

HW Characterize stochastic domination and FKG property on $\mathbb{R}$.

Theorem 1 (straightforward)

Let $\mathbb{P} = \bigotimes_i \mu_i \qquad \mu_i(1) = p_i = 1 - \mu_i(0)$

$\mathbb{P}' = \bigotimes_i \mu_i' \qquad \mu_i'(1) = p_i' = 1 - \mu_i'(0)$

with $\quad p_i' \geq p_i \quad , \quad i = 1, \dots, N$

Then $\quad \mathbb{P}'$ stoch. dominates $\mathbb{P}$

Proof: coupling. Construct probab. measure $\mathbb{Q}$ on $\Omega \times \Omega$ such that

① $\forall A \subseteq \Omega \quad \mathbb{Q}(A \times \Omega) = \mathbb{P}(A)$

$\mathbb{Q}(\Omega \times A) = \mathbb{P}'(A)$

② $\quad \mathbb{Q}\left(\{(\underline{\omega}, \underline{\omega}') : \underline{\omega} \leq \omega'\}\right) = 1$

Let $f : \Omega \to \mathbb{R}$ be increasing and

$F : \Omega \times \Omega \to \mathbb{R} : F(\omega, \omega') := f(\omega') - f(\omega)$

then: $E_{P'}(f(\underline{\omega}')) - E_P(f(\omega)) = E_{\mathbb{Q}}(F(\omega, \omega')) \geq 0.$

Construction of Q: $Q = \bigotimes_i \nu_i$

Where

$$\nu_i(1,1) = p_i = \mu_i(1)$$
$$\nu_i(1,0) = 0 = 0$$
$$\nu_i(0,1) = p_i' - p_i = \mu_i'(1) - \mu_i(1)$$
$$\nu_i(0,0) = 1 - p_i' = \mu_i'(0)$$

$\square$

Theorem 2 (Harris, 1960)

Let $\mathbb{P} = \bigotimes_i \mu_i$ $\qquad \mu_i(1) = p_i = 1 - \mu_i(0)$

be (inhomogeneous) Bernoulli measure

Then $\mathbb{P}$ has the FKG property

Proof by induction on N:

$N = 1$: $f, g : \{0,1\} \to \mathbb{R}$; $f(1) \geq f(0)$
$g(1) \geq g(0)$

$$E(f \cdot g) = p_1 f(1) g(1) + (1 - p_1) f(0) g(0)$$

$$\mathbb{E}_1(f)\mathbb{E}(g) = \left(p_1 f(1) + (1-p_1) f(0)\right) \cdot$$

$$\left(p_1 g(1) + (1-p_1) g(0)\right) =$$

$$p_1^2 f(1) g(1) + p_1(1-p_1)\left(f(1) g(0) + f(0) g(1)\right)$$

$$+ (1-p_1)^2 f(0) g(0) =$$

$$\mathbb{E}_1(f \cdot g) - p_1(1-p_1)\left(f(1) - f(0)\right)\left(g(1) - g(0)\right)$$

Induction: $f, g : \Omega_{N+1} \to \mathbb{R}$

Note

$$\mathbb{E}_{N+1}(f \mid \underline{\omega}_1^N) = p_{N+1} f(\underline{\omega}_1^N 1) + q_{N+1} f(\underline{\omega}_1^N 0) \quad \boxed{\text{is increasing on } \Omega_N}$$

$$\mathbb{E}_{N+1}(f \cdot g) = \mathbb{E}_N\left(\mathbb{E}_{N+1}(f \cdot g \mid \underline{\omega}_1^N)\right)$$

as $N=1$

$$= \mathbb{E}_N\Big(\mathbb{E}_{N+1}(f \mid \underline{\omega}_1^N)\,\mathbb{E}_{N+1}(g \mid \underline{\omega}_1^N) +$$

$$p_{N+1}(1-p_{N+1})\left(f(\underline{\omega}_1^N 1) - f(\underline{\omega}_1^N 0)\right)\left(g(\underline{\omega}_1^N 1) - g(\underline{\omega}_1^N 0)\right)\Big)$$

induction

$$\geq \mathbb{E}_N\left(\mathbb{E}_{N+1}(f \mid \underline{\omega}_1^N)\,\mathbb{E}_{N+1}(g \mid \underline{\omega}_1^N)\right) \geq$$

$$\mathbb{E}_N\left(\mathbb{E}_{N+1}(f|\underline{\omega}_1^N)\right)\cdot\mathbb{E}_N\left(\mathbb{E}_{N+1}(g|\underline{\omega}_1^N)\right) =$$

$$\mathbb{E}_{N+1}(f)\cdot\mathbb{E}_{N+1}(g) \qquad \square$$

<u>Remarks</u> on the two theorems - outlook:

1) <u>Theorem</u> Let Ω be a poset and $\mathbb{P}, \mathbb{P}'$ two probability measures on it. Then $\mathbb{P} \leq \mathbb{P}'$ iff $\exists$ a probab measure $\mathbb{Q}$ on $\Omega\times\Omega$ such that 1 & 2 in the proof of Thm 1 hold.

