SELF-INTERACTING RANDOM MOTIONS — A SURVEY

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Abstract. We present a survey of results concerning self-interacting random walks and self-repelling continuous random motions. A self-interacting random walk (SIRW) is a nearest neighbour walk on the one-dimensional integer lattice $\mathbb{Z}$ which starts from the origin and at each step jumps to a neighbouring site, the probability of jumping along a bond being proportional to w(number of previous jumps along that lattice bond), where $w : \mathbb{N} \to \mathbb{R}_+$ is a monotone weight function. We consider exponential [$w(n) = \exp\{-\beta n\}$, $\beta > 0$], subexponential [$w(n) = \exp\{-\beta n^\kappa\}$, $\beta > 0$, $0 < \kappa < 1$], polynomially decaying [$w(n) \sim n^{-\alpha}$, $\alpha > 0$], asymptotically constant [$w(n) \sim 1$] and weakly increasing [$w(n) \sim n^\alpha$, $0 < \alpha < 1$] weight functions. These weight functions define variants of the so-called 'myopic self-repelling' and 'reinforced' random walk. We present functional limit theorems for the local time processes of these random walks and limit theorems for the position of the random walker at late times. A generalization of the Ray-Knight theory of local time is in the background of these results.

In the second part of the paper we present recent results concerning the construction and primary properties of a continuous, locally self-repelling process X_t. The process is a.s. continuous and recurrent, it has a regular occupation time density (local time) denoted $L_t(x)$, and the self-repellence of its trajectories is achieved by the dynamical driving mechanism formally expressed as $dX_t = -\text{grad } L_t(X_t)dt$. This means that the process X_t is instantaneously pushed in the direction of the decrease of its local time. The constructed process is self-similar with scale-exponent $\nu = 2/3$ and has non-trivial local variation of order $3/2$ (in contrast with the finite quadratic variation of semi-martingales).

We do not give proofs in this survey. The full proofs can be found in the papers [T1-T4] and [TW].

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0. Introduction

We survey recent results concerning one-dimensional self-interacting random motions. The survey is based on the papers [T1-T4] and [TW], where the complete detailed proofs can be found.

The paper consists of two larger chapters (further divided in subsections): In the first chapter we present the results related to *self-interacting random walks* (abreviated SIRW) on $\mathbb{Z}$. These are essentially limit theorems on the asymptotics of the local times and of the late-time positions of these random walks. The second chapter is entirely devoted to the construction and primary properties of the so-called *true self-repelling motion* on $\mathbb{R}$ (abreviated TSRM). This stochastic process, constructed in [TW], is of completely new type with striking analytic and dynamical properties and it is the natural upshot of the investigations started with [T1].

The need for investigation of late-time asymptotics of random walks with long memory arose naturally in the probability and statistical physics literature. See e.g. [APP], [CD], [D1], [DR], [PP], etc. and the survey chapters of [Lw] and [MS] for the historical origins of the problems. An SIRW is a nearest neighbour random walk X_i on $\mathbb{Z}$ defined as follows: the walk starts from the origin of the lattice and at time $i+1$ it jumps to one of the two neighbouring sites of X_i, so that the probability of jumping along a bond of the lattice is proportional to

$$w(\text{number of previous jumps along that bond})$$

where

$$w : \mathbb{N} \to \mathbb{R}_+$$

is a weight function to be specified later. It turned out very early that the more or less standard (nevertheless sophisticated) techniques of asymptotic analysis of probability theory, such as different mixing criteria, control of decay of correlations, martingale approximations etc, are simply not applicable to these phenomena. This technical obstruction is, of course, closely related to and caused by the essential phenomenological or physical features of these processes. Namely, by the complicated long-time memory effects built up by the self-interaction mechanisms driving these random walks. In papers [T1-T4] we developed a method which allowed us to prove limit theorems for the distributions of the late time position of a class of SIRWs. These results will be presented in Chapter 1. It is well-known that there are very few mathematically rigorous results, which describe the *low-dimensional anomalous behaviour caused by the physically relevant interactions*. In our case anomalous diffusion (overdiffusive spreading) is caused by self-repellence of the RW trajectories, [T1, T2, TW].

