

LIMIT THEOREMS FOR WEAKLY REINFORCED RANDOM WALKS ON \mathbb{Z}

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ABSTRACT. The weakly reinforced random walk (WRRW) on the one-dimensional integer lattice \mathbb{Z} starts from the origin of the lattice and at each step it jumps to a neighbouring site, the probability of jumping along a bond being proportional to w (number of previous jumps along that lattice bond), where $w : \mathbb{N} \rightarrow \mathbb{R}_+$, with $w(n) \sim n^\alpha$ for large n , and $\alpha \in (0, 1)$ is a fixed parameter. We prove that the properly scaled local time process of WRRW converges *in probability* to a deterministic function. Using this result we also prove a limit theorem for the position of the random walker at late times.

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Dedicated to Endre Csáki on his 60-th birthday

1. Introduction

We continue to investigate the long time asymptotic behaviour of self-interacting random walks on the one-dimensional integer lattice \mathbb{Z} . The walk X_i , $i = 0, 1, 2, \dots$ starts from the origin of the lattice and at time $i + 1$ it jumps to one of the two neighbouring sites of X_i , so that the probability of jumping along a bond of the lattice is proportional to

$$w(\text{number of previous jumps along that bond})$$

where

$$w : \mathbb{N} \rightarrow \mathbb{R}_+$$

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is a weight function to be specified later. Formally, for a nearest neighbour walk

$\underline{x}_0^i = (x_0, x_1, \dots, x_i)$ we define

(1.1)

$$r(\underline{x}_0^i) = \#\{0 \leq j < i : (x_j, x_{j+1}) = (x_i, x_i + 1) \text{ or } (x_i + 1, x_i)\}$$

(1.2)

$$l(\underline{x}_0^i) = \#\{0 \leq j < i : (x_j, x_{j+1}) = (x_i, x_i - 1) \text{ or } (x_i - 1, x_i)\}.$$

That is: the number $r(\underline{x}_0^i)$ (respectively, $l(\underline{x}_0^i)$) shows how many times had the walk \underline{x}_0^i visited the edge adjacent from the right (respectively, from the left) to the terminal site x_i . The random walk X_i is governed by the law:

$$\begin{aligned} \mathbf{P}\left(X_{i+1} = X_i + 1 \mid \underline{X}_0^i = \underline{x}_0^i\right) &= \frac{w(r(\underline{x}_0^i))}{w(r(\underline{x}_0^i)) + w(l(\underline{x}_0^i))} \\ (1.3) \qquad \qquad \qquad &= 1 - \mathbf{P}\left(X_{i+1} = X_i - 1 \mid \underline{X}_0^i = \underline{x}_0^i\right). \end{aligned}$$

The long time asymptotic behaviour of the random walk X_i depends strongly on the choice of the weight function $w(\cdot)$. In three previous papers we analysed the following cases:

(1) The so-called ‘*true self avoiding walk*’, with $w(n) = \exp(-g \cdot n)$, $g > 0$, was studied in TÓTH (1995). There we showed that for long times X_n scales as $n^{2/3}$ and we proved a limit theorem for $A^{-2/3} X_{\theta_{s/A}}$, as $A \rightarrow \infty$, where $\theta_{s/A}$ is a geometrically distributed random variable with distribution $\mathbf{P}(\theta_{s/A} = n) = (1 - \exp\{-s/A\}) \exp\{-ns/A\}$, independent of the random walk X_n .

(2) The *generalized ‘true self-avoiding walk’*, a generalization of the previous model, with subexponential self-repulsion $w(n) = \exp(-g \cdot n^\kappa)$, $g > 0$, $\kappa \in (0, 1)$ was investigated in TÓTH (1994). In this case we found that X_n scales as $n^{(\kappa+1)/(\kappa+2)}$ and we proved a limit theorem for $A^{-(\kappa+1)/(\kappa+2)} X_{\theta_{s/A}}$, as $A \rightarrow \infty$.

(3) Finally, in TÓTH (1996) weight functions with power-law asymptotics were considered: the so-called *polynomially self-repelling walks*, with $w(n) \sim n^{-\alpha}$, $\alpha > 0$, respectively, the *asymptotically free walks*, with $w(n) = 1 + \mathcal{O}(n^{-1})$ asymptotically, for $n \gg 1$. In these cases the correct scaling of X_n was $n^{-1/2}$ (as for ordinary random walks) but the scaling limit was *not* gaussian. A particular case of

asymptotically free walks, the *once reinforced random walk* or *random walk partially reflected/attracted at its extrema* has also been considered in DAVIS (1995).

In the present paper we consider self-interacting random walks with *polynomial self-attraction*:

$$(1.4) \quad w(n) = (1 - \alpha)^{-1} \left(\frac{n}{2}\right)^\alpha - B(1 - \alpha)^{-2} \left(\frac{n}{2}\right)^{\alpha-1} + \mathcal{O}(n^{\alpha-2}),$$

or, equivalently

$$(1.5) \quad w(n)^{-1} = (1 - \alpha) \left(\frac{n}{2}\right)^{-\alpha} + B \left(\frac{n}{2}\right)^{-1-\alpha} + \mathcal{O}(n^{-2-\alpha}),$$

where $\alpha \in (0, 1)$ and $B \in \mathbb{R}$ are fixed constant parameters. Since in the definition (1.3) of jump probabilities only ratios of w -s play any role, the constant factor in front of the leading term is chosen for convenience only. Note, that the next-to-leading term is assumed asymptotically ‘smooth’.

We call these walks *weakly reinforced random walks* (WRRW) since the self-attraction of trajectories is slightly weaker than in the linearly reinforced case (with $w(n) = 1 + Bn$, $B > 0$). According to DAVIS (1990) self-interacting random walks on \mathbb{Z} are recurrent if and only if $\sum_{n=0}^{\infty} w(n)^{-1} = \infty$, otherwise the random walker eventually sticks to one (randomly selected) edge of the lattice, jumping back and forth on it indefinitely. PEMANTLE (1988) proved that the linearly reinforced random walk has an asymptotic distribution on \mathbb{Z} without any scaling. These two remarks explain why we confine our investigations to $\alpha \in (0, 1)$ in (1.4), (1.5).