2) FKG measures on associative lattices are relevant. Their characterization: Holley's ineq.

3) Relevance in stat phys. Correlation inequalities: Ising, F-K models etc.

<u>Immediate consequence</u> of these simple theorems: well defined critical point(s):

let $x, y \in V$, then:

$$\{\theta_x(p) > 0\} \Longleftrightarrow \{\theta_y(p) > 0\}$$

$$\{\chi_x(p) = \infty\} \Longleftrightarrow \{\chi_y(p) = \infty\}$$

$$\mathbb{P}_p(|C_x| = \infty) \geq \mathbb{P}_p(\{x \leftrightarrow y\} \cap \{|C_x| = \infty\})$$

$$\underset{\text{Harris ineq}}{\geq} \underbrace{\mathbb{P}_p(x \leftrightarrow y)}_{> 0} \mathbb{P}_p(|C_x| = \infty)$$

$$\mathbb{E}_p(|C_x|) \geq \mathbb{E}_p(\mathbb{1}(x \leftrightarrow y) \cdot |C_y|)$$

$$\underset{\text{Harris ineq}}{\geq} \underbrace{\mathbb{P}_p(x \leftrightarrow y)}_{> 0} \mathbb{E}_p(|C_y|)$$

Def

$$p_H := \sup\{p : \theta_x(p) = 0\}$$ no matter what x

$$= \inf\{p : \theta_x(p) > 0\}$$

$$p_T := \sup\{p : \chi_x(p) < \infty\}$$ no matter what x

$$= \inf\{p : \chi_x(p) = \infty\}$$

H for Hammersley

T for Temperley

a priori : $p_T \leq p_H$

Q : Is $p_T = p_H$? far not trivial question

"sharpness of phase transition"

existence of ∞ cluster(s) is a <u>tail event</u>. Measurable w.r.t.

$$\mathcal{T} := \bigcap_{\substack{\Lambda \subset V \\ |\Lambda| < \infty}} \mathcal{F}_{\Lambda^c}$$

the tail σ-algebra

Due to Kolmogorov's 0-1 law

$$\{\theta(p) = 0\} \Leftrightarrow \mathbb{P}_p(\exists\, \infty \text{ cluster(s)}) = 0$$

$$\{\theta(p) > 0\} \Leftrightarrow \mathbb{P}_p(\exists\, \infty \text{ cluster(s)}) = 1$$

Q: $\theta(p_H) = 0$? ; > 0 ?

Is there ∞ cluster at the critical density?

far not trivial question.

Not even known in important cases.

First estimates of p_c on $\mathbb{Z}^d$:

the phase transition is not trivial ($0 < p_c < 1$) for $d \geq 2$

Hammersley 1957

bond percolation on $\mathbb{Z}^d$:

The connectivity constant:

$w_n :=$ # simple (self-avoiding) walks (on $\mathbb{Z}^d$) of length $\underline{n}$ starting at 0

Straightforward: $w_{n+m} \leq w_n \cdot w_m$

or $\log w_{n+m} \leq \log w_n + \log w_m$

From the subadditive convergence lemma it follows that

$$\lambda(d) := \lim_{n \to \infty} (w_n)^{1/n} \in [1, 2d-1] \text{ exists}$$

Digression: Subadditive Convergence

Lemma: Let $a_n \in \mathbb{R}$ $n \in \mathbb{N}$ be a subadditive sequence:

$$a_{n+m} \leq a_n + a_m$$

Then $\exists \lim_{n\to\infty} \frac{a_n}{n} = \inf \frac{a_n}{n} \in [-\infty, +\infty)$

Proof: Fix $N < \infty$ and write $n = kN + r$

$$k = \left\lfloor \frac{n}{N} \right\rfloor \; ; \; r = n - kN \in \{0, \dots, N-1\}$$

$$\frac{a_n}{n} = \frac{a_{kN+r}}{kN+r} \leq \frac{k a + a_r}{kN + r} =$$

$$\frac{a_N + \frac{a_r}{k}}{N + \frac{r}{k}}$$

Thus $\overline{\lim_{n\to\infty}} \frac{a_n}{n} \leq \frac{a_N}{N}$ this holds for <u>any</u> N