The construction and investigation of the (continuous space-time) stochastic process called *true self-repelling motion* was motivated by the need of a deeper understanding, on the level of *invariance principle* of the asymptotics of the myopic self-avoiding walk considered in [T1]. It is also worth noting that the urge for construction of such a locally self-repelling continuous space-time random motion had been a major challange in the theory of stochastic processes in the last decade. For some earlier attempts see e.g. [DR], [NRW] and references cited therein. The stochastic process constructed in [TW] meets all the phenomenological and analytical requirements, it seems to be the most natural object deserving the name *true self-repelling motion*. The primary properties of the process, denoted by X_t, are the following:

• Continuity and recurrence;

• Existence and due regularity of the local time, denote it $L_t(x)$;

• Dynamics: the process X_t is driven by the negative gradient of its local time. This property is formally expressed by the equation

$$dX_t = -\frac{\partial L_t(X_t)}{\partial x} dt. \tag{0.1}$$

This equation phenomenologically expresses the fact that the moving particle 'feels' a pressure pushing in the direction where it had spent less time in the past. The unevennes of the past occupation time measure is felt only locally. This crucial property of our process entitles us to call it 'truely self-repelling". We warn the reader that equation (0.1) should be properly regularized in order to get proper mathematical meaning.

• Scaling and local variation: the process is self-similar under the scaling $\alpha^{-2/3}X_{\alpha t}$, $\alpha > 0$ and has non-trivial local variation of order $3/2$ (in contrast with the finite quadratic variation of semimartingales).

These primary properties clearly show that the process X_t constructed is not a semimartingale and it is not solution of a stochastic differential equation. We face a new type of stochastic phenomenon.

1. Self-Interacting Random Walks on $\mathbb{Z}$

1.1. Setup.

Let $\mathbb{N} \ni i \mapsto X_i \in \mathbb{Z}$ be a nearest neighbour walk, starting from the origin of the lattice. I.e. $X_0 = 0$ and $|X_{i+1} - X_i| = 1$ for any $i \in \mathbb{N}$. For such a walk we define the upcrossing, downcrossing and (edge) local time processes in the most natural usual way:

$$\begin{aligned}
U_i(x) &:= \#\{\, j \in [0,i) \,|\, X_i = x,\ X_{i+1} = x+1 \,\}, && i \in \mathbb{N},\ x \in \mathbb{Z},\\
D_i(x) &:= \#\{\, j \in [0,i) \,|\, X_i = x+1,\ X_{i+1} = x \,\}, && i \in \mathbb{N},\ x \in \mathbb{Z},\\
L_i(x) &:= U_i(x) + D_i(x), && i \in \mathbb{N},\ x \in \mathbb{Z}.
\end{aligned}$$

It is straightforward that for any $i \geq 0$ we have

$$i = \sum_{y \in \mathbb{Z}} L_i(y). \tag{1.1}$$

The inverse local time processes are

$$\begin{aligned}
T^{\mathrm{U}}_{x,m} &:= \inf\{\, i \in \mathbb{N} \,|\, U_i(x) \geq m \,\}, && x \in \mathbb{Z},\ m \in \mathbb{N},\\
T^{\mathrm{D}}_{x,m} &:= \inf\{\, i \in \mathbb{N} \,|\, D_i(x) \geq m \,\}, && x \in \mathbb{Z},\ m \in \mathbb{N}.
\end{aligned}$$

Finally, the local time processes stopped at inverse local times are

$$\Lambda^*_{x,m}(y) := L_{T^*_{x,m}}(y), \qquad x \in \mathbb{Z},\ m \in \mathbb{N},\ y \in \mathbb{Z}.$$

Hereafter the superscript $*$ stands for either U or D. In $\Lambda^*_{x,m}(y)$ one should think about x and m as fixed parameters and y variable. From (1.1) it follows that for any $x \in \mathbb{Z}$ and $m \in \mathbb{N}$

$$T^*_{x,m} = \sum_{y \in \mathbb{Z}} \Lambda^*_{x,m}(y). \tag{1.2}$$

The quantities defined above make perfectly good sense for any nearest neighbour walk on $\mathbb{Z}$.