The paper is organized as follows: In Section 2 we formulate our main results: Theorem 1 describes the asymptotics of the local process of WRRW, Theorem 2 is a limit theorem for the position of the WRRW at late times. In Section 3 we give a representation of the local time process of our random walks in terms of generalized Pólya Urn Schemes. Section 4 is devoted to the proof of Theorem 1. As the proof of Theorem 2 is identical to a similar proof in TÓTH (1996), we don’t repeat those details here.

In order to keep the paper self-contained we had to include parts of our previous paper TÓTH (1996). These overlapping parts are typed with `petite` and thus, they are clearly distinguishable from the genuinely new parts.

2. Results

The present section is divided in two subsections: in subsection 2.1. we formulate the limit theorems referring to the local time processes and hitting times of the WRRW. In subsection 2.2. we formulate the limit theorems for the position of the SIRW at late times.

2.1. The Local Time Process and Hitting Times

We define the following (bond) local time process:

$$(2.1.1) \quad L(l, i) = \#\{0 \leq j < i : X_j = l, X_{j+1} = l - 1\}, \quad l \in \mathbb{Z}, i \in \mathbb{N}$$

and stopping times

$$(2.1.2) \quad T_{k,-1}^> = 0, \quad T_{k,m}^> = \inf\{i > T_{k,m-1}^> : X_{i-1} = k - 1, X_i = k\} \quad k > 0, m \geq 0.$$

$$(2.1.3) \quad T_{k,0}^< = 0, \quad T_{k,m}^< = \inf\{i > T_{k,m-1}^< : X_{i-1} = k + 1, X_i = k\} \quad k \geq 0, m \geq 1.$$

In plain words: $L(l, i)$ is the number of leftwards jumps on the bond $l \rightarrow l - 1$ performed by the random walk up to time i . $T_{k,m}^>$ is the time of the $m + 1$ -th arrival to the lattice site k coming from left, $T_{k,m}^<$ is the time of the m -th arrival to the lattice site k coming from right.

In formula (2.1.4) below and thereafter the superscript $*$ stands for either $<$ or $>$. We consider the following shifted (bond) local time processes of the walk stopped at $T_{k,m}^*$:

$$(2.1.4) \quad S_{k,m}^*(l) = L(k - l, T_{k,m}^*)$$

$S_{k,m}^*(l)$ is roughly half of the total number of jumps across the bond $\{k - l - 1, k - l\}$:

$$(2.1.5) \quad \#\{0 \leq j < T_{k,m}^* : \{X_j, X_{j+1}\} = \{k - l - 1, k - l\}\} = 2S_{k,m}^*(l) + \mathbb{1}_{[0,k)}(l)$$

Denote

$$(2.1.6) \quad \omega_{k,m}^{*-} = \omega^- \left(S_{k,m}^* \right) = \inf \{ l \leq 0 : S_{k,m}^*(l) > 0 \}$$

$$(2.1.7) \quad \omega_{k,m}^{*+} = \omega^+ \left(S_{k,m}^* \right) = \sup \{ l \geq k : S_{k,m}^*(l) > 0 \}.$$

In plain words: $k - \omega_{k,m}^{*+}$, respectively $k - \omega_{k,m}^{*-} - 1$, is the leftmost, respectively rightmost, site visited by the stopped walk $\underline{X}_0^{T_{k,m}^*}$.

From (2.1.5) it clearly follows that

$$(2.1.8) \quad T_{k,m}^* = 2 \sum_{l=\omega_{k,m}^{*-}}^{\omega_{k,m}^{*+}} S_{k,m}^*(l) + k = 2 \sum_{l=-\infty}^{\infty} S_{k,m}^*(l) + k.$$

Looking at the formal definitions only, in principle, these local times or hitting times might be infinite, i.e. it could happen that the site $k \in \mathbb{Z}$ is never hit. From the results of DAVIS (1990) it follows that in case of WRRW-s considered in the present paper, with probability one, this does *not* happen: all the random variables defined above are finite almost surely.

The following theorem and its corollary describes the precise asymptotics of the *local time processes* $S_{k,m}^*(\cdot)$ and *hitting times* $T_{k,m}^*$ of WRRW:

Theorem 1. *The limit*

$$(2.1.9) \quad D = \sum_{j=0}^{\infty} \left(\frac{1}{w(2j)} - \frac{1}{w(2j+1)} \right)$$

exists and $D \in (0, \infty)$.

Let $x \in [0, \infty)$, $h \geq 0$ *and* $*$ *=* $<$ *or* $>$ *be fixed.*

$$(2.1.10) \quad \frac{\omega_{[Ax],[A^{1/(1-\alpha)h}] }^{*-}}{A} \xrightarrow{\mathbf{P}} -D^{-1}h^{1-\alpha}$$

$$(2.1.11) \quad \frac{\omega_{[Ax],[A^{1/(1-\alpha)h}] }^{*+}}{A} \xrightarrow{\mathbf{P}} 2x + D^{-1}h^{1-\alpha}$$

$$(2.1.12) \quad \sup_y \left| \frac{S_{[Ax],[A^{1/(1-\alpha)h}]}^*([Ay])}{A^{1/(1-\alpha)}} - \{h^{1-\alpha} + D(x - |y-x|)\}_+^{1/(1-\alpha)} \right| \xrightarrow{\mathbf{P}} 0$$

Remarks: Note that the non-trivial scaling of the local time process provides *convergence in probability to a deterministic function rather than convergence in distribution to a genuinely stochastic process.*

From the previous theorem and (2.1.8) follows immediately:

Corollary 1. *Let x , h and $*$ and δ be as in Theorem 1.*

$$(2.1.13) \quad \frac{T_{[Ax],[A^{1/(1-\alpha)}h]}^*}{2A^{(2-\alpha)/(1-\alpha)}} \xrightarrow{\mathbf{P}} \frac{2-2\alpha}{2-\alpha} D \left(x + (D^{-1}h)^{1-\alpha} \right)^{(2-\alpha)/(1-\alpha)}.$$

as $A \rightarrow \infty$.

