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Relaxed Sector Condition and Random Walk in Divergence Free Random Drift Field

based on joint work with
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Oberwolfach, 28 Oct – 1 Nov 2013

Notation:

$(\Omega, \pi, \tau_z : z \in \mathbb{Z}^d)$ — probability space with ergodic $\mathbb{Z}^d$-action

$\mathcal{E} = \{k \in \mathbb{Z}^d : |k| = 1\}$ — possible steps of the rw

$v_k : \Omega \to [-1, 1], \quad k \in \mathcal{E}$

- $v_k(\omega) + v_{-k}(\tau_k\omega) \equiv 0$ — vector field
- $\sum_{k\in\mathcal{E}} v_k(\omega) \equiv 0$ — divergence-free
- $\int_\Omega v_k(\omega) d\pi(\omega) = 0,$ — no overall drift

Lift it to a stationary and divergence free vector field over $\mathbb{Z}^d$:

$$V_k(\omega, x) := v_k(\tau_x \omega)$$

$$V_{-k}(x+k) + V_k(x) \equiv 0, \qquad \sum_{k \in \mathcal{E}_d} V_k(x) \equiv 0, \qquad \mathbf{E}\Big(V_k(x) \Big) = 0.$$

The random walk:

$$\mathbf{P}_\omega\Big(X(t+dt) = x+k \;\Big|\; X(t) = x \Big) = (1 + V_k(\omega, x))\, dt + \mathcal{O}((dt)^2).$$

The diffusion analogue: $V : \mathbb{R}^d \to \mathbb{R}^d$ stationary, divergence-free vector field,

$$dX(t) = dB(t) + V(X(t))dt,$$

Question: Diffusive asymptotics of $X(T)/\sqrt{T}$ as $T \to \infty$?

Drift field and its covariances:

$$\varphi(\omega) := \sum_{k \in \mathcal{E}_d} k v_k(\omega), \qquad \Phi(\omega, x) := \varphi(\tau_x \omega)$$

$$C_{i,j}(x) := \mathbf{E}\Big(\Phi_i(x) \Phi_j(0) \Big), \qquad \widehat{C}_{i,j}(p) := \sum_{x \in \mathbb{Z}^d} e^{\sqrt{-1} p \cdot x} C_{i,j}(x)$$

H_{-1}**-condition − assumed:**

$$(2\pi)^{-d} \int_{[-\pi,\pi]^d} \frac{\sum_{i=1}^d \widehat{C}_{i,i}(p)}{\sum_{i=1}^d (1 - \cos(p \cdot e_i))} dp < \infty$$

Equivalently:

$$\lim_{T \to \infty} T^{-1} \mathbf{E}\Big(\left(\int_0^T \Phi(S(t)) dt \right)^2 \Big) < \infty,$$

The environment process: $\eta_n := \tau_{X_n}\omega$. Stationary and ergodic pure jump Markov process on (Ω, π). (Stationarity due to div-freeness.)

Martingale decomposition of the walk: $X(t) = Y(t) + Z(t)$,

$$Y(t) := X(t) - \int_0^t \varphi(\eta(s))ds, \qquad Z(t) := \int_0^t \varphi(\eta(s))ds,$$

$Y(\cdot)$ and $X(\cdot)$ are uncorrelated (easy), and

$$\mathbf{E}\Big(X_i(t)X_j(t) \Big) = \delta_{i,j}(2d)t + \mathbf{E}\Big(Z_i(t)Z_j(t) \Big).$$

Diffusive bounds:

$$2d \leq \overline{\lim_{T\to\infty}} T^{-1}\mathbf{E}\Big(|X(T)|^2 \Big) \leq 2d + \sum_{i=1}^{d} \widetilde{C}_{i,i} < \infty$$

Upper bound from H_{-1}.