Let $w : \mathbb{N} \to \mathbb{R}_+$ be a weight function, either monotone non-decreasing or monotone non-increasing. The SIRW defined by the weight function $w(\cdot)$ is a nearest neighbour walk X_i on $\mathbb{Z}$ with $X_0 = 0$ and governed by the law

$$\begin{aligned}
\mathbf{P}\Big(X_{i+1} = X_i + 1 \,\Big|\, \underline{X}^i_0\Big) &= \frac{w\big(L_i(X_i)\big)}{w\big(L_i(X_i)\big) + w\big(L_i(X_i - 1)\big)}\\
&= 1 - \mathbf{P}\Big(X_{i+1} = X_i - 1 \,\Big|\, \underline{X}^i_0\Big),
\end{aligned} \tag{1.3}$$

where we used the shorthand notation $\underline{X}_0^i := (X_0, X_1, \dots, X_i)$. In plain words: the random walker jumps to one of the two nearest neighbour sites so that the probability of jumping across an edge of the lattice is always proportional to the weight associated to the number of previous jumps across that edge. It is intuitively clear that monotone non-increasing weight functions define self-repelling walks while monotone non-decreasing weight functions define self-attracting walks. It is also clear that in the conditional jump probabilities (1.3) the full past history of the walk plays an important role, so these walks have extremely long memory, they are strongly non-Markovian. So, one should expect interesting, non-trivial long time asymptotic behaviour.

It turns out that the asymptotic behaviour of the SIRW is very sensitive to the choice of the weight function $w(\cdot)$. The following particular cases of the weight function have been investigated:

(1) In [T1] we considered the weight function

$$w(n) = \exp(-\beta n)$$

where $\beta > 0$ is a fixed parameter. This weight function imposes exponentially strong self-repellence on the random walk trajectories. The model was originally defined on $\mathbb{Z}^d$ in [APP] and has a sort of notoriety in the physics literature, where it is usually called 'true self-avoiding random walk'. We shall use the term 'exponentially self-repelling walk' (abreviated ESRW). In [APP] the authors argue that the critical dimension of this RW model is 2. This means that in three and more dimensions the analogous RW behaves diffusively, exhibiting Gaussian limit under standard diffusive scaling. In two-dimensions logarithmic corrections to the diffusive scaling limit are due. In [OP] and [PP] the authors argue that in one-dimension the position of the ESRW scales with the 2/3-rd power of time. (No attempt is made there to identify the scaling limit.) These physical statements are based on so-called renormalization group arguments, which are appealing and thought-inspiring but far from mathematical rigour. In [T1] we stated and proved limit theorems for the ESRW on $\mathbb{Z}^1$, in agreement with the physicists' prediction regarding the scaling power, also identifying explicitly the limiting distribution: the main result of that paper is essentially a limit theorem for (the Laplace transform of) the distribution of $N^{-2/3}X_{[Nt]}$

(2) In [T2] we considered the sub-exponentially self-repelling walks (SESRW) governed by the weight functions

$$w(n) = \exp(-\beta n^\kappa),$$

where now $\beta > 0$ and $\kappa \in (0, 1)$ are both fixed parameters. These models interpolate between the ordinary diffusive behaviour of simple symmetric random walks ($\kappa = 0$)

and the super-diffusive ESRW-s considered earlier ($\kappa = 1$). The main result here is a limit theorem for (the Laplace transform of) the distribution of $N^{-(\kappa+1)/(\kappa+2)} X_{[Nt]}$.
(3) In [T3] we considered the so-called polynomially self-repelling walks (PSRW) defined by monotone decreasing weight functions of the form