2.2. Limit Theorem for the Position at Late Times

The second result concerns the limiting distribution of the WRRW X_n for late times. We denote by $P(n, k)$, $n \in \mathbb{N}$, $k \in \mathbb{Z}$ the distribution of our WRRW at time n :

$$(2.2.1) \quad P(n, k) = \mathbf{P}(X_n = k)$$

and by $R(s, k)$, $s \in \mathbb{R}_+$, $k \in \mathbb{Z}$ the distribution of the walk observed at an independent random time θ_s , of geometric distribution

$$(2.2.2) \quad \mathbf{P}(\theta_s = n) = (1 - e^{-s}) e^{-sn},$$

$$(2.2.3) \quad R(s, k) = \mathbf{P}(X_{\theta_s} = k) = (1 - e^{-s}) \sum_{n=0}^{\infty} e^{-sn} P(n, k).$$

We define the following rescaled ‘densities’ of the above distributions

$$(2.2.4) \quad \pi_A(t, x) = A^{(1-\alpha)/(2-\alpha)} P([At], [A^{(1-\alpha)/(2-\alpha)}x])$$

$$(2.2.5) \quad \hat{\pi}_A(s, x) = A^{(1-\alpha)/(2-\alpha)} R(A^{-1}s, [A^{(1-\alpha)/(2-\alpha)}x])$$

$t, s \in \mathbb{R}_+$, $x \in \mathbb{R}$.

Theorem 2. *For any $s \in \mathbb{R}_+$ and $x \in \mathbb{R}$*

$$(2.2.6) \quad \hat{\pi}_A(s, x) \rightarrow \hat{p}_D^{(\alpha)}(s, x)$$

as $A \rightarrow \infty$, where

$$(2.2.7) \quad \hat{p}_D^{(\alpha)}(s, x) = s \int_0^{\infty} e^{-st} p_D^{(\alpha)}(t, x) dt$$

and

(2.2.8)

$$p_D^{(\alpha)}(t, x) = \frac{1}{2-2\alpha} \left(\frac{2-2\alpha}{2-\alpha} \cdot \frac{D}{t} \right)^{1/(2-\alpha)} \left\{ \left(\frac{2-\alpha}{2-2\alpha} \cdot \frac{t}{D} \right)^{(1-\alpha)/(2-\alpha)} - |x| \right\}_+^{\alpha/(1-\alpha)}.$$

This is of course a *local limit theorem* for the WRRW, observed at an independent random time $\theta_{s/A}$ of geometric distribution with mean $e^{-s/A} (1 - e^{-s/A})^{-1} \sim A/s$.

In particular the (integral) limit law follows:

$$(2.2.9) \quad \mathbf{P} \left(A^{-(1-\alpha)/(2-\alpha)} X_{\theta_{s/A}} < x \right) \rightarrow \int_{-\infty}^x \widehat{p}_D^{(\alpha)}(s, y) dy.$$

This is a little bit short of stating the limit theorem for *deterministic time*:

$$(2.2.10) \quad \mathbf{P} \left(A^{-(1-\alpha)/(2-\alpha)} X_{[At]} < x \right) \rightarrow \int_{-\infty}^x p_D^{(\alpha)}(t, y) dy$$

But, of course, we can conclude that the sequence $A^{-(1-\alpha)/(2-\alpha)} X_{[At]}$, with $t \in \mathbb{R}_+$ fixed and $A \rightarrow \infty$, is tight and, if it converges in distribution then (2.2.10) also holds.

Remark: On the other hand we have good reason to expect that the sequence of *random processes* $t \mapsto X^{(A)}(t) = A^{-(1-\alpha)/(2-\alpha)} X_{[At]}$ is not tight in the function space $D[0, 1]$ and there is no continuous limit *process*.

Given Corollary 1, the proof of Theorem 2 is formally identical to the proof of Theorem 3 in TÓTH (1996). We omit the repetition of those details here.

3. Representation of the Local Time Process in Terms of Pólya Urns

3.1. Generalized Pólya Urn Schemes

Given two weight functions

$$(3.1.1) \quad r : \mathbb{N} \rightarrow \mathbb{R}_+$$

$$(3.1.2) \quad b : \mathbb{N} \rightarrow \mathbb{R}_+,$$

a generalized Pólya Urn Scheme is a Markov chain (ρ_i, β_i) on $\mathbb{N} \times \mathbb{N}$ with transition probabilities

$$(3.1.3) \quad \mathbf{P} \left((\rho_{i+1}, \beta_{i+1}) = (k+1, l) \mid (\rho_i, \beta_i) = (k, l) \right) = \frac{r(k)}{r(k) + b(l)}$$

$$(3.1.4) \quad \mathbf{P} \left((\rho_{i+1}, \beta_{i+1}) = (k, l+1) \mid (\rho_i, \beta_i) = (k, l) \right) = \frac{b(l)}{r(k) + b(l)},$$

and no other transitions allowed. Usually the initial values $(\rho_0, \beta_0) = (0, 0)$ are assumed and β_i and ρ_i are interpreted as the number of blue, respectively red marbles drawn from the urn up to time i . Denote by τ_m the time when the m -th red marble is drawn and by $\mu(m)$ the number of blue marbles drawn before the m -th red one:

$$(3.1.5) \quad \tau_m = \min \{ i \mid \rho_i = m \}$$

$$(3.1.6) \quad \mu(m) = \beta_{\tau_m}.$$

The functions defined below are essential in the study of the Pólya Urn Scheme defined above

$$(3.1.7) \quad R_p(n) = \sum_{j=0}^{n-1} (r(j))^{-p}, \quad p \in \mathbb{N}$$

$$(3.1.8) \quad B_p(n) = \sum_{j=0}^{n-1} (b(j))^{-p}, \quad p \in \mathbb{N}.$$

We shall be particularly interested in $p = 1, 2$.

Lemma 1. *For any $m \in \mathbb{N}$ and $\lambda < \min\{r(j) : 0 \leq j \leq m-1\}$ the following identity holds:*

$$(3.1.9) \quad \mathbf{E} \left(\prod_{j=0}^{\mu(m)-1} \left(1 + \frac{\lambda}{b(j)} \right) \right) = \prod_{j=0}^{m-1} \left(1 - \frac{\lambda}{r(j)} \right)^{-1}.$$

In particular

$$(3.1.10) \quad \mathbf{E} \left(B_1(\mu(m)) \right) = R_1(m)$$

$$(3.1.11) \quad \mathbf{E} \left(\left[B_1(\mu(m)) - \mathbf{E} B_1(\mu(m)) \right]^2 \right) = R_2(m) + \mathbf{E} \left(B_2(\mu(m)) \right)$$

Proof. The proof of (3.1.9) follows from standard martingale considerations, using the representation of the generalized Pólya Urn scheme in terms of two independent renewal processes with exponentially distributed waiting times (see e.g. the Appendix of DAVIS (1990)). Expanding (3.1.9) to second order in λ yields (3.1.10) and (3.1.11). We leave the standard details of this proof as an exercise for the reader. \square

3.2. The Local Time Process

For sake of definiteness we consider the case of superscript $>$, i.e. we stop the SIRW at the hitting time $T_{k,m}^>$. The case of superscript $<$ is done in a very similar way, with straightforward slight changes.

