Lecture 4 : Applications:

- tagged particle diffusion in SEP
  symmetric & zero-mean asymm.
- rw/diffusion in div-free drift field
- self repelling Brownian polymers.

Tagged particle diffusion in SSEP on $\mathbb{Z}^d$

(Kipnis-Varadhan, 1986)

Recall definitions & notation from lecture 1 ....

Symmetric: $p_z = p_{-z}$, $t \mapsto \eta_t$ reversible, $G = G^*$

$$Y(t) := X(t) - \int_0^t \varphi(\eta_s)\,ds \;; \qquad \varphi(\omega) := \sum_{y \in \mathbb{Z}^d_*} \omega_y\, y\, p_y$$

$Y(t)$ mtg... $E(Y(t)) = 0$; $E(Y(t)^2) = t(1-\rho)\sum y^2 p_y$

Needed

① $\lim_{t\to\infty} \frac{1}{t} E\left(\left(\int_0^t \varphi(\eta_s)\,ds\right)^2\right) < \infty$; ② $Y(t)$ & $\int_0^t \varphi(\eta_s)\,ds$ don't cancel asymptotically

1 time reversal:

fix $t < \infty$, let $\eta^*_s := \eta_{t-s}$ $\quad 0 \leq s \leq t$

$$\left(\eta^*_s\right)_{0\leq s\leq t} \overset{\text{distr.}}{\sim} \left(\eta_s\right)_{0\leq s\leq t} \quad \boxed{\text{reversibility}}$$

$$\left(X^*(t), \int_0^t \varphi(\eta^*_s)\,ds\right) = \left(-X(t), \int_0^t \varphi(\eta_s)\,ds\right)$$

Thus $\quad \mathbb{E}\left(X^*(t)\cdot\int_0^t \varphi(\eta_s)\,ds\right) = 0$

Hence $\quad \mathbb{E}\left(X(t)^2\right) + \mathbb{E}\left(\left(\int_0^t \varphi(\eta_s)\,ds\right)^2\right) = \mathbb{E}\left(Y(t)^2\right) = t(1-\rho)\sum_{y\in\mathbb{Z}_*} y^2 p_y$

$< \infty$

Note: asymptotic cancellation of $Y(t)$ and

$\int_0^t \varphi(\eta_s)\,ds$ $\boxed{\text{may}}$ occur!

<u>Theorem</u> (R. Arratia 1983, conjectured by F. Spitzer 1970s)

$d=1$, $p_{+1}=p_{-1}=\frac{1}{2}$ (nearest neighbour, symmetric)

$$N^{-1/4}\,Y(Nt) \Longrightarrow \text{FBM}$$

We prove that this doesn't happen if

either ($d>1$) or ($d=1$ and $p_z=p_{-z}$ is <u>not</u> n.n.)

$\mathbb{Z}^d_*$ connected for $(p_{x-y})_{x,y\in\mathbb{Z}^d_*}$.

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$G = G_1 + G_2$: G_1 jumps of the tagged particle

G_2 jumps of the other particles

$G_1 = G_1^*, \quad G_2 = G_2^*$

$u_\lambda := (\lambda I - G)^{-1} \varphi$

$$M_\lambda(t) = u_\lambda(\eta_t) - u_\lambda(\eta_0) - \int_0^t G u_\lambda(\eta_s) ds$$ the K-V approximating mtg

$$X(t) = Y(t) + M_\lambda(t) + \mathrm{error}_\lambda(t)$$

compute the bracket (quadr. var.) processes: $\langle N(t)^2 \rangle, \langle M_\lambda(t)^2 \rangle, \langle N(t), M_\lambda(t) \rangle$

... straightforward...