Hence the Lemma $\square$

<u>Theorem</u> (Hammersley, 1957)

$d \geq 2$, bond percolation

$$\frac{1}{2d-1} \leq \lambda(d)^{-1} \leq p_c(\mathbb{Z}^d, \text{bond-perc}) \leq 1 - \lambda(2)^{-1} \leq \frac{2}{3}$$

<u>Proof</u>:

<u>lower bound</u>:

$$\mathbb{P}_p(0 \leftrightarrow \partial B_{0,n}) \overset{\text{union bound}}{\leq} w_n p^n$$

If $p \leq \lambda(d)^{-1}$, then $w_n p^n \longrightarrow 0$ ✓

<u>upper bound</u>

$$\mathcal{G}' \subseteq \mathcal{G} \implies p_c(\mathcal{G}') \geq p_c(\mathcal{G})$$

Thus, for $d \geq 2$: $p_c(\mathbb{Z}^2) \geq p_c(\mathbb{Z}^d)$

on $\mathbb{Z}^2$ apply Peierls' argument

Let: $K_n :=$ # contours on dual lattice of length n surrounding 0.

Contour = circuit on the Whitney dual.

$\mathcal{K}_n \leq n \cdot w_n$ — straightforward.

$\Downarrow$

$$\overline{\lim_{n\to\infty}} (\mathcal{K}_n)^{1/n} \leq \lambda(2)$$

Let $p > 1 - \lambda(2)^{-1}$

$q = 1 - p < \lambda(2)^{-1}$

and L so large that $\sum_{n \geq 8L} q^n \mathcal{K}_n < 1$

Then: $\mathbb{P}_p\big(\partial([-L,L]\times[-L,L]) \leftrightarrow \infty\big) =$

$\mathbb{P}_p$(no dual contour surrounding $[-L,L]\times[-L,L]$ is blocked)

union bound $\geq 1 - \sum_{n>8L} w_n q^n > 0$

$\mathbb{P}_p(\omega_e = 1,\ e \in [-L,L]\times[-L,L]) > 0$

$\Rightarrow$ (by Harris ineq): $\mathbb{P}_p(0 \leftrightarrow \infty) > 0$ $\square$

__Remark__ The exact value of p_c can be determined in some exceptional 2d cases:

p_c (square lattice, bond) $= \frac{1}{2}$

p_c (triangular lattice, bond) $= 2 \sin \frac{\pi}{18}$

p_c (hexagonal lattice, bond) $= 1 - 2 \sin \frac{\pi}{18}$

p_c (triangular lattice, site) $= \frac{1}{2}$

and some more.

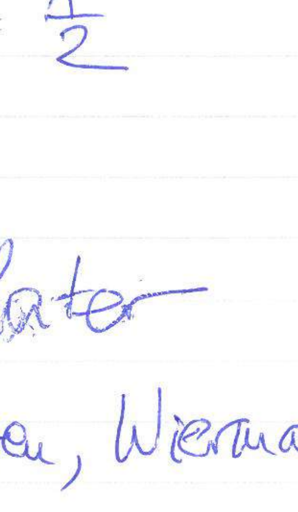
Long way... we'll see later

Sykes-Essam, Russo, Kesten, Wierman...

Otherwise: clever bounds on p_c

HW: • compute $\theta(p), \chi(p)$ for T_d

and for the cactus graph:



• compute $\theta(p), \chi(p)$ for Sierpinski graphs in various dimension

Comparing critical densities of bond vs. site percolation (on the same graph)

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ of bdd degree.

<u>Theorem</u> (Hammersley, 1957)

$$p_c(\mathcal{G}, \text{site}) \geq p_c(\mathcal{G}, \text{bond}).$$

<u>Proof</u>: fix $p \in (0,1)$ and

$v \in \mathcal{V}$, $V \subseteq \mathcal{V}$,

$\beta = \beta(v, V, \mathcal{G}) := \mathbb{P}_p^b(v \leftrightarrow V, \text{ in } \mathcal{G})$