Let $(\rho_i^{(l)}, \beta_i^{(l)})$, $l \in \mathbb{Z}$ be *independent* Pólya Urn Schemes with weight functions

$$(3.2.1) \quad r^{(l)}(j) = w(2j+1) \quad b^{(l)}(j) = w(2j) \quad \text{for } l \in (-\infty, 0] \cup [k+1, \infty)$$

$$(3.2.2) \quad r^{(l)}(j) = w(2j) \quad b^{(l)}(j) = w(2j+1) \quad \text{for } l \in [1, k-1]$$

$$(3.2.3) \quad r^{(l)}(j) = w(2j) \quad b^{(l)}(j) = w(2j) \quad \text{for } l = k.$$

Denote by $\mu^{(l)}(m)$ the random variables defined in (3.1.6), the superscript l showing to which of the Urn Schemes it belongs.

The extension to *self-interacting walks* of F. Knight's description of the local time process $S_{k,m}^>(l)$, $l \in \mathbb{Z}$ as a *Markov chain* is formally exhaustively explained in TÓTH (1995). According to these arguments $S_{k,m}^>(l)$, $l \in \mathbb{Z}$ is obtained by patching together three homogeneous Markov chains in the following way:

(I.) In the interval $l \in (0, k-2)$, that is steps $0 \rightarrow 1, 1 \rightarrow 2, \dots, (k-2) \rightarrow (k-1)$:

$$(3.2.4) \quad S_{k,m}^>(0) = m, \quad S_{k,m}^>(l+1) = \mu^{(l+1)}(S_{k,m}^>(l) + 1), \quad l = 0, 1, \dots, k-2.$$

(II.) The single step $(k-1) \rightarrow k$ is exceptional

$$(3.2.5) \quad S_{k,m}^>(k-1) = \text{given by (3.2.4)}, \quad S_{k,m}^>(k) = \mu^{(k)}(S_{k,m}^>(k-1) + 1).$$

(III.) In the intervals $l \in (-\infty, 0)$, respectively $l \in (k+1, \infty)$, that is steps $0 \rightarrow -1, -1 \rightarrow -2, -2 \rightarrow -3, \dots$, respectively $k \rightarrow (k+1), (k+1) \rightarrow (k+2), (k+2) \rightarrow (k+3), (k+3) \rightarrow (k+4), \dots$:

$$(3.2.6) \quad S_{k,m}^>(0) = m, \quad S_{k,m}^>(l-1) = \mu^{(l)}(S_{k,m}^>(l)), \quad l = 0, -1, -2, \dots$$

respectively

$$(3.2.7) \quad S_{k,m}^>(k) = \text{given by (3.2.5)}, \quad S_{k,m}^>(l+1) = \mu^{(l+1)}(S_{k,m}^>(l)), \quad l = k, k+1, k+2, \dots$$

Due to (3.2.1) these last two Markov chains have the same transition laws.

4. Proof of Theorem 1.

4.1. Preparations.

As suggested by the representation of the local times given in the previous section, we consider two homogeneous Markov chains $\mathcal{Z}(l)$ and $\tilde{\mathcal{Z}}(l)$, $l = 0, 1, 2, \dots$ on the state space \mathbb{N} , defined as follows:

$$(4.1.1) \quad \mathcal{Z}(l+1) = \mu^{(l+1)}(\mathcal{Z}(l) + 1), \quad \tilde{\mathcal{Z}}(l+1) = \tilde{\mu}^{(l+1)}(\tilde{\mathcal{Z}}(l))$$

where the processes $\{\mu^{(l)}(\cdot)\}_{l \in \mathbb{N}}$, are those defined in (3.1.5)-(3.1.6), belonging to i.i.d. Pólya Urn Schemes $\{(\rho_i^{(l)}, \beta_i^{(l)})\}_{l \in \mathbb{N}}$, with weight functions

$$(4.1.2) \quad r(j) = w(2j) \quad b(j) = w(2j+1)$$

and similarly, the processes $\{\tilde{\mu}^{(l)}(\cdot)\}_{l \in \mathbb{N}}$ belong to i.i.d. Pólya Urn Schemes $\{(\tilde{\rho}_i^{(l)}, \tilde{\beta}_i^{(l)})\}_{l \in \mathbb{N}}$, with weight functions

$$(4.1.3) \quad \tilde{r}(j) = w(2j+1) \quad \tilde{b}(j) = w(2j).$$

We shall also need the hitting time

$$(4.1.4) \quad \tilde{\sigma}_0 = \tilde{\sigma}_0(\tilde{\mathcal{Z}}(\cdot)) = \inf\{l : \tilde{\mathcal{Z}}(l) = 0\}.$$

From (4.1.1) and (3.1.5)-(3.1.6) we see that $\tilde{\sigma}_0$ is actually the extinction time of $\tilde{\mathcal{Z}}(\cdot)$:

$$(4.1.5) \quad \tilde{\mathcal{Z}}(l) \equiv 0 \quad \text{for } l \geq \tilde{\sigma}_0.$$

Lemma 1 suggests the introduction of the following functions

$$(4.1.6) \quad U_p(n) = \sum_{j=0}^{n-1} (w(2j))^{-p}, \quad p = 1, 2.$$

$$(4.1.7) \quad V_p(n) = \sum_{j=0}^{n-1} (w(2j+1))^{-p}, \quad p = 1, 2.$$

Using formulas (3.1.10) and (3.1.11) of Lemma 1 and the functions introduced above we get the following identities:

(4.1.8)

$$\mathbf{E}\left(V_1(\mathcal{Z}(l+1)) \middle| \mathcal{Z}(l) = n\right) = U_1(n+1)$$

