Lecture 2 & 3: Martingale approximation, Kipnis-Varadhan theory

Martingale approximation, Kipnis - Varadhau theory

(Ω, π) probability space, $\mathcal{H} = \mathcal{L}^2(\Omega, \pi)$

$t \mapsto \eta_t$ stationary & ergodic Markov process on (Ω, π)

$P_t : \mathcal{H} \longmapsto \mathcal{H}$, $P_t f(\omega) := \mathbb{E}_\omega(f(\eta_t))$

the semigroup, assumed: strong continuity.

G: the infinitesimal generator (possibly unbounded)

G^* the adjoint generator

Assume : $\mathcal{C} \subset \mathcal{H}$ common core of G & G^*

$$S := -\frac{1}{2}\left(G+G^*\right); \quad A = \frac{1}{2}\left(G-G^*\right)$$

$$\forall \varphi, \psi \in \mathcal{C} : (\varphi, S\psi) = (S\varphi, \psi); \quad (\varphi, S\varphi) \geq 0$$

$$(\varphi, A\psi) = -(A\varphi, \psi)$$

Assume: $\left.\begin{array}{l} S \text{ essentially self-adjoint} \\ A \text{ essentially skew self-adjoint} \end{array}\right\}$ on $\mathcal{C}$

<u>Question:</u> let $f \in \mathcal{L}^2(\Omega, \pi)$, $\int_\Omega f \, d\pi = 0$

¿ CLT / invariance principle for $N^{-1/2} \int_0^{Nt} f(\eta_s) ds$?

<u>The variance:</u>

$$c(s) := \mathbb{E}\left(f(\eta_{t_0}) f(\eta_{t_0+s})\right) = (f, P_s f)$$

$$D(t) := \mathrm{Var}\left(t^{-1/2} \int_0^t f(\eta_s) ds\right) = 2 \int_0^t \frac{t-s}{t} c(s) ds$$

$$\hat{D}(\lambda) := 2 \int_0^\infty e^{-\lambda s} c(s) ds = 2 \int_0^\infty e^{-\lambda s} (f, P_s f) = 2 (f, R_\lambda f)$$

where $R_\lambda := \int_0^\infty e^{-\lambda s} P_s \, ds = (\lambda I - G)^{-1}$ is the resolvent

$$\hat{D}(\lambda) = \lambda^2 \int_0^\infty e^{-\lambda s} s\, D(s)\, ds = \int_0^\infty e^{-s} \cdot s\, D(s/\lambda)\, ds$$

Hence:

$$\underline{\lim}_{t\to\infty} D(t) \leq \underline{\lim}_{\lambda\to 0} \hat{D}(\lambda) \leq \overline{\lim}_{\lambda\to 0} \hat{D}(\lambda) \leq \overline{\lim}_{t\to\infty} D(t)$$

If $G = G^*$, then $C(s) > 0$ and automatically

$$\exists \lim_{t\to\infty} D(t) =: \sigma^2 \in (0, \infty]$$

Variational formula:

$$(f, R_\lambda f) = \sup_{g\in\mathcal{H}} \left\{ 2(f,g) - (g,(\lambda I+S)g) - (Ag,(\lambda I+S)^{-1}Ag) \right\}$$

Proof:

$$(f, R_\lambda f) = (R_\lambda f, (\lambda I - G^*)R_\lambda f) = (R_\lambda f, (\lambda I+S)R_\lambda f)$$

$$= \sup_{g\in\mathcal{H}} \left\{ 2(R_\lambda f, g) - (g, (\lambda I+S)^{-1} g) \right\}$$

$$= \sup_{g\in\mathcal{H}} \left\{ (f,g) - ((\lambda I - G^*)g, (\lambda I+S)^{-1}(\lambda I - G^*)g) \right\}$$

$$= \sup_{g\in\mathcal{H}} \left\{ 2(f,g) - (g,(\lambda I+S)g) - (Ag,(\lambda I+S)^{-1}Ag) \right\}$$

In particular:

$$0 < \left\{ \begin{matrix} \text{lower bound from} \\ \text{a cleverly chosen} \\ g \in \mathcal{H} \end{matrix} \right\} \leq (f, R_\lambda f) \leq (f, (\lambda I + S)^{-1} f)$$

The $\mathcal{H}_{-1}$ – condition (for $f \in \mathcal{H}$):

$$\lim_{\lambda \downarrow 0} (f, (\lambda I + S)^{-1} f) < \infty, \quad \text{or equivalently}$$

$$f \in \mathrm{Ran}(S^{1/2}) = \mathrm{Dom}(S^{-1/2})$$

$S^{1/2}$, $S^{-1/2}$ defined in terms of the spectral theorem

Probabilistic meaning of $\mathcal{H}_{-1}$:

let $t \mapsto \xi_t$ be Markov process on (Ω, π) with infinitesimal generator $-S = \frac{1}{2}(G + G^*)$ ["symmetrized"], reversible with π.