$$w(n) = (1+\alpha)^{-1}(n/2)^{-\alpha} - B(1+\alpha)^{-2}(n/2)^{-\alpha-1} + \mathcal{O}(n^{-\alpha-2}),$$

where now $\alpha \in (0,\infty)$ and $B \in \mathbb{R}$ are fixed parameters. Here we prove limit a theorem for (the Laplace transform of) the distribution of $N^{-1/2} X_{[Nt]}$. The scaling power is diffusive, but the limit law is *non-Gaussian.*
(4) In the same paper [T3] we also considered the asymptotically free walks (AFW), where the weight function is:

$$w(n) = 1 - 2Bn^{-1} + \mathcal{O}(n^{-2}),$$

whith $B \in \mathbb{R}$ a fixed parameter. Now $w(\cdot)$ could be either monotone non-decreasing or monotone non-increasing. (Thus defining either a self-attracting or a self-repelling walk.) Here the scaling limit to be taken is again $N^{-1/2} X_{[Nt]}$, but the limiting distribution depends quite subtly on the details of the weight functions. In particular on the value of

$$\delta := 2w(0)^{-1} + 2\sum_{j=1}^{\infty} \left(w(2j)^{-1} - w(2j-1)^{-1} \right).$$

Note, that $\delta \in (0,2]$ for the self-repelling case and $\delta \in [2,\infty)$ for the self-attracting case. Note also that the *once reinforced random walk* or *random walk partially reflected at extrema* considered in detail in connection with the construction of *Brownian motion perturbed at extrema*, is very particular case of these asymptotically free walks, having weight function

$$w(n) = \begin{cases} 2/\delta & \text{for} \quad n = 0 \\ 1 & \text{for} \quad n \neq 0 \end{cases}$$

See e.g. [CPY], [D2], [D3], [D4], [PW], [W], etc.
(5) Finally, in [T4] we define the weakly reinforced random walks (WRRW), governed by the weight function

$$w(n) = (1-\alpha)^{-1}(n/2)^{\alpha} - B(1-\alpha)^{-2}(n/2)^{\alpha-1} + \mathcal{O}(n^{\alpha-2}),$$

where $\alpha \in (0,1)$ and $B \in \mathbb{R}$ are fixed parameters. Here we prove limit theorem for (the Laplace transform of) the distribution of $N^{-(1-\alpha)/(2-\alpha)} X_{[Nt]}$.

1.2. Conjugate Diffusions and Generalized Ray-Knight Processes.

Before stating explicitly the limit theorems for the SIRW-s we have to make a short detour to define the notions of *conjugate diffusions on* $\mathbb{R}_+$, *generalized Ray-Knight processes* and state a theorem about them. These results might be interesting on their own, from a purely diffusion-theoretical point of view.

Let

$$a:(0,\infty)\to(0,\infty),\qquad b:(0,\infty)\to(-\infty,\infty)$$

be smooth functions and define the second order differential operators

$$\begin{aligned}\left[Gf\right](x) &:= \frac{1}{2}a(x)f''(x)+\left(\frac{1}{4}a'(x)+b(x)\right)f'(x)\\ \left[Hf\right](x) &:= \frac{1}{2}a(x)f''(x)+\left(\frac{1}{4}a'(x)-b(x)\right)f'(x).\end{aligned}$$

We call the operators G and H a *conjugate pair* of diffusion generators on $\mathbb{R}_+$. The *analytic content* of this conjugacy is the following (equivalent) pair of commutation relations

$$\frac{d}{dx}G=H^*\frac{d}{dx},\qquad \frac{d}{dx}H=G^*\frac{d}{dx}$$

where G^* and H^* are the formal Lebesgue adjoints of G, respectively H:

$$\begin{aligned}\left[G^*f\right](x) &= \frac{1}{2}a(x)f''(x)+\left(\frac{3}{4}a'(x)-b(x)\right)f'(x)+\left(\frac{1}{4}a''(x)-b'(x)\right)f(x)\\ \left[H^*f\right](x) &= \frac{1}{2}a(x)f''(x)+\left(\frac{3}{4}a'(x)+b(x)\right)f'(x)+\left(\frac{1}{4}a''(x)+b'(x)\right)f(x).\end{aligned}$$