Theorem. *If the* H_{-1}-condition *holds then the asymptotic co-variance matrix*

$$(\sigma^2)_{i,j} := \lim_{T\to\infty} T^{-1}\mathbf{E}\Big(X_i(T)X_j(T) \Big)$$

exists, and is bounded as follows

$$2dI_d \le \sigma^2 \le 2dI_d + \tilde{C}.$$

For any $m \in \mathbb{N}$, $t_1, \dots, t_m \in \mathbb{R}_+$ *and* $f : \mathbb{R}^{md} \to \mathbb{R}$ *continuous and bounded*

$$\int_\Omega \Big|\mathbf{E}_\omega\Big(f(\dots, T^{-1/2}X(Tt_j), \dots)\Big) - \mathbf{E}\Big(f(\dots, W(t_j), \dots) \Big)\Big| d\pi(\omega) \to 0,$$

as $T \to \infty$, *where* $t \mapsto W(t) \in \mathbb{R}^d$ *is a Brownian motion with*

$$\mathbf{E}\Big(W_i(t) \Big) = 0, \qquad \mathbf{E}\Big(W_i(s)W_j(t) \Big) = \min\{s,t\}(\sigma^2)_{i,j}$$

Historical comments:

[Kozlov (1985)] claims similar result for $V(x)$ *finitely dependent*.

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[Komorowski, Landim, Olla (2012)]: Same result – now fully proved – with restrictive conditions: H_{-1} ($\mathcal{L}^2$-condition on *stream-field*) replaced by $\mathcal{L}^{\max\{2+\varepsilon,d\}}$. More later .... Proof very technical.

$\mathcal{L}^{\infty}$ stream-field: [Komorowski, Olla (2003)]

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The continuous space diffusion problem: $\mathcal{L}^\infty$ stream-field: [Papanicolaou, Varadhan (1981)], [Osada (1983)]; $\mathcal{L}^2$ stream-field: [Oelschläger (1988)] – very technical proof.

Helmholtz's Theorem, stream field: $\mathbb{Z}^d_* := \mathbb{Z}^d + (1/2, \dots, 1/2)$

$\mathbf{d} = 2$:
There exists a scalar field (*height function*): $H : \Omega \times \mathbb{Z}^2_* \to \mathbb{R}$ with stationary increments such that

$$V = \operatorname{curl} H, \qquad V_k(x) = H(x + \frac{k + \tilde{k}}{2}) - H(x + \frac{k - \tilde{k}}{2})$$

$\mathbf{d} = 3$:
There exists a vector field (*stream field*) $H_k : \Omega \times \mathbb{Z}^3_* \to \mathbb{R}$, $k \in \mathcal{E}$, with stationary increments such that

$$V = \operatorname{curl} H, \qquad V_k(\omega, x) = \dots \text{explain in plain words}$$

The H_{-1} condition equiv.: The height function/stream field is **stationary** (not just of stationary increments!) and $\mathcal{L}^2$.

Examples and essentially different cases:

◦ H stationary + ergodic + bounded [S.M. Kozlov (1985)]:

$$h \in \mathcal{L}^{\infty}, \qquad H(\omega, x) = h(\tau_x \omega) - h(\omega)$$

◦ H stationary + ergodic + unbounded + curl H bounded:

$$h \in \mathcal{L}^{2} \setminus \mathcal{L}^{\infty}, \qquad H(\omega, x) = h(\tau_x \omega) - h(\omega)$$

this is the case discussed today. H_{-1}-condition ✓

◦ H {stationary + ergodic} increments (but not stationary).

+ curl H bounded. No H_{-1}-condition, superdiffusive.

◦◦ randomly oriented Manhattan-lattice

◦◦ six-vertex / square ice ($d = 2$)

◦◦ dimer tiling ($d = 2$)

Some **operators** on the Hilbert space $\mathcal{L}^2(\Omega, \pi)$:

$\mathcal{L}^2(\Omega, \pi)$-gradient : $\nabla_k f(\omega) := f(\tau_k \omega) - f(\omega)$

$\nabla_k^* = \nabla_{-k}$

$\mathcal{L}^2(\Omega, \pi)$-Laplacian : $\Delta f(\omega) := \sum_{k \in \mathcal{E}} (f(\tau_k \omega) - f(\omega))$