$\mathcal{H}_{-1}$: $\lim_{t\to\infty} \operatorname{Var}\left(t^{-1/2} \int_0^t f(\xi_s)\, ds\right) < \infty.$

Theorem (M. Gordin, B.A. Lifshitz 1978):

If $f \in \mathrm{Ran}(G)$ then

$$\frac{1}{\sqrt{T}} \int_0^T f(\eta_s)\,ds \Rightarrow \mathcal{N}(0, \sigma^2), \quad \sigma^2 = -2(f, G^{-1} f) \in (0, \infty).$$

Proof: essentially same as the $|\Omega| < \infty$ case.

let $g \in \mathcal{H}$: $f = Gg$.

$$M(t) := \int_0^t Gg(\eta_s)\,ds + g(\eta_0) - g(\eta_t), \text{ mtg}, \quad \mathbb{E}(M(t)^2) = \sigma^2 t$$

$$\int_0^t f(\eta_s)\,ds = M(t) - g(\eta_0) + g(\eta_t)$$

$$\frac{g(\eta_t) - g(\eta_0)}{\sqrt{t}} \xrightarrow{\mathcal{L}^2} 0 \quad \checkmark$$

actually: invariance principle holds

Theorem (C. Kipnis, SRS Varadhan 1986(1984)) 9

$$G = G^* =: -S \quad (\text{That is: if } t \mapsto \eta_t \text{ is reversible.})$$

If $\lim_{\lambda \downarrow 0} (f, (\lambda+S)^{-1} f) = \| S^{-1/2} f \|^2 < \infty$ $\quad \mathcal{H}_{-1}$

then there exists an $\mathcal{L}^2$ martingale (adapted to $\mathcal{F}_t$) with stationary & ergodic increments and variance

$$\mathbb{E}(M(t)^2) = \sigma^2 t, \qquad \sigma^2 = 2\| S^{-1/2} f \|^2$$

such that

$$\lim_{N \to \infty} \frac{1}{N} E\left(\left(\int_0^N f(\eta_s)\,ds - M(N) \right)^2 \right) = 0$$

Corollary

The finite dimensional marginal distributions of

$$t \longmapsto \frac{1}{\sigma\sqrt{N}} \int_0^{Nt} f(\eta_s)\, ds$$

converge to those of a 1d. standard BM.

Actually: tightness is also proved in K-V.

Proof: spectral calculus — later, ...

Applications: tagged particle diffusion in SSEP
RW among Random Conductancies
.... later...

Non-reversible case

Recall notation:

$$G = -S + A, \quad S = S^* \geq 0, \quad A = -A^*$$

$$R_\lambda := \int_0^\infty e^{-\lambda s} P_s \, ds, \quad \| R_\lambda \| \leq \lambda^{-1}$$

$(\lambda I + S)^{\pm 1/2}$ defined in terms of the spectral thm.

$$\| (\lambda + S)^{-1/2} \| \leq \lambda^{-1/2}, \quad \lambda > 0$$

$$\mathcal{H}_{-1} := \mathrm{Dom}(S^{-1/2}) = \mathrm{Ran}(S^{1/2}) = \{ \varphi \in \mathcal{H} : \lim_{\lambda \downarrow 0} \| (\lambda I + S)^{-1/2} \varphi \| < \infty \}$$

$f \in \mathcal{H}$, $\int f \, d\pi = 0$; $u_\lambda := R_\lambda f$. (12)

Theorem KV (B. Tóth 1986, following Kipnis–Varadhan)

If (A) $\lim_{\lambda \downarrow 0} \lambda^{1/2} u_\lambda = 0$ & (B) $\lim_{\lambda \downarrow 0} S^{1/2} u_\lambda =: v \in \mathcal{H}$

then $\sigma^2 := 2 \lim_{\lambda \downarrow 0} (u_\lambda, f) = 2\|v\|^2 \in (0, \infty)$ exists

and there exists a 0-mean, $\mathcal{L}^2$ martingale $M(t)$ adapted to $\mathcal{F}_t$ with stationary & ergodic increments

$$\mathbb{E}\left(M(t)^2\right) = \sigma^2 t$$

such that

$$\lim_{N \to \infty} \frac{1}{N} \mathbb{E}\left(\left(\int_0^N f(\eta_s)\, ds - M(N)\right)^2\right) = 0.$$