Define the functions

$$\begin{aligned}u(x) &:= \sqrt{\frac{2}{a(x)}}\exp\{-\int_1^x\frac{2b(y)}{a(y)}dy\}\\ v(x) &:= \sqrt{\frac{2}{a(x)}}\exp\{\int_1^x\frac{2b(y)}{a(y)}dy\}.\end{aligned}$$

With the help of these functions we can express the differential operators G, H, G^* and H^* as follows:

$$\begin{aligned}G &= \frac{1}{v}\frac{d}{dx}\frac{1}{u}\frac{d}{dx}, & G^* &= \frac{d}{dx}\frac{1}{u}\frac{d}{dx}\frac{1}{v}\\ H &= \frac{1}{u}\frac{d}{dx}\frac{1}{v}\frac{d}{dx}, & H^* &= \frac{d}{dx}\frac{1}{v}\frac{d}{dx}\frac{1}{u}.\end{aligned}$$

We consider two diffusion processes on $\mathbb{R}_+$: $\xi(t)$ and $\eta(t)$ with generators G, respectively H. More precisely: the generators of $\xi(t)$, respectively $\eta(t)$, *restricted to smooth functions with compact support in* $(0,\infty)$ *act as* G, *respectively* H. The diffusions $\xi(t)$ and $\eta(t)$ are uniquely determined by these generators as long as they do not hit the boundary $\{0\}=\partial\mathbb{R}_+$. The ambiguity in the behaviour of the processes

$\xi(t)$ and $\eta(t)$ at 0 is eliminated in the following way: $\xi(t)$ is *reflected instantaneously at* 0 (see definition VII.3.11. in [RY]) and $\eta(t)$ is *stopped at*

$$\tau_0 := \inf\{t : \eta(t) = 0\}.$$

We call the processes $\xi(t)$ and $\eta(t)$ *conjugate diffusions*. Note that the conjugacy refers to the *law* of the processes $\xi(t)$ and $\eta(t)$ and not to their path-wise realization. The *scale functions* and *speed measures* of the processes $\xi(t)$, respectively $\eta(t)$ are $r(x)$ and $n(dx)$, respectively $s(x)$ and $m(dx)$. According to standard results about one-dimensional diffusions (see exercise VII.3.20. in [RY]), on $(0, \infty)$ we have:

$$r(x) = \int^x u(y)dy, \qquad n(dx) = v(x)dx \tag{1.4}$$

$$s(x) = \int^x v(y)dy, \qquad m(dx) = u(x)dx. \tag{1.5}$$

The lower limits in the integrals defining r and s are irrelevant. Formulas (1.4) and (1.5) give *the probabilistic content* of conjugacy of the diffusions $\xi(t)$ and $\eta(t)$: the derivative of the scale function of one is the Radon-Nikodym derivative of the speed measure of the other, and vice versa. In accordance to the behaviour at the boundary described in the previous paragraph we set

$$n(\{0\}) = 0, \qquad m(\{0\}) = \infty.$$

The conjugacy of a pair of diffusions is invariant under diffeomorphisms of $\mathbb{R}_+$: let

$$\Gamma : (0, \infty) \to (0, \infty)$$

be a C^2 bijection which has a C^2 inverse Γ^{-1}, and preserves the orientation of the half-line $\mathbb{R}_+$:

$$\lim_{x \searrow 0} \Gamma(x) = 0, \qquad \lim_{x \nearrow \infty} \Gamma(x) = \infty.$$