(4.1.9)

$$\mathbf{D}^2\left(V_1(\mathcal{Z}(l+1)) \middle| \mathcal{Z}(l) = n\right) = U_2(n+1) + \mathbf{E}\left(V_2(\mathcal{Z}(l+1)) \middle| \mathcal{Z}(l) = n\right)$$

(4.1.10)

$$\mathbf{E}\left(U_1(\tilde{\mathcal{Z}}(l+1)) \middle| \tilde{\mathcal{Z}}(l) = n\right) = V_1(n)$$

(4.1.11)

$$\mathbf{D}^2\left(U_1(\tilde{\mathcal{Z}}(l+1)) \middle| \tilde{\mathcal{Z}}(l) = n\right) = V_2(n) + \mathbf{E}\left(U_2(\tilde{\mathcal{Z}}(l+1)) \middle| \tilde{\mathcal{Z}}(l) = n\right)$$

As both functions $n \mapsto U_1(n)$ and $n \mapsto V_1(n)$ are *bijections* between \mathbb{N} and their ranges it is more convenient to consider the Markov chains

$$(4.1.12) \quad \mathcal{Y}(l) = V_1(\mathcal{Z}(l)), \quad \tilde{\mathcal{Y}}(l) = U_1(\tilde{\mathcal{Z}}(l)), \quad l = 0, 1, 2, \dots$$

instead of $\mathcal{Z}(l)$, respectively $\tilde{\mathcal{Z}}(l)$. With this change of variable formulas (4.1.8)-(4.1.11) transform as follows:

(4.1.13)

$$\mathbf{E}\left(\mathcal{Y}(l+1) \middle| \mathcal{Y}(l) = x\right) = U_1(V_1^{-1}(x) + 1)$$

(4.1.14)

$$\mathbf{D}^2\left(\mathcal{Y}(l+1) \middle| \mathcal{Y}(l) = x\right) = U_2(V_1^{-1}(x) + 1) + \mathbf{E}\left(V_2 \circ V_1^{-1}(\mathcal{Y}(l+1)) \middle| \mathcal{Y}(l) = x\right)$$

(4.1.15)

$$\mathbf{E}\left(\tilde{\mathcal{Y}}(l+1) \middle| \tilde{\mathcal{Y}}(l) = x\right) = V_1 \circ U_1^{-1}(x)$$

(4.1.16)

$$\mathbf{D}^2\left(\tilde{\mathcal{Y}}(l+1) \middle| \tilde{\mathcal{Y}}(l) = x\right) = V_2 \circ U_1^{-1}(x) + \mathbf{E}\left(U_2 \circ U_1^{-1}(\tilde{\mathcal{Y}}(l+1)) \middle| \tilde{\mathcal{Y}}(l) = x\right)$$

We introduce the functions $F, G : \text{Ran}(V_1) \rightarrow \mathbb{R}$ and $\tilde{F}, \tilde{G} : \text{Ran}(U_1) \rightarrow \mathbb{R}$ defined below

(4.1.17)

$$\begin{aligned} F(x) &= \mathbf{E}\left(\mathcal{Y}(l+1) \middle| \mathcal{Y}(l) = x\right) - x \\ &= U_1(V_1^{-1}(x) + 1) - x \end{aligned}$$

(4.1.18)

$$\begin{aligned} G(x) &= \mathbf{E}\left(\left[\mathcal{Y}(l+1) - \mathbf{E}\left(\mathcal{Y}(l+1) \middle| \mathcal{Y}(l) = x\right)\right]^2 \middle| \mathcal{Y}(l) = x\right) \\ &= U_2(V_1^{-1}(x) + 1) + \mathbf{E}\left(V_2 \circ V_1^{-1}(\mathcal{Y}(1)) \middle| \mathcal{Y}(0) = x\right) \end{aligned}$$

$$(4.1.19) \quad \begin{aligned} \tilde{F}(x) &= \mathbf{E} \left(\tilde{\mathcal{Y}}(l+1) \middle| \tilde{\mathcal{Y}}(l) = x \right) - x \\ &= V_1 \circ U_1^{-1}(x) - x \end{aligned}$$

$$(4.1.20) \quad \begin{aligned} \tilde{G}(x) &= \mathbf{E} \left(\left[\tilde{\mathcal{Y}}(l+1) - \mathbf{E} \left(\tilde{\mathcal{Y}}(l+1) \middle| \tilde{\mathcal{Y}}(l) = x \right) \right]^2 \middle| \tilde{\mathcal{Y}}(l) = x \right) \\ &= V_2 \circ U_1^{-1}(x) + \mathbf{E} \left(U_2 \circ U_1^{-1}(\tilde{\mathcal{Y}}(1)) \middle| \tilde{\mathcal{Y}}(0) = x \right). \end{aligned}$$

Since $\mathcal{Y}(\cdot)$ and $\tilde{\mathcal{Y}}(\cdot)$ are *Markov chains*, from (4.1.13)-(4.1.20) it follows that the processes

$$(4.1.21) \quad \mathcal{M}(l) = \mathcal{Y}(l) - \mathcal{Y}(0) - \sum_{j=0}^{l-1} F(\mathcal{Y}(j)), \quad \tilde{\mathcal{M}}(l) = \tilde{\mathcal{Y}}(l) - \tilde{\mathcal{Y}}(0) - \sum_{j=0}^{l-1} \tilde{F}(\tilde{\mathcal{Y}}(j))$$

are *martingales* with quadratic variation processes

$$(4.1.22) \quad \langle \mathcal{M}, \mathcal{M} \rangle(l) = \sum_{j=0}^{l-1} G(\mathcal{Y}(j)), \quad \langle \tilde{\mathcal{M}}, \tilde{\mathcal{M}} \rangle(l) = \sum_{j=0}^{l-1} \tilde{G}(\tilde{\mathcal{Y}}(j)).$$

4.2. Asymptotics of the Relevant Functions.

In the present subsection we give the asymptotics of the relevant functions, F , G , \tilde{F} , \tilde{G} to be used in the proof of Theorem 1. All formulas are valid for *large values* of the variable and are obtained from (1.4) and (1.5) in a straightforward way.