Remarks:

1. A & B $\Leftrightarrow$ C: $\lim_{\lambda,\mu\to 0} (\lambda+\mu)(u_\lambda, u_\mu) = 0$ (B. Tóth 1986)
   applied to randomized periodic Lorentz gas

2. Cheap: $\lim_{\lambda\downarrow 0} \lambda \|u_\lambda\| = 0$

   $\left.\begin{array}{l} \overline{\lim}\, \lambda^{1/2} \|u_\lambda\| < \infty \\ \overline{\lim}\, \|S^{1/2} u_\lambda\| < \infty \end{array}\right\}$ if $f \in \mathcal{H}_{-1}$

3. If $G = G^*$ then A $\Leftrightarrow$ B $\Leftrightarrow$ $\mathcal{H}_{-1}$ Kipnis-Varadhan

   (spectral theorem)

Proof (follow main lines of Kipnis-Varadhan, replacing spectral calculus with resolvent calculus) (14)

$$(u_\lambda, f) = (u_\lambda, (\lambda I - G) u_\lambda) = \underbrace{\lambda \|u_\lambda\|^2}_{\downarrow \atop 0} + \underbrace{(u_\lambda, S u_\lambda)}_{\downarrow \atop \|v\|^2}$$

$$M_\lambda(t) := u_\lambda(\eta_t) - u_\lambda(\eta_0) - \int_0^t (G u_\lambda)(\eta_s)\, ds, \quad \text{mtg.}$$

$$\mathbb{E}(M_\lambda(t)) = 0 \, ; \quad \mathbb{E}(M_\lambda(t)^2) = \ldots = 2 \|S^{1/2} u_\lambda\|^2$$

15

$$\mathbb{E}\left(\sup_{0 \le s \le t} \left(M_\lambda(s) - M_\mu(s) \right)^2 \right) \le 2\,\mathbb{E}\left(\left(M_\lambda(t) - M_\mu(t) \right)^2 \right) =$$

Doob

$$= 4t \, \| S^{1/2} u_\lambda - S^{1/2} u_\mu \|^2$$

by B : $\exists$ $t \mapsto M(t)$ mtg, such that

$$\mathbb{E}\left(\sup_{0 \le s \le t} \left(M_\lambda(s) - M(s) \right)^2 \right) \le 4t \, \| S^{1/2} u_\lambda - v \|^2$$

$$\mathbb{E}\left(M(t)^2 \right) = \lim_{\lambda \to 0} \mathbb{E}\left(M_\lambda(t)^2 \right) = \dots = 2 \|v\|^2 t$$

$$\int_0^t f(\eta_s)\,ds = M(t) + \big(M_\lambda(t) - M(t)\big)$$

$$- u_\lambda(\eta_t) + u_\lambda(\eta_0) + \int_0^t \lambda u_\lambda(\eta_s)\,ds$$

The error terms:

① $$\mathbb{E}\Big(\sup_{0\le s\le t}\big(M_\lambda(s)-M(s)\big)^2\Big) \le 4t\,\|S^{1/2}u_\lambda - v\|^2$$

②,③ $$\mathbb{E}\big(|u_\lambda(\eta_0)|^2\big) = \mathbb{E}\big(|u_\lambda(\eta_t)|^2\big) = t\,(t\lambda)^{-1}\cdot\lambda\|u_\lambda\|^2$$

④ $$\mathbb{E}\Big(\sup_{0\le s\le t}\Big(\int_0^s \lambda u_\lambda(\eta_r)\,dr\Big)^2\Big) \le t\,\mathbb{E}\Big(\int_0^t \lambda^2|u_\lambda(\eta_r)|^2 dr\Big)$$

$$= t\,(t\lambda)\,\lambda\|u_\lambda\|^2 \quad \boxed{\text{choose } \lambda = t^{-1}}$$

$\square$

Conditions A & B $\Leftrightarrow$ C too abstract, 17
difficult to check (however: directly used in (BT'86))

The Strong Sector Condition (Varadhan, 1995)

$$(\exists C<\infty):(\forall \varphi, \psi \in \mathcal{C}): |(\psi, A\varphi)|^2 \leq C^2 (\psi, S\psi)(\varphi, S\varphi)$$

or, equivalently: $\| S^{-1/2} A S^{-1/2} \| \leq C < \infty$.

