Lecture 1: Introduction, setup, notation, the problem, motivating examples.

Setup

Ω : complete separable metric sp.

$k \mapsto \eta_k$ Markov chain on Ω

or

$t \mapsto \eta_t$ Markov process on Ω, a.s. cadlag, Feller

π : probability measure on Ω stationary & ergodic for η_k / η_t

$$\mathcal{H} := \mathcal{L}^2(\Omega, \pi)$$

Transition operator

discrete time: $P : \mathcal{H} \to \mathcal{H}$, $(Pf)(\omega) := \mathbb{E}_\omega(f(\eta_1))$

positivity preserving, contraction

ergodicity: $Pf = f \Leftrightarrow f = c \cdot \mathbb{1}$

Chapman-Kolmogorov
Markov property: $\mathbb{E}_\omega(f(\eta_k)) = P^k f(\omega)$

Continuous time : $P_t : \mathcal{H} \to \mathcal{H}$, $P_t f(\omega) := \mathbb{E}_\omega(f(\eta_t))$ 3

positivity preserving, contraction

Ch-K, Markov : $P_{t+s} = P_t P_s$ semigroup

$\boxed{\text{assumed}}$ Feller property : $\forall f \in \mathcal{H}$ $t \mapsto P_t f$ is continuous

$t \mapsto P_t$ strongly continuous contraction semigroup

Infinitesimal generator

$$Gf := \lim_{h \downarrow 0} h^{-1} (P_h f - f) \,, \quad P_t f = e^{tG} f$$

densely defined in $\mathcal{H}$... see Hille-Yosida...

$$Gf(\omega) = \lim_{h \downarrow 0} h^{-1} \left(\mathbb{E}_\omega (f(\eta_h)) - f(\omega) \right)$$

discrete time MC embedded in continuous time MP:

$$P_t := e^{-\alpha t} \sum_{k=0}^{\infty} \frac{(\alpha t)^k}{k!} P^k = e^{t\alpha(P-I)} \,, \quad G = P - I$$

vice versa iff $\|G\| < \infty$, or η_t pure jump process with bdd total jump rate.

The problem:

5

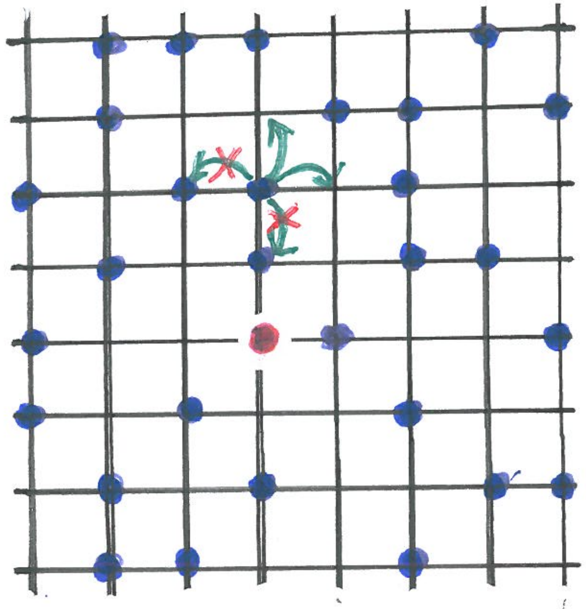
Let $f \in \mathcal{H}$, $\int f \, d\pi = 0$ (fixed)

Prove Central Limit Theorem / diffusive limit for

$$N^{-1/2} \sum_{k=0}^{N-1} f(\eta_k) \qquad T^{-1/2} \int_0^T f(\eta_s)\, ds$$

Motivating examples:

Ex1 Tagged particle motion in interacting particle system: the Simple Exclusion Process



* 0 or 1 particle / lattice site

* independently of each-other particles attempt r.w. jumps with distribution $(p_z)_{z \in \mathbb{Z}^d}$

assume finite range

* if target site is empty: jumps performed
* if target site is occupied: jump suppressed
* follow trajectory of $\boxed{\text{tagged}}$ particle: $t \longmapsto X(t)$

System observed from the position of the tagged part.: $t \mapsto \eta_t \in \Omega = \{0,1\}^{\mathbb{Z}^d_*}$, $\mathbb{Z}^d_* = \mathbb{Z}^d \setminus \{0\}$ 7

$$Gf(\omega) = \sum_{x,y \in \mathbb{Z}^d_*} p_{y-x}\, \omega_x (1-\omega_y)\left(f(\omega^{xy}) - f(\omega)\right)$$

$$+ \sum_{y \in \mathbb{Z}^d_*} p_y (1-\omega_y) \left(f(\omega^y) - f(\omega)\right)$$

$$(\omega^{xy})_z = \begin{cases} \omega_y & z = x \\ \omega_x & z = y \\ \omega_z & z \neq x,y \end{cases} \qquad (\omega^y)_z = \begin{cases} 0 & z = -y \\ \omega_{z+y} & z \neq -y \end{cases}$$

$f : \Omega \to \mathbb{R}$, finite cylinder function.