$\Delta^* = \Delta \leq 0$

multiplication ops. : $M_k f(\omega) := v_k(\omega) f(\omega)$

$M_k^* = M_k$

A commutation relation – due to div-freeness of v:

$$\sum_{k\in\mathcal{E}} M_k \nabla_k + \sum_{k\in\mathcal{E}} \nabla_{-k} M_k = 0$$

The **infinitesimal generator** of the environment process:

$$L = P - I = \frac{1}{2}\Delta + \sum_{k\in\mathcal{E}} M_k \nabla_k =: -S + A$$

Relaxed Sector Condition [I. Horváth, B. Tóth, B. Vető (2012)]

Theorem: Efficient martingale approximation (a la Kipnis-Varadhan) holds for $\int_0^t \varphi(\eta_s) ds$ if

(1)	" $S^{-1/2} A S^{-1/2}$ "	is *skew self-adjoint* (not just skew symmetric).
(2)	$\varphi \in \mathtt{Ran}(S^{-1/2})$	H_{-1}-condition

Remarks:

(1) Extends Varadhan et al.'s *Graded Sector Condition*.

(2) Proof: partly reminiscent of Trotter-Kurtz.

Two possible definitions of " $S^{-1/2}AS^{-1/2}$ ":

$$B := \sum_{k\in\mathcal{E}} \left((-\Delta)^{-1/2}\nabla_{-k}\right) M_k \, (-\Delta)^{-1/2} = \text{" } S^{-1/2}AS^{-1/2} \text{ "}$$

on $\mathcal{C} := \mathtt{Dom}(-\Delta)^{-1/2} = \mathtt{Ran}(-\Delta)^{1/2}$

$$\widetilde{B} := (-\Delta)^{-1/2} \sum_{k\in\mathcal{E}} M_k \left((-\Delta)^{-1/2}\nabla_k\right) = \text{" } S^{-1/2}AS^{-1/2} \text{ "}$$

on $\widetilde{\mathcal{C}} := \{f \in \mathcal{L}^2 : \sum_{k\in\mathcal{E}} M_k \left((-\Delta)^{-1/2}\nabla_k\right) f \in \mathtt{Dom}(-\Delta)^{-1/2}\}$

Facts (easy): (1) B is *skew symmetric* on $\mathcal{C}$.

(2) $\mathcal{C} \subset \widetilde{\mathcal{C}}$ and $\widetilde{B}\big|_{\mathcal{C}} = B$.

(3) $\widetilde{B} = \overline{\widetilde{B}}$ and $\widetilde{B} = -B^*$.

Wanted: $\overline{B} = \widetilde{B}$, or, equivalently $\overline{B} = -B^*$

What is missing from skew self-adjointmess of " $S^{-1/2}AS^{-1/2}$ "?

von Neumann's criterion:

$$\begin{pmatrix} B \text{ skew symmetric, and} \\ \overline{\mathtt{Ran}(B \pm I)} = \mathcal{H} \end{pmatrix} \Leftrightarrow \begin{pmatrix} B \text{ essentially} \\ \text{skew self-adjoint} \end{pmatrix}$$

Needed:

$$\sum_{k \in \mathcal{E}} M_k \Big((-\Delta)^{-1/2} \nabla_k \Big) \psi = (-\Delta)^{1/2} \psi \qquad \Rightarrow \qquad \psi = 0.$$

Warning: Formal manipulation deceives: $\psi \notin \mathtt{Dom}(-\Delta)^{-1/2}$!

Raise it to the lattice $\mathbb{Z}^d$: change of notation: from now on: $\nabla, \Delta, \ldots$ = lattice gradient, lattice Laplacian, . . .