Theorem (Varadhan, 1995):

$$\text{SSC} \;\&\; f \in \mathcal{H}_{-1} \Rightarrow \text{KV}$$

Applied to:

- ☆ tagged particle diffusion in 0-mean ASEP (Varadhan, 1998)
- ☆ RW/diffusion in div-free drift field with bounded stream tensor field (T. Komorowski, S. Olla, 2003)

The Graded Sector Condition

(S. Sethuraman, SRS. Varadhan, H-T Yau, 2000)

$$\mathcal{H} = \overline{\bigoplus_{n=0}^{\infty} \mathcal{H}_n} \quad , \quad \mathcal{C}_n = \mathcal{C} \cap \mathcal{H}_n$$

$$S = \sum_{n=0}^{\infty} S_{n,n} \quad ; \quad A = \sum_{\substack{m,n \geq 0 \\ |m-n| \leq r}} A_{m,n}$$

$$S_{n,n} : \mathcal{H}_n \to \mathcal{H}_n \quad , \quad S_{n,n} = S_{n,n}^* \geq 0$$

$$A_{\ n} : \mathcal{H}_n \to \mathcal{H}_m \ ; \quad A_{nm}^* = -A_{m,n}$$

Remark: extension ...

$\left(\exists\, K<\infty,\ \beta<1,\ C<\infty\right):\left(\forall \varphi\in \mathcal{C}_n,\ \psi\in\mathcal{C}_m\right):$ 20

$$\left\|(\psi, A_{m,n}\varphi)\right\|^2 \leq C^2\left(\delta_{m,n} n^{2K} + (1-\delta_{m,n}) n^{2\beta}\right)(\psi, S_{mm}\psi)(\varphi, S_{n,n}\varphi)$$

or, equivalently

$$\left\| S_{mm}^{-1/2} A_{mn} S_{nn}^{-1/2}\right\| \leq C\left(\delta_{m,n} n^{K} + (1-\delta_{mn}) n^{\beta}\right)$$

<u>Theorem</u> (Sethuraman, Varadhan, Yau, 2000)

$$\text{GSC} \ \&\ f\in\mathcal{H}_{-1} \Longrightarrow \text{KV}$$

[also for $\beta=1$, C small]

Applied to:

- tagged particle diffusion in ASEP, $d \geq 3$
  (S. Sethuraman, SRS Varadhan, H-T Yau, 2000)

- diffusion in divergence-free Gaussian drift field $d \geq 3$
  (T. Komorowski, S. Olla, 2003)

- self-repelling Brownian polymer and myopic self avoiding random walk in $d \geq 3$
  (I. Horváth, B. Tóth, B. Vető, 2012)

Relaxed Sector Condition

(I. Horváth, B. Tóth, B. Vető, 2012)

$$B_\lambda : (\lambda I + S)^{1/2}\mathcal{C} \to \mathcal{H}$$

$$B_\lambda := (\lambda I + S)^{-1/2} A \,(\lambda I + S)^{-1/2}$$

essentially skew self adjoint

$$K_\lambda := (I + B_\lambda)^{-1}, \qquad \|K_\lambda\| \leq 1.$$

$$R_\lambda = (\lambda I + S)^{-1/2} K_\lambda \,(\lambda I + S)^{-1/2}$$

If by some miracle

$$K_\lambda \xrightarrow{st} K \qquad \text{[in the strong operator topology]}$$

then, for $f \in \mathcal{H}_{-1}$, $f = S^{1/2} g$, $g \in \mathcal{H}$

$$\lambda^{1/2} u_\lambda = \lambda^{1/2} (\lambda I + S)^{-1/2} K_\lambda (\lambda I + S)^{-1/2} S^{1/2} g \longrightarrow 0 \qquad \text{A}$$

$$S^{1/2} u_\lambda = S^{1/2} (\lambda I + S)^{-1/2} K_\lambda (\lambda I + S)^{-1/2} S^{1/2} g \longrightarrow K g \qquad \text{B}$$

$\exists\ \tilde{\mathcal{C}} \subseteq \bigcap_{\lambda>0} \mathrm{Dom}(B_\lambda)$, dense in $\mathcal{H}$

$B : \tilde{\mathcal{C}} \to \mathcal{H}$, essentially skew self-adjoint.