From (1.5) we get

$$(4.2.1) \quad U_1(n) = n^{1-\alpha} + u + \mathcal{O}(n^{-\alpha})$$

$$(4.2.2) \quad V_1(n) = n^{1-\alpha} + v + \mathcal{O}(n^{-\alpha})$$

$$(4.2.3) \quad V_2(n), U_2(n) = \begin{cases} \mathcal{O}(n^{1-2\alpha}) & \text{if } 0 < \alpha < \frac{1}{2} \\ \mathcal{O}(\log n) & \text{if } \alpha = \frac{1}{2} \\ \mathcal{O}(1) & \text{if } \frac{1}{2} < \alpha < 1 \end{cases}$$

u and v in (4.2.1) and (4.2.2) are two real constants. We define

$$(4.2.4) \quad D = \lim_{n \rightarrow \infty} (U_1(n) - V_1(n)) = u - v$$

Clearly,

$$(4.2.5) \quad \begin{aligned} D &= \sum_{j=0}^{\infty} \left(\frac{1}{w(2j)} - \frac{1}{w(2j+1)} \right) \\ &= \frac{1}{w(0)} - \sum_{j=1}^{\infty} \left(\frac{1}{w(2j-1)} - \frac{1}{w(2j)} \right) \end{aligned}$$

and hence, due to (1.4),

$$(4.2.6) \quad 0 < D < w(0)^{-1} < \infty.$$

The asymptotics of the functions F , \tilde{F} , G , and \tilde{G} is given in the next Lemma:

Lemma 2. *The following asymptotics hold for $x \gg 1$:*

$$(4.2.7) \quad F(x) = D + \mathcal{O}(x^{-\alpha/(1-\alpha)} \vee x^{-1})$$

$$(4.2.8) \quad \tilde{F}(x) = -D + \mathcal{O}(x^{-\alpha/(1-\alpha)} \vee x^{-1})$$

$$(4.2.9) \quad G(x), \tilde{G}(x) = \begin{cases} \mathcal{O}(x^{(1-2\alpha)/(1-\alpha)}) & \text{if } 0 < \alpha < \frac{1}{2} \\ \mathcal{O}(\log x) & \text{if } \alpha = \frac{1}{2} \\ \mathcal{O}(1) & \text{if } \frac{1}{2} < \alpha < 1. \end{cases}$$

Proof. Note first that (4.2.1) and (4.2.2) imply

$$(4.2.10) \quad U_1^{-1}(x) = x^{1/(1-\alpha)} - \frac{u}{1-\alpha} x^{\alpha/(1-\alpha)} + \mathcal{O}(1)$$

and

$$(4.2.11) \quad V_1^{-1}(x) = x^{1/(1-\alpha)} - \frac{v}{1-\alpha} x^{\alpha/(1-\alpha)} + \mathcal{O}(1),$$

respectively. Inserting (4.2.1) and (4.2.11) into (4.1.17) [respectively, (4.2.2) and (4.2.10) into (4.1.19)] we readily get (4.2.7) [respectively, (4.2.8)].

In order to prove (4.2.9) we note first that, due to (4.2.10), (4.2.11) and (4.2.3) we have:

$$(4.2.12) \quad \begin{aligned} & U_2(V_1^{-1}(x) + 1), V_2 \circ V_1^{-1}(x), V_2 \circ U_1^{-1}(x), U_2 \circ U_1^{-1}(x) = \\ & = \begin{cases} \mathcal{O}(x^{(1-2\alpha)/(1-\alpha)}) & \text{if } 0 < \alpha < \frac{1}{2} \\ \mathcal{O}(\log x) & \text{if } \alpha = \frac{1}{2} \\ \mathcal{O}(1) & \text{if } \frac{1}{2} < \alpha < 1 \end{cases} \end{aligned}$$

Inserting these into (4.1.18) and (4.1.20), and applying Jensen's inequality (note that the functions $x \mapsto x^{(1-2\alpha)/(1-\alpha)}$ and $x \mapsto \log x$ are *concave*), we get eventually

(4.2.9). \square

Note also that the functions $x \mapsto F(x)$ and $x \mapsto \tilde{F}(x)$ are monotone decreasing, with

$$(4.2.13) \quad D = \lim_{x \rightarrow \infty} F(x) \leq F(x) \leq F(0) = \frac{1}{w(0)}, \quad -D = \lim_{x \rightarrow \infty} \tilde{F}(x) \leq \tilde{F}(x) \leq \tilde{F}(0) = 0.$$

4.3. Scaling.

The proper scaling of the processes $\mathcal{Y}(\cdot)$ and $\tilde{\mathcal{Y}}(\cdot)$ is determined by the dominant terms in the asymptotics of the functions F, G , respectively \tilde{F}, \tilde{G} . The scaling of the processes $\mathcal{Z}(\cdot)$ and $\tilde{\mathcal{Z}}(\cdot)$ is determined by the functional relations (4.1.12).

(4.2.9)-(4.2.11) suggest the following scaling:

$$(4.3.1) \quad Y_A(t) = A^{-1} \mathcal{Y}([At]) \quad \tilde{Y}_A(t) = A^{-1} \tilde{\mathcal{Y}}([At]).$$

The rescaled martingales $M_A(\cdot)$, $\tilde{M}_A(\cdot)$ and their quadratic variation processes will be

$$(4.3.2) \quad M_A(t) = A^{-1} \mathcal{M}([At]) = Y_A(t) - Y_A(0) - \int_0^{A^{-1}[At]} F(AY_A(s)) ds$$

$$(4.3.3) \quad \tilde{M}_A(t) = A^{-1} \tilde{\mathcal{M}}([At]) = \tilde{Y}_A(t) - \tilde{Y}_A(0) - \int_0^{A^{-1}[At]} \tilde{F}(A\tilde{Y}_A(s)) ds$$

$$(4.3.4) \quad \langle M_A, M_A \rangle(t) = A^{-2} \langle \mathcal{M}, \mathcal{M} \rangle([At]) = \int_0^{A^{-1}[At]} A^{-1} G(AY_A(s)) ds$$

$$(4.3.5) \quad \langle \tilde{M}_A, \tilde{M}_A \rangle(t) = A^{-2} \langle \tilde{\mathcal{M}}, \tilde{\mathcal{M}} \rangle([At]) = \int_0^{A^{-1}[At]} A^{-1} \tilde{G}(A\tilde{Y}_A(s)) ds$$