$\pi_\rho = \text{Bernoulli}(\rho)$, $\rho \in (0,1)$, stationary: ⑧

$\forall f$ cylinder fnct. : $\int_\Omega Gf(\omega)\, d\pi_\rho(\omega) = 0$ ✓

ergodic: $f \in \mathcal{L}^2(\Omega, \pi_\rho)$; $Gf = 0 \Leftrightarrow f = c\mathbb{1}$ ✓

Let $\varphi : \Omega \to \mathbb{R}^d$, $\varphi(\omega) := \sum_{y \in \mathbb{Z}^d_*} (\rho - \omega_y)\, y \cdot p_y$

Then: $Y(t) := \tilde{X}(t) - \int_0^t \varphi(\eta_s)\, ds$

is an $\mathcal{L}^2$ martingale with stationary & ergodic increments.

$$\tilde{X}(t) = Y(t) + \int_0^t \varphi(\eta_s)\,ds$$

Wanted: CLT for $X(t)$.

Route: CLT for $Y(t)$ by Martingale CLT ✓

CLT for $\int_0^t \varphi(\eta_s)\,ds$: our problem.

---

* symmetric: $p_{-y} = p_y$, $t \mapsto \eta_t$ reversible, $G = G^*$
* zero mean, asymmetric: $p_{-y} \neq p_y$, but $\sum_y y p_y = 0$
* asymmetric with drift: $\sum_y y p_y \neq 0$

E.2 Random walk in bistochastic random environment: (10)

$(\Omega, \pi, \tau_z : z \in \mathbb{Z}^d)$ probability space with ergodic $\mathbb{Z}^d$-action: $\tau_z : \Omega \to \Omega$.

$p_e : \Omega \to [0,1]$ $\qquad \{e \in \mathbb{Z}^d : |e| = 1\} =: \mathcal{E}$

$$\sum_{e \in \mathcal{E}} p_{-e}(\tau_e \omega) = \sum_{e \in \mathcal{E}} p_e(\omega) \leq 1$$

$t \mapsto X(t)$ continuous time RW on $\mathbb{Z}^d$: given $\omega \in \Omega$

$$\mathbb{P}_\omega\big(X(t+dt) = x+e \,\big|\, X(t) = x\big) = p_e(\tau_x \omega)\, dt + o(dt)$$

$\eta_t := \tau_{X(t)}\omega \in \Omega$ Markov process (pure jump, with bdd rates)

$f : \Omega \to \mathbb{R}$ bdd, measurable

$$Gf(\omega) = \sum_{e \in \mathcal{E}} p_e(\omega)\left(f(\tau_e\omega) - f(\omega)\right)$$

π stationary measure – due to bistochasticity.

$$\int Gf(\omega)\, d\pi(\omega) = \dots = \int \Big(\sum_e p_e(\tau_{-e}\omega) - \sum_e p_e(\omega)\Big) f(\omega)\, d\pi(\omega) = 0$$

also ergodic

$\varphi : \Omega \to \mathbb{R}^d, \quad \varphi(\omega) := \sum_{e \in \mathcal{E}} e\, p_e(\omega)$

$$Y(t) := X(t) - \int_0^t \varphi(\eta_s)\, ds$$

is $\mathcal{L}^2$- martingale with stationary & ergodic increments

$$X(t) = Y(t) + \int_0^t \varphi(\eta_s)\, ds$$

martingale CLT

our problem

$$c_e(\omega) := \frac{1}{2}\Big(p_e(\omega) + p_{-e}(\tau_e \omega)\Big) = c_{-e}(\tau_e \omega)$$

$$v_e(\omega) := \frac{1}{2}\Big(p_e(\omega) - p_{-e}(\tau_e \omega)\Big) = -v_{-e}(\tau_e \omega)$$

$$Gf(\omega) = \sum_{e \in \mathcal{E}} c_e(\omega)\Big(f(\tau_e \omega) - f(\omega)\Big) + \sum_{e \in \mathcal{E}} v_e(\omega)\Big(f(\tau_e \omega) - f(\omega)\Big)$$

$$= -Sf(\omega) + Af(\omega) \quad , \quad -S = \frac{G + G^*}{2}, \quad A = \frac{G - G^*}{2}$$

ergodicity:

$$-(f, Gf) = \cdots = \frac{1}{2} \sum_{e \in \mathcal{E}} \int c_e(\omega)\Big(f(\tau_e \omega) - f(\omega)\Big)^2 d\pi(\omega) = 0$$

iff $f = c \cdot \mathbb{1}$.

Two extremes:

① $v_e(\omega) \equiv 0$, $\forall e \in \mathcal{E}$; $G = G^*$

r.w. among random conductancies.