Wanted:
NO nontrivial scalar field $\Psi : \Omega \times \mathbb{Z}^d \to \mathbb{R}$ with stationary increment, and $\mathbf{E}\Big(\Psi \Big) = 0$ solves the PDE

$$\Delta \Psi = V \cdot \nabla \Psi. \tag{1}$$

$$\text{"} \ \Psi = (-\Delta)^{-1/2} \psi \text{ ",} \qquad \nabla \Psi(\omega, x) = (-\Delta)^{-1/2} \nabla \psi(\tau_x \omega)$$

Note similarity: No sublinearly growing harmonic function on $\mathbb{Z}^d$.

Let Ψ be solution of (1). Then $t \mapsto R(t) := \Psi(X(t))$ is a martingale with stationary and ergodic increments.

$$\varrho^2 := t^{-1}\mathbf{E}\Big(R(t)^2 \Big) = \cdots = 2 \, \| \, \psi \, \|^2 .$$

and

$$t^{-1/2} R(t) \Rightarrow \mathcal{N}(0, \rho^2)$$

We prove $\varrho = 0$, and hence $\psi = 0$.

Let $\delta > 0$ and $K < \infty$. Then

$$\mathbf{P}\Big(\,|R(t)\,| > \delta\sqrt{t}\Big) \leq \;\; \mathbf{P}\Big(\{|R(t)\,| > \delta\sqrt{t}\} \wedge \{|X(t)\,| \leq K\sqrt{t}\} \Big) + \mathbf{P}\Big(\{\{|X(t)\,| > K\sqrt{t}\} \Big).$$

By diffusive upper bound (due to H_{-1}):

$$\lim_{K\to\infty} \overline{\lim_{t\to\infty}}\, \mathbf{P}\Big(\{|X(t)\,| > K\sqrt{t}\} \Big) = 0.$$

Remains to prove:

$$\lim_{t\to\infty} \mathbf{P}\Big(\{|R(t)\,| > \delta\sqrt{t}\} \wedge \{|X(t)\,| \leq K\sqrt{t}\} \Big) = 0.$$

$\mathbf{d = 2}$ (with bare hands):

$$\lim_{N\to\infty} N^{-1} \max_{x\in[-N,N]^2} |\Psi(x)| = 0 \qquad \text{in probeb.}$$

Maximum principle: $$\max_{x\in\Lambda} |\Psi(x)| = \max_{x\in\partial\Lambda} |\Psi(x)|$$

By ergodic thm . . . : $$N^{-1} \max_{x\in\partial[-N,N]^2} |\Psi(x)| \xrightarrow{\mathbf{P}} 0,$$

Thus: $$N^{-1} \max_{x\in[-N,N]^2} |\Psi(x)| \xrightarrow{\mathbf{P}} 0,$$

$\mathbf{d} \geq 2$, using heat kernel (upper) bound of [Morris, Peres (2005)]:

$$\mathbf{P}\Big(\{|R(t)\,| > \delta\sqrt{t}\} \wedge \{|X(t)\,| \leq K\sqrt{t}\} \Big) \leq \delta^{-2} t^{-1} \mathbf{E}\Big(\, |R(t)\,|^2 \, 1\!\!1_{\{|X(t)\,| \leq K\sqrt{t}\}} \Big)$$

[Morris, Peres (2005)]: There exists $C = C(d) < \infty$ such that if $X(t)$ is nearest neighbour *bi-stochastic* rw on $\mathbb{Z}^d$ with total jump rate 1, then

$$\sup_{x \in \mathbb{Z}^d} \mathbf{P}\Big(X(t) = x \Big) \leq C t^{-d/2}.$$

Also: for $\Psi(x)$ *with stationary increments*

$$\lim_{|x\,| \to \infty} |x\,|^{-2} \mathbf{E}\Big(\, |\Psi(x)\,|^2 \, \Big) = 0.$$

From these two:

$$t^{-1} \mathbf{E}\Big(\, |R(t)\,|^2 \, 1\!\!1_{\{|X(t)\,| \leq K\sqrt{t}\}} \Big) \leq C t^{-d/2-1} \sum_{|x\,| \leq K\sqrt{t}} \mathbf{E}\Big(\, |\Psi(x)\,|^2 \, \Big) \to 0,$$

as $t \to \infty$. $\square$