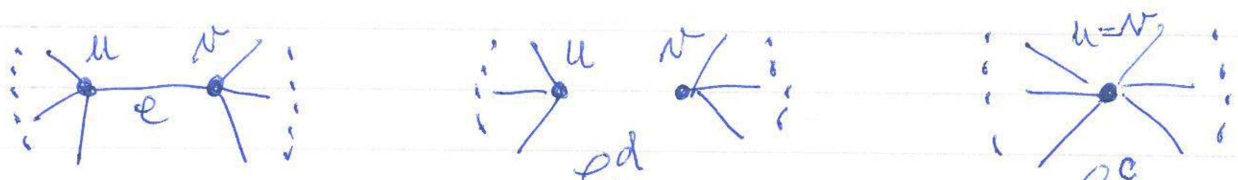
$\sigma = \sigma(v, V, \mathcal{G}) := \mathbb{P}_p^s(v \leftrightarrow V, \text{ in } \mathcal{G})$

we prove by induction on $|\mathcal{E}|$: $\boxed{\sigma \leq p \cdot \beta}$

$|\mathcal{E}| = 1$: $\mathcal{G} = \overset{v}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{u}{\bullet}$, $V = \{u\}$

$$\beta = p \;;\; \sigma = p^2 = p \cdot \beta \quad \checkmark$$

$|\mathcal{E}| > 1$:



$\mathcal{G}$ $\quad$ $\mathcal{G}^d$ $\quad$ $\mathcal{G}^c$

bond (u,v) deleted $\quad$ bond (u,v) contracted

$$\beta(v, V, \mathcal{G}) = \mathbb{P}^b(v \leftrightarrow V, \omega_e = 1) + \mathbb{P}^b(v \leftrightarrow V, \omega_e = 0)$$

$$= p\,\beta(v, V, \mathcal{G}^c) + (1-p)\,\beta(v, V, \mathcal{G}^d)$$

induction step $\rightarrow$
$$\geq \frac{1}{p}\left(p\,\sigma(v, V, \mathcal{G}^c) + (1-p)\,\sigma(v, V, \mathcal{G}^d)\right)$$

$$p\,\sigma(v, V, \mathcal{G}^c) = \mathbb{P}^s(v \leftrightarrow V, \omega_u = 1, \text{ on } \mathcal{G})$$

$$(1-p)\,\sigma(v, V, \mathcal{G}^d) \geq \mathbb{P}^s(v \leftrightarrow V, \omega_u = 0, \text{ on } \mathcal{G})$$

(both by conditioning on ω_u)

Hence: $$\beta(v, V, \mathcal{G}) \geq \frac{1}{p}\,\sigma(v, V, \mathcal{G})$$

q.e.d. $\square$

Hence the Thm...

Main questions to come:

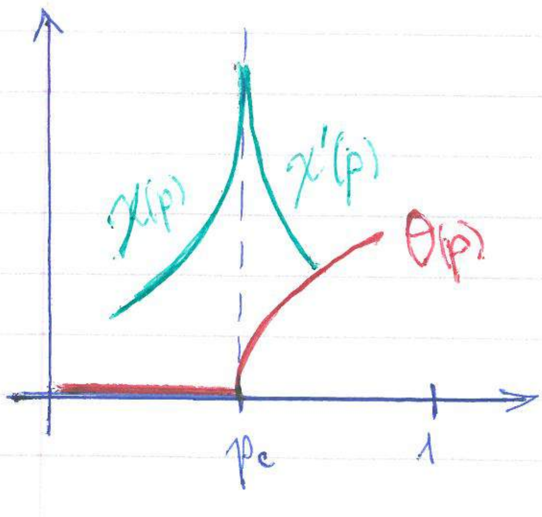
- number of ∞ clusters ($p > p_H$)
  continuity of $(p_H, 1) \ni p \mapsto \theta(p)$

- sharpness of the phase transition:
  $p_T = p_H$, actually, for $p < p_H$
  $\mathbb{P}_p(|C_0| > n)$ decays exponentially fast as $n \to \infty$

  Ingredients: more sophisticated tools

- Finer analysis of $p \mapsto \theta(p)$, $p \mapsto \chi(p)$

- critical behaviour



$$\theta(p) \sim (p - p_c)^{\beta} \qquad p \downarrow p_c$$
$$\chi(p) \sim (p_c - p)^{-\gamma} \qquad p \uparrow p_c$$
$$\chi'(p) \sim (p - p_c)^{-\gamma'} \qquad p \downarrow p_c$$
$$\mathbb{P}_{p_c}(|C| > n) \sim n^{-1/\delta} \qquad p = p_c$$

etc

The numerical value of p_c is incidental: depending on local details of the underlying graph.
The critical exponents ($\delta, \gamma, \beta, \dots$) are robust, universal: depend only on global features, as. e.g. <u>dimension</u>

Not even on the dimension above the critical dimension $\bar{d} = 6$ (conjectured)

- $\boxed{d=2}$ due to topological reasons (Whitney...): very special features, reasonings, results
- $\boxed{d=2}$ conformal invariance at p_c
- $\boxed{d > \bar{d}}$ high dimension: "mean field behaviour" as in Erdős-Rényi or on $\mathbb{T}_d$.

...