Consider the diffusions

$$\widetilde{\xi}(t) = \Gamma(\xi(t)), \qquad \widetilde{\eta}(t) = \Gamma(\eta(t)).$$

It is easy to check that, if $\xi(t)$ and $\eta(t)$ are a conjugate pair of diffusions on $\mathbb{R}_+$ then so are $\widetilde{\xi}(t)$ and $\widetilde{\eta}(t)$, with

$$\widetilde{a} = \left[(\Gamma')^2 \cdot a \right] \circ \Gamma^{-1}$$
$$\widetilde{b} = \left[\Gamma' \cdot b \right] \circ \Gamma^{-1}$$

The following stopping times will be used: for $y \in [0, \infty)$

$$\sigma_y := \inf\{t : \xi(t) = y\}$$
$$\tau_y := \inf\{t : \eta(t) = y\}$$

with the usual convention $\inf \emptyset = \infty$.

We impose some conditions on the behaviour of the diffusions $\xi(t)$ and $\eta(t)$ near 0 and ∞:

<u>*Condition at 0:*</u> We give two equivalent formulations — one referring to the diffusion $\xi(t)$ the other one to $\eta(t)$ — of the same single condition:

$$\int_0^1 \left(\int_y^1 u(z)dz\right) v(y)dy = \int_0^1 \big[r(1) - r(y)\big] n(dy) < \infty \tag{1.6}$$

$$\int_0^1 \left(\int_0^z v(y)dy\right) u(z)dz = \int_0^1 \big[s(z) - s(0)\big] m(dz) < \infty. \tag{1.7}$$

(We could have chosen any positive number instead of 1 as upper limit of integration in (1.6)/(1.7).) The left hand sides in the two formulas (1.6) and (1.7) are clearly the same, so we emphasize again that these are just two different formulations of the same condition. In probabilistic terms, these conditions are equivalent to the following:

$$\left(\sigma_x \,\middle|\, \xi(0) = 0\right) \xrightarrow{\mathbf{P}} 0 \qquad \text{as } x \to 0 \tag{1.8}$$

$$\left(\tau_0 \,\middle|\, \eta(0) = x\right) \xrightarrow{\mathbf{P}} 0 \qquad \text{as } x \to 0. \tag{1.9}$$

In plain words (1.8), respectively (1.9), means that $\xi(t)$ does not stick to 0, respectively $\eta(t)$ can hit 0 in finite time.

In particular, from (1.7) it also follows that

$$s(x) - s(0) = \int_0^x v(y)dy < \infty$$

and we can choose

$$s(x) = \int_0^x v(y)dy \qquad \text{i.e.} \qquad s(0) = 0.$$

<u>*Conditions at ∞:*</u> Again, we give two equivalent formulations of the boundary condition at infinity: the first formulation refers to the diffusion $\xi(t)$, the second one to $\eta(t)$:

$$\int_1^\infty \left(\int_y^\infty u(z)dz\right) v(y)dy = \int_1^\infty \big[r(\infty) - r(y)\big] n(dy) = \infty \tag{1.10}$$

$$\int_1^\infty \left(\int_1^z v(y)dy\right) u(z)dz = \int_1^\infty \big[s(z) - s(1)\big] m(dz) = \infty. \tag{1.11}$$

(Now, we could have chosen any positive number instead of 1 as lower limit of integration in (1.10)/(1.11).) Clearly, the left hand sides of (1.10) and (1.11) are the same again. The probabilistic content of these conditions is the following: for any fixed $y \geq 0$

$$\left(\sigma_x \,\middle|\, \xi(0) = y\right) \xrightarrow{\mathbf{P}} \infty \qquad \text{as } x \to \infty \tag{1.12}$$

$$\left(\tau_y \,\middle|\, \eta(0) = x\right) \xrightarrow{\mathbf{P}} \infty \qquad \text{as } x \to \infty. \tag{1.13}$$

Condition (1.12), respectively (1.13), means that $\xi(t)$ does not escape to infinity, respectively $\eta(t)$ does not come in from infinity, in finite time.