② $c_e(\omega) \equiv \frac{1}{2d}$ (constant)

r.w. in div-free random drift field.

diffusion analogue: $V : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ stationary divergence-free vector field

$$dX(t) = dB(t) + V(X(t))\,dt$$

drifting in incompressible turbulent flow

Ex3 Self-repelling Brownian polymer

$V : \mathbb{R}^d \longrightarrow \mathbb{R}$, C^∞, fast decay

$$\hat{V}(p) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ip\cdot x} V(x)\,dx \geq 0$$ "positive type"

e.g. $V(x) = \exp(-|x|^2)$

$$dX(t) = dB(t) - \operatorname{grad}(V * \ell_t)(X(t))\,dt$$

where $\ell_t(A) = \ell_0(A) + |\{0 < s \leq t : X(s) \in A\}|$

$X(\cdot)$ pushed by negative gradient of its own occupation time measure.

$$\eta_t(x) := -\operatorname{grad}(V * l_t)(X(t)+x)$$

$t \longmapsto \eta_t$ is Markov process with a.s. continuous sample path in

$$\Omega = \{\omega \in C^\infty(R^d \to R^d) : \omega \text{ gradient field}, \|\omega\|_{k,m,r} < \infty\}$$

$$\|\omega\|_{k,m,r} := \sup_{x \in R^d} (1+|x|)^{-1/r} \left| \partial^{|m|}_{m_1,\dots,m_d} \omega_k(x) \right|$$

partial derivatives of ω increase slower than any power of $|x|$, as $|x| \to \infty$.

infinitesimal generator ... later

stationary & ergodic measure π on Ω:

Gaussian

$$\langle \omega_k(x)\,\omega_\ell(y)\rangle = -\partial^2_{k\ell}\, V * \Delta^{-1}(x-y) =: K_{k\ell}(x-y)$$

$$\hat{K}_{k\ell}(p) = \frac{p_k p_\ell}{|p|^2}\,\hat{V}(p)$$

$$X(t) = B(t) + \int_0^t \varphi(\eta_s)\,ds$$

$$\varphi : \Omega \to \mathbb{R}^d, \quad \varphi(\omega) := \omega(0)$$

analysis in Gaussian Hilbert space.

Warming up : $|\Omega| < \infty$

(classroom exercise)

fix $x_0 \in \Omega$ "the origin"

$R_0 := 0$, $R_n := \min\{k > R_{n-1} : \eta_k = x_0\}$

$$\left.\begin{array}{l} T_n := R_{n+1} - R_n \\ X_n := \displaystyle\sum_{k=R_n}^{R_{n+1}-1} f(\eta_k) \end{array}\right\} \left(T_n, X_n\right)_{n \geq 1} \quad \text{i.i.d.}$$

$\nu_n := \min\{m : R_m \geq n\}$

$$\sum_{k=0}^{n-1} f(\eta_k) =$$

$$\sum_{k=0}^{R_1-1} f(\eta_k) + \sum_{k=1}^{\nu_n} X_k - \sum_{k=n}^{R_{\nu_n}-1} f(\eta_k)$$

$$\mathbb{E}(T_k) = b, \quad \mathbb{E}(X_k) = 0, \quad \mathbb{E}(X_k^2) = \tilde{\sigma}^2$$

$$n^{-1/2} \sum_{k=0}^{R_1-1} f(\eta_k) \xrightarrow{\mathbb{P}} 0 \; ; \quad n^{-1/2} \sum_{k=1}^{\nu_n} X_k \Longrightarrow \mathcal{N}\left(0, \frac{\tilde{\sigma}^2}{b}\right)$$

$$n^{-1/2} \sum_{k=n}^{R_{\nu_n}-1} f(\eta_k) \xrightarrow{\mathbb{P}} 0$$

compute the variance : later

This proof can be saved in positive recurrent cases with $\mathbb{E}(\exp(\lambda T)) < \infty$. 20

Alternative approach: martingale approx.

$$f = (I-P)g \,; \quad \text{i.e.} \quad g := (I-P)^{-1} f.$$

$$\sum_{k=0}^{n-1} f(\eta_k) = \sum_{k=0}^{n-1} g(\eta_k) - \sum_{k=0}^{n-1} Pg(\eta_k)$$

$$= g(\eta_0) - g(\eta_n) + \underbrace{\sum_{k=1}^{n} \big(g(\eta_k) - Pg(\eta_{k-1}) \big)}_{=: M_n}$$

$n \mapsto M_n$ martingale with stationary and ergodic increments, $\mathcal{L}^2$.

$$\sigma^2 := \mathbb{E}\left(\left(g(\eta_k) - Pg(\eta_{k-1})\right)^2\right) = \mathbb{E}(g(\eta_k)^2) - \mathbb{E}(Pg(\eta_k))^2$$

$$= \|g\|^2 - \|Pg\|^2 = 2(f,g) - \|f\|^2$$

Theorem:

$$n^{-1/2} \sum_{k=0}^{n-1} f(\eta_k) \Longrightarrow \mathcal{N}(0, \sigma^2)$$

Proof: use the Martingale CLT.