$$\langle (N(t)+M_\lambda(t))^2 \rangle =$$ (5)

$$\int_0^t \sum_{z\in\mathbb{Z}^d_*} p_z (1-\eta_{s,z}) \left\{ z + (u_\lambda(\eta_s^z) - u_\lambda(\eta_s)) \right\}^2 ds +$$

$$\int_0^t \sum_{x,y\in\mathbb{Z}^d_*} p_{y-x}\, \eta_{s,x} (1-\eta_{s,y}) \left\{ u_\lambda(\eta_s^{xy}) - u_\lambda(\eta_s) \right\}^2 ds =: A_\lambda(t) + B_\lambda(t)$$

$$\mathbb{E}(B_\lambda(t)) = t\cdot (u_\lambda, -G_2 u_\lambda)$$

Wanted: $\lim_{\lambda\downarrow 0} (u_\lambda, -G_2 u_\lambda) > 0.$

Sufficient: $(\varphi, -G_2^{-1}\varphi) < \infty$

Indeed:

$$0 < \|(-G)^{-1/2}\varphi\|^2 = \lim_{\lambda \downarrow 0} \left(\varphi, (\lambda I - G)^{-1}\varphi\right) =$$

$$= \lim_{\lambda \downarrow 0} \left((\mu I - G_2)^{-1/2}\varphi, (\mu I - G_2)^{1/2}(\lambda I - G)^{-1}\varphi\right) \leq$$

$$\leq \|(\mu I - G_2)^{-1/2}\varphi\| \lim_{\lambda \downarrow 0} \|(\mu I - G_2)^{1/2}(\lambda I - G)^{-1}\varphi\| =$$

$$= \sqrt{\left(\varphi, (\mu I - G_2)^{-1}\varphi\right)} \lim_{\lambda \downarrow 0} \sqrt{\left(u_\lambda, (\mu I - G_2) u_\lambda\right)}$$

$$\xrightarrow{\mu \downarrow 0} \sqrt{\left(\varphi, -G_2^{-1}\varphi\right)} \lim_{\lambda \downarrow 0} \sqrt{\left(u_\lambda, -G_2 u_\lambda\right)}$$

all limits exist by monotonicity

Let $\tilde{\eta}_s$ be SSEP with same $(p_z)_z$ on $\mathbb{Z}^d_*$ (17)

$$(\varphi, (-G_2)^{-1}\varphi) < \infty \iff \lim_{t\to\infty} \frac{1}{t} \mathbb{E}\left(\left(\int_0^t \varphi(\tilde{\eta}_s)\, ds \right)^2 \right) < \infty.$$

Lemma Let $\tilde{\eta}_s$ be SSEP on arbitrary connected graph, stationary with π_ϱ. Then, for any two vertices x & y:

$$\lim_{t\to\infty} \frac{1}{t} \mathbb{E}\left(\left(\int_0^t (\eta_{s,x} - \eta_{s,y})\, ds \right)^2 \right) < \infty.$$

Proof sufficient to prove for $x \sim y$.

$$\mathcal{J}_{xy}(t) := \#\{x \to y \text{ jumps in } [0,t]\}$$

(8)

$$\underbrace{K_{xy}(t)}_{\text{mtg}} := \mathcal{J}_{xy}(t) - \mathcal{J}_{yx}(t) - p_{x,y}\int_0^t \left(\eta_{sx}(1-\eta_{sy}) - \eta_{sy}(1-\eta_{sx})\right) ds$$

$$= \underbrace{\mathcal{J}_{xy}(t) - \mathcal{J}_{yx}(t)}\; - \underbrace{p_{xy}\int_0^t (\eta_{sx} - \eta_{sy})\, ds}$$

uncorrelated, by time reversal.

Hence $\mathbb{E}\left(\left(\int_0^t (\eta_{sx} - \eta_{sy})\, ds\right)^2\right) \leq \frac{1}{p_{xy}^2}\mathbb{E}\left(K_{xy}(t)^2\right) = \frac{2\rho(1-\rho)}{p_{xy}}\, t$

$\square$

Tagged particle diffusion in 0-mean ASEP 9

(Varadhan, 1995)

setup & notation: same as before

but $p_{-z} \neq p_z$, $\sum z p_z = 0$

$G \neq G^*$: reversible/self adjoint version of K-V doesn't apply.