Beside condition (1.12)/(1.13) we also impose

$$\lim_{x \nearrow \infty} s(x) = s(\infty) = \infty. \tag{1.14}$$

Our main class of examples consists of *pairs of Bessel processes* of dimension δ, respectively, $2-\delta$, raised to some power $q > 0$. Fix $\delta > 0$ and let

$$a(x) = 4x, \qquad b(x) \equiv \delta - 1.$$

The corresponding generators G and H will be

$$G = 2x\frac{\partial^2}{\partial x^2} + \delta\frac{\partial}{\partial x}, \qquad H = 2x\frac{\partial^2}{\partial x^2} + (2-\delta)\frac{\partial}{\partial x}.$$

I.e. $\xi(t)$, respectively $\eta(t)$ will be the squared Bessel processes of dimension δ, respectively, the squared Bessel process of dimension $2-\delta$, stopped at first hitting of 0. (Note that this second process — stopped at first hitting of 0 — is well defined even for $2-\delta < 0$.) The various functions arising in this case are:

$$u(x) = \frac{x^{-\delta/2}}{\sqrt{2}} \qquad\qquad v(x) = \frac{x^{(\delta-2)/2}}{\sqrt{2}}$$

$$r(x) = \begin{cases} \dfrac{\sqrt{2}}{2-\delta} x^{(2-\delta)/2} & \text{if} \quad \delta \neq 2 \\[2ex] \dfrac{\ln x}{\sqrt{2}} & \text{if} \quad \delta = 2 \end{cases} \qquad\qquad s(x) = \frac{\sqrt{2}}{\delta} x^{\delta/2}.$$

Conditions (1.6)/(1.7), (1.10)/(1.11) and (1.14) are easily checked.

Using diffeomorphisms $\Gamma : \mathbb{R}_+ \to \mathbb{R}_+$ we get a wider class of examples. E.g. $\Gamma(x) = x^{q/2}$, $q > 0$ transforms the pair of *squared* Bessel processes into a pair of Bessel processes of the same generalized dimensions δ, respectively $2-\delta$, raised to power q. In particular, with the choice $\delta = 1 = 2-\delta$ and $q = 1$ we get a pair of Brownian motions reflecting, respectively stopped at 0: $\xi(t) = |W_t|$, $\eta(t) = |W_{t\wedge\tau_0}|$. In the sequel we shall use the shorthand notation BES^q_δ, respectively, $\widetilde{\mathrm{BES}}^q_{2-\delta}$ for the Bessel process of generalized dimension $\delta \in \mathbb{R}$, respectively, the Bessel process of generalized dimension $2-\delta$, *stopped at first hitting of* 0, both raised to power $q>0$.

Remark: If we also require *self-similarity* of the pair of conjugate difusions, i.e. that there is a $\gamma>0$ such that for any $\alpha>0$

$$\alpha^{-\gamma}\xi(\alpha t)\overset{\text{law}}{=}\xi(t), \qquad \alpha^{-\gamma}\eta(\alpha t)\overset{\text{law}}{=}\eta(t),$$

then (up to a common constant multiplier) we *must* have $\xi=\text{BES}^q_\delta$, $\eta=\widetilde{\text{BES}}^q_{2-\delta}$, with $q=2\gamma$ and $\delta>0$ arbitrary. This follows from [Lm].