$\forall \varphi \in \tilde{\mathcal{C}}$, $\lim_{\lambda \to 0} \| B_\lambda \varphi - B \| = 0$.

or, equivalently: $S^{-1/2} A S^{-1/2}$ is skew self-adjoint (not just skew-Hermitian)

<u>Theorem RSC</u> (Horváth, Tóth, Vető, 2012)

$$\text{RSC} \ \& \ f \in \mathcal{H}_{-1} \Longrightarrow \text{KV}$$

Applied to:

RW / diffusion in divergence-free drift field

Remarks

One can naturally define $B = S^{-1/2} A S^{-1/2} : \tilde{\mathcal{E}} \to \mathcal{H}$ as skew-Hermitian. The point is to prove

$$\overline{\mathrm{Ran}(I \pm B)} = \mathcal{H}$$ – von Neumann's condition

$$\mathrm{SSC} \Rightarrow \mathrm{GSC} \Rightarrow \mathrm{RSC}$$

see next Thm

Theorem Setup of GSC.

If there exist positive, nondecreasing sequences

$$d_n < \infty \, , \quad \sum_{n=1}^{\infty} c_n^{-1} = \infty$$

and

$$\left\| S_{mm}^{-1/2} A_{m,n} S_{nn}^{-1/2} \right\| \leq \delta_{m,n} d_n + (1-\delta_{mn}) c_n$$

then RSC holds with $B = S^{-1/2} A S^{-1/2}$ defined on $\tilde{\mathcal{C}} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$ (no closure!)

Proof: w. l. o. g. $r = 1$.

Notation:

$$\varphi = (\varphi_1, \varphi_2, \dots) \,;\; \varphi^n = (\varphi_1, \varphi_2, \dots, \varphi_n, 0, 0, \dots) \,,\quad \varphi_k \in \mathcal{H}_k$$

Clearly $B := S^{-1/2} A S^{-1/2} : \bigoplus_{n=0}^{\infty} \mathcal{H}_n \to \mathcal{H}$ is skew-Hermitian.

needed: $\overline{\mathrm{Ran}\,(1 \pm B)} = \mathcal{H}$

Assume: $\varphi \perp \mathrm{Ran}\,(1-B)$. Then

$$0 = (\varphi, (1-B)\varphi^n) = \|\varphi^n\|^2 - (\varphi_{n+1}, B_{n+1,n}\,\varphi_n)$$

By Schwarz:

$$\|\varphi_{n+1}\|^2 + \|\varphi_n\|^2 \geq \frac{2}{c_n} \left|(\varphi_{n+1}, B_{n+1,n}\,\varphi_n)\right| = \frac{2}{c_n} \|\varphi^n\|^2 \geq \frac{1}{c_n} \|\varphi\|^2$$

for n suff. large

sum over n : $\|\varphi\|^2 \geq \infty \cdot \|\varphi\|^2$ $\square$

Proof of Thm RSC:

Lemma B_n, $n \geq 1$, and $B = B_\infty$ densely defined closed operators over $\mathcal{H}$. Assume:

(i) $\mu \in \bigcap_{1 \leq n \leq \infty} \operatorname{Res}(B_n) \subseteq \mathbb{C}$, and $\sup_{1 \leq n \leq \infty} \|(\mu I - B)^{-1}\| < \infty$.

(ii) $\tilde{\mathcal{E}} \subseteq \mathcal{H}$, dense subspace.
$\tilde{\mathcal{E}} \subseteq \bigcap_{1 \leq n < \infty} \operatorname{Dom}(B_n)$ and $\tilde{\mathcal{E}}$ is a core for B_∞
such that $(\forall \tilde{h} \in \tilde{\mathcal{E}})$: $\lim_{n \to \infty} \|B_n \tilde{h} - B \tilde{h}\| = 0$

Then

$$(\mu I - B_n)^{-1} \xrightarrow{st} (\mu I - B)^{-1}$$

Proof of the Lemma:

$\hat{\mathcal{C}} := \{ \hat{h} = (\mu I - B)\tilde{h},\ \tilde{h} \in \tilde{\mathcal{C}} \}$ is dense in $\mathcal{H}$.

for $\hat{h} \in \hat{\mathcal{C}}$:

$$\{ (\mu I - B_n)^{-1} - (\mu I - B)^{-1} \} \hat{h} = (\mu I - B_n)^{-1} (B_n \tilde{h} - B \tilde{h}) \longrightarrow 0$$

... $\square$

Proof of the Theorem:

apply the Lemma with B_λ, B, $\mu = -1$ ...

get $K_\lambda \xrightarrow{st} K$ as $\lambda \to 0$.

$\square$