The Strong Sector Condition (Thm SSC) will be used.

Needed:

1. $\varphi \in \mathcal{H}_{-1}$: done in the symmetric case

2. check the SSC:

$\forall f, g \in \mathcal{H}$: $\quad |(f, Ag)|^2 \leq C^2 \cdot (f, Sf) \cdot (g, Sg)$!

or, equiv: $\quad |(f, Gg)|^2 \leq C^2 (f, Gf)(g, Gg)$!

3. $Y(t)$ & $\int_0^t \varphi(\eta_s)\,ds$ don't cancel asymptotically: done similarly as in the symmetric case

<u>Cycles</u>: $C = (0 = y_0, y_1, y_2, \dots, y_n = 0)$

$y_j \in \mathbb{Z}^d$, $0 \le i < j < n$: $y_i \neq y_j$

$z_j := y_j - y_{j-1}$, $j = 1, 2, \dots, n$

<u>Walk based on cycle C</u>:

$$p_z^C := \frac{1}{n} \sum_{j=1}^{n} \mathbb{1}(z_j = z); \quad \sum_z p_z = 1, \quad \sum_z z\, p_z = 0$$

<u>Proposition</u>: All finite range, 0-mean walks are convex combinations of finitely many cycle-walks

$$p_z = \sum_{k=1}^{r} \alpha_k \, p_z^{C_k}$$

(the representation is <u>not</u> unique)

Given C (finite cycle)

$$\mathbb{Z}^d_C := \{x \in \mathbb{Z}^d : 0 \notin x + C\} = \{x \in \mathbb{Z}^d : x \neq -y_j,\ j = 1, \dots, n\}$$

C cycle, $x \in \mathbb{Z}^d_C$:

$$G_{1,C,x} f(\omega) = \sum_{j=0}^{n-1} \omega_{x+y_j} (1 - \omega_{x+y_{j+1}}) \left(f(\omega^{x+y_j,\, x+y_{j+1}}) - f(\omega) \right)$$

$$G_{2,C} f(\omega) = \sum_{k=0}^{n-1} \Big\{ \sum_{j=1}^{n-2} \omega_{y_{k+j}} (1 - \omega_{y_{k+j+1}}) \left(f(\omega^{y_{k+j},\, y_{k+j+1}}) - f(\omega) \right) +$$

$$(1 - \omega_{z_{k+1}}) \left(f(\tau_{z_{k+1}} \omega) - f(\omega) \right) \Big\}$$

In plain words:

$G_{1,C,x}$: is the $\boxed{\text{TASEP}}$ generator $\boxed{\text{on}}$ the cycle $\boxed{x+C}$

$G_{2,C}$: is $\boxed{\text{TASEP seen from the tagged particle}}$ generator $\boxed{\text{on}}$ the cycle $\boxed{C}$

For these the SSC holds, with constant depending only on $|C|$.

= since these are Markov processes on finite state space =

If
$$p_\bullet = \sum_{k=1}^{r} \alpha_k \, p_\bullet^{c_k}$$

then
$$G = \sum_{k=1}^{r} \frac{\alpha_k}{|C_k|} \left\{ \sum_{x \in \mathbb{Z}^d_{C_k}} G_{1, C_k, x} + G_{2, C_k} \right\}$$

Lemma: Let
$$G = \sum_k G_k$$

and $\exists C$: $\forall k \, \forall f, g \quad |(f, A_k g)| \leq C \sqrt{(f, S_k f)} \sqrt{(g, S_k g)}$

then $\forall f, g \quad |(f, A g)| \leq C \sqrt{(f, Sf)} \sqrt{(g, Sg)}$

Proof: Schwarz. $\square$

Hence: SSC for G, and the result follows $\blacksquare$