The functional relations (4.1.12), the asymptotics (4.2.1), respectively (4.2.2), and the scaling (4.3.1) determine the proper scaling of the processes $\mathcal{Z}(\cdot)$ and $\tilde{\mathcal{Z}}(\cdot)$:

$$(4.3.6) \quad Z_A(t) = A^{-1/(1-\alpha)} \mathcal{Z}([At]) \quad \tilde{Z}_A(t) = A^{-1/(1-\alpha)} \tilde{\mathcal{Z}}([At]).$$

4.4. Convergence of the processes

We assume that the initial conditions converge in probability to the deterministic constants y_0 , respectively \tilde{y}_0 : denoting the events

$$(4.4.1) \quad \mathcal{A}_{\delta, A} = \left\{ |Y_A(0) - y_0| < \delta \right\}, \quad \tilde{\mathcal{A}}_{\delta, A} = \left\{ \left| \tilde{Y}_A(0) - \tilde{y}_0 \right| < \delta \right\}$$

we have for any fixed $\delta > 0$

$$(4.4.2) \quad \mathbf{P}\left(\mathcal{A}_{\delta, A}\right) \rightarrow 1, \quad \mathbf{P}\left(\tilde{\mathcal{A}}_{\delta, A}\right) \rightarrow 1.$$

First we show that the martingales $M_A(\cdot)$ and $\tilde{M}_A(\cdot)$ converge to zero in probability,

uniformly on compact intervals $s \in [0, t]$. Indeed:

$$\begin{aligned}
\mathbf{E}\left(\langle M_A, M_A \rangle(t)\right) &= \int_0^{A^{-1}[At]} A^{-1} \mathbf{E}\left(G(AY_A(s))\right) ds \\
(4.4.3) \quad &\leq \int_0^{A^{-1}[At]} A^{-1} \mathbf{E}\left(C_1 (AY_A(s))^{1-\alpha} + C_2\right) ds \\
&\leq A^{-\alpha} C_1 \int_0^{A^{-1}[At]} \left(\mathbf{E}Y_A(s)\right)^{1-\alpha} ds + A^{-1} C_2 t \\
&\leq A^{-\alpha} C_1 \int_0^{A^{-1}[At]} (Y_A(0) + C_3 s)^{1-\alpha} ds + A^{-1} C_2 t \rightarrow 0
\end{aligned}$$

In the first inequality the asymptotics (4.2.9) of the function G is used, the second one follows from Jensen's inequality, finally in the last inequality we have used (4.3.2) and the fact that the function F is bounded. Define the events

$$(4.4.4) \quad \mathcal{B}_{t,\delta,A} = \left\{ \sup_{0 \leq s \leq t} |M_A(s)| < \delta \right\}, \quad \tilde{\mathcal{B}}_{\delta,A} = \left\{ \sup_{0 \leq s \leq D^{-1}\tilde{y}_0} \left| \tilde{M}_A(s) \right| < \delta \right\}.$$

From (4.4.3) and a similar argument applied to the martingale $\tilde{M}_A(\cdot)$ we conclude that for any $t \in [0, \infty)$ and any $\delta > 0$ fixed

$$(4.4.5) \quad \mathbf{P}\left(\mathcal{B}_{t,\delta,A}\right) \rightarrow 1, \quad \mathbf{P}\left(\tilde{\mathcal{B}}_{\delta,A}\right) \rightarrow 1$$

as $A \rightarrow \infty$. Due to (4.3.2) (respectively, (4.3.3)) and (4.2.13), on the sets $\mathcal{A}_{\delta,A} \cap \mathcal{B}_{t,\delta,A}$ (respectively, on the sets $\tilde{\mathcal{A}}_{\delta,A} \cap \tilde{\mathcal{B}}_{\delta,A}$) we have for $s \in [0, t]$ (respectively, for $s \in [0, D^{-1}\tilde{y}_0]$)

$$(4.4.6) \quad Y_A(s) \geq \{y_0 + Ds - 2\delta\}_+,$$

respectively,

$$(4.4.7) \quad \tilde{Y}_A(s) \geq \{\tilde{y}_0 - Ds - 2\delta\}_+.$$

Consequently, given any $t \in [0, \infty)$ fixed, on the set $\mathcal{A}_{\delta,A} \cap \mathcal{B}_{t,\delta,A}$

$$(4.4.8) \quad Y_A(s) \geq \begin{cases} 0 & \text{for } 0 \leq s \leq D^{-1}\{3\delta - y_0\}_+ \wedge t \\ \delta & \text{for } D^{-1}\{3\delta - y_0\}_+ \wedge t < s \leq t. \end{cases}$$

On the other hand, on the set $\tilde{\mathcal{A}}_{\delta,A} \cap \tilde{\mathcal{B}}_{\delta,A}$

$$(4.4.9) \quad \tilde{Y}_A(s) \geq \begin{cases} \delta & \text{for } 0 \leq s < D^{-1}\{\tilde{y}_0 - 3\delta\}_+ \\ 0 & \text{for } D^{-1}\{\tilde{y}_0 - 3\delta\}_+ \leq s \leq D^{-1}\tilde{y}_0. \end{cases}$$

Now, choose A big enough to have

$$(4.4.10) \quad F(A\delta) - D < \delta, \quad \tilde{F}(A\delta) + D < \delta.$$

From (4.3.2) and (4.4.8) it follows that for any $t \in [0, \infty)$, on $\mathcal{A}_{\delta,A} \cap \mathcal{B}_{t,\delta,A}$

$$(4.4.11) \quad \begin{aligned} & \sup_{0 \leq s \leq t} |Y_A(s) - (y_0 + Ds)| \\ & \leq |Y_A(0) - y_0| + \int_0^{A^{-1}[At]} (F(AY_A(s)) - D) ds + \sup_{0 \leq s \leq t} |M_A(s)| \\ & \leq \delta + 3(w(0)^{-1} - D)D^{-1}\delta + (t + A^{-1})\delta + \delta \leq (t + 3(w(0)D)^{-1})\delta, \end{aligned}$$

and hence for any $t \in [0, \infty)$

$$(4.4.12) \quad \sup_{0 \leq s \leq t} |Y_A(s) - (y_0 + Ds)| \xrightarrow{\mathbf{P}} 0$$

On the other hand, from (4.3.3) and (4.4.9) it follows that, on the set $\tilde{\mathcal{A}}_{\delta,A} \cap \tilde{\mathcal{B}}_{\delta,A}$

$$(4.4.13) \quad \begin{aligned} & \sup_{0 \leq s \leq D^{-1}\tilde{y}_0} |\tilde{Y}_A(s) - (\tilde{y}_0 - Ds)| \\ & \leq |\tilde{Y}_A(0) - \tilde{y}_0| + \int_0^{A^{-1}[AD^{-1}\tilde{y}_0]} (\tilde{F}(A\tilde{Y}_A(s)) + D) ds \\ & \quad + \sup_{0 \leq s \leq D^{-1}\tilde{y}_0} |\tilde{M}_A(s)| \\ & \leq \delta + 3\delta + (t + A^{-1})\delta + \delta \leq (D^{-1}\tilde{y}_0 + 6)\delta \end{aligned}$$

and hence:

$$(4.4.14) \quad \sup_{0 \leq s \leq D^{-1}\tilde{y}_0} |\tilde{Y}_A(s) - (\tilde{y}_0 - Ds)| \xrightarrow{\mathbf{P}} 0$$

4.5. Convergence of the extinction time

The forthcoming argument is a repeat of the proof presented in subsection 5.7/A1 of TÓTH (1996).

For $x \in \mathbb{R}_+$ we denote

$$(4.5.1) \quad \tilde{\sigma}_x = \inf\{l \geq 0 : \tilde{\mathcal{Y}}(l) \leq x\}$$

$$(4.5.2) \quad \tilde{\sigma}_{x,A} = \inf\{t \geq 0 : \tilde{Y}_A(t) \leq x\}$$

We prove now that for any $\eta > 0$:

$$(4.5.3) \quad \lim_{y \rightarrow 0} \lim_{A \rightarrow \infty} \mathbf{P} \left(\tilde{\sigma}_{0,A} > \eta \middle| \tilde{Y}_A(0) = y \right) = 0,$$

which is equivalent to

$$(4.5.4) \quad \lim_{y \rightarrow 0} \lim_{A \rightarrow \infty} \mathbf{P} \left(\tilde{\sigma}_0 > A\eta \middle| \tilde{\mathcal{Y}}(0) = Ay \right) = 0.$$

From (4.2.13) it follows that there exists an $x_0 < \infty$ such that for $x \geq x_0$

$$(4.5.5) \quad \tilde{F}(x) \leq -\frac{D}{2} < 0$$

and thus

$$(4.5.6) \quad \mathcal{N}(l) = \tilde{\mathcal{Y}}(l) + \frac{D}{2}l$$

is *supermartingale* as long as $\mathcal{Y}(l) \geq x_0$. Applying the optional sampling theorem to the supermartingale $\mathcal{N}(l)$ we get for $y > x_0$

$$(4.5.7) \quad \mathbf{E} \left(\tilde{\sigma}_{x_0} \middle| \tilde{\mathcal{Y}}(0) = y \right) \leq \frac{2}{D}y.$$

Now, we prove (4.5.4):

$$(4.5.8) \quad \begin{aligned} & \lim_{y \rightarrow 0} \lim_{A \rightarrow \infty} \mathbf{P} \left(\tilde{\sigma}_0 > A\eta \middle| \tilde{\mathcal{Y}}(0) = Ay \right) \leq \\ & \lim_{y \rightarrow 0} \lim_{A \rightarrow \infty} \mathbf{P} \left(\tilde{\sigma}_{x_0} > A\eta/2 \middle| \tilde{\mathcal{Y}}(0) = Ay \right) + \lim_{A \rightarrow \infty} \sup_{0 \leq x \leq x_0} \mathbf{P} \left(\tilde{\sigma}_0 > A\eta/2 \middle| \tilde{\mathcal{Y}}(0) = x \right). \end{aligned}$$

Applying Markov's inequality and (4.5.7) we get

$$(4.5.9) \quad \lim_{y \rightarrow 0} \lim_{A \rightarrow \infty} \mathbf{P} \left(\tilde{\sigma}_{x_0} > A\eta/2 \middle| \tilde{\mathcal{Y}}(0) = Ay \right) \leq \lim_{y \rightarrow 0} \frac{4y}{D\eta} = 0.$$

On the other hand, since x_0 is constant independent of A , the second limit on the right hand side of (4.5.8) clearly vanishes. Hence (4.5.4), or equivalently (4.5.3), follows.

From (4.5.3) and (4.4.14) it follows that

$$(4.5.10) \quad \tilde{\sigma}_{A,0} \xrightarrow{\mathbf{P}} D^{-1}\tilde{y}_0.$$

4.6. End of the proof

Collecting the results of subsections 4.1–4.5 we conclude that, provided that $Z_A(0) \xrightarrow{\mathbf{P}} z_0$, for any fixed $t \in [0, \infty)$:

$$(4.6.1) \quad \sup_{0 \leq s \leq t} \left| Z_A(s) - \{z_0^{1-\alpha} + Ds\}^{1/(1-\alpha)} \right| \xrightarrow{\mathbf{P}} 0$$

and, provided that $\tilde{Z}_A(0) \xrightarrow{\mathbf{P}} \tilde{z}_0$,

$$(4.6.2) \quad \tilde{\sigma}_{0,A} \xrightarrow{\mathbf{P}} \tilde{D}^{-1}\tilde{z}_0^{1-\alpha}$$

and

$$(4.6.3) \quad \sup_{0 \leq s \leq \tilde{D}^{-1}\tilde{z}_0^{1-\alpha}} \left| \tilde{Z}_A(s) - \{\tilde{z}_0^{1-\alpha} - Ds\}^{1/(1-\alpha)} \right| \xrightarrow{\mathbf{P}} 0$$

Given the representation of the local time process described in subsection 3.2., Theorem 1 follows directly from (4.6.1)-(4.6.3), after noting that due to (3.1.11) it is easily seen that the single exceptional step (3.2.5) does not spoil the continuity of the limit process at $y = x$. \square

Corollary 1 follows directly from Theorem 1. Note that the *joint convergence* of the processes $S_{[Ax],[A^{1/(1-\alpha)h}]}^*([A\cdot])/A^{1/(1-\alpha)}$ and extinction times $\omega_{[Ax],[A^{1/(1-\alpha)h}]}^{*\pm}/A$ is needed in this proof. \square

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