Let now $\xi(t)$ and $\eta(t)$ be a pair of conjugate diffusions and define

(1.15)

$$\phi(y):=\mathbf{E}\Big(\exp\{-\int_0^{\tau_0}\eta(s)ds\}\,\Big|\,\eta(0)=y\Big)$$

(1.16)

$$\psi(y):=\int_0^\infty\Big\{\int_0^\infty\mathbf{E}\Big(\exp\{-\int_0^t\xi(s)ds\}\;1\!\!1\{\xi(t)>y\}\,\Big|\,\xi(0)=z\Big)dt\Big\}\,\phi(z)dz.$$

Theorem 1.1. *Let $\xi(t)$ and $\eta(t)$ be a pair of conjugate diffusions on $\mathbb{R}_+$ with the coefficients $a(x)$ and $b(x)$ satisfying conditions (1.6)/(1.7), (1.10)/(1.11) and (1.14). Then the functions ϕ and ψ defined in (1.15), respectively (1.16), are identical:*

$$\phi\equiv\psi.$$

Remark: This theorem is closely related to the generalized Cieselski-Taylor identities, see [Bn] and [T3]. Its direct consequence, Theorem 1.2 below is of more importance for us now.

Fix the two parameters $x\le 0$ and $h\in\mathbb{R}_+$. The *generalized Ray-Knight process* $\mathbb{R}\ni y\mapsto\Lambda_{x,h}(y)\in\mathbb{R}_+$ is patched together as follows:

$$\Lambda_{x,h}(y)=\begin{cases}\Big(\eta_{\mathrm{l}}(x-y)\,\Big|\,\eta_{\mathrm{l}}(0)=h\Big) & \text{for}\quad y\in(-\infty,x]\\ \Big(\xi(y+x)\,\Big|\,\xi(0)=h\Big) & \text{for}\quad y\in[x,0]\\ \Big(\eta_{\mathrm{r}}(y)\,\Big|\,\eta_{\mathrm{r}}(0)=\xi(-x)\Big) & \text{for}\quad y\in[x,\infty)\end{cases}$$

where the diffusions $\eta_{\mathrm{l}}(\cdot)$ and $\eta_{\mathrm{r}}(\cdot)$ are conjugate to the diffusion $\xi(\cdot)$ in the sense of the previous paragraphs, and the three bits are independent, conditionally on the values at the patching points. For $x\ge 0$ we define

$$\Lambda_{x,h}(y)\overset{\text{law}}{=}\Lambda_{-x,h}(-y).$$

Since the processes $\eta_{\mathrm{l}}(\cdot)$ and $\eta_{\mathrm{r}}(\cdot)$ almost surely hit 0 in finite time and they are stopped at this hitting time, the process $\Lambda_{x,h}(\cdot)$ almost surely has compact support and the total area under $\Lambda_{x,h}(\cdot)$

(1.17) $$T_{x,h}:=\int_{-\infty}^{\infty}\Lambda_{x,h}(y)dy.$$

is almost surely finite. The random variable $T_{x,h}$ defined in (1.17) has an absolutely continuous distribution. Let

$$\varrho(t,x,h) := \frac{\partial}{\partial t}\mathbf{P}\Big(T_{x,h} < t\Big)$$

be the density of the distribution of $T_{x,h}$. Define $\mathbb{R}_+ \times \mathbb{R} \ni (t,x) \mapsto \pi(t,x) \in \mathbb{R}_+$ as follows

$$\pi(t,x) := \int_0^\infty \varrho(t,x,h)dh.$$

We denote by $\widehat{\varrho}$ and $\widehat{\pi}$ the Laplace transforms of ϱ, respectively, π:

$$\begin{aligned}
\widehat{\varrho}(s,x,h) &:= s\int_0^\infty e^{-st}\varrho(t,x,h)dt = s\mathbf{E}\Big(\exp\{-sT_{x,h}\}\Big)\\
\widehat{\pi}(s,x) &:= s\int_0^\infty e^{-st}\pi(t,x)dt = \int_0^\infty \widehat{\varrho}(s,x,h)dh.
\end{aligned}$$

The following assertion is a direct corollary of Theorem 1.1, but due to its importance in our context, we prefer to formulate it as a separate theorem:

Theorem 1.2. *Let $\Lambda_{x,h}(y)$ be a generalized Ray-Knight process. For any fixed $t >$ 0, respectively $s > 0$, the functions $(-\infty,\infty) \ni x \mapsto \pi(t,x) \in (0,\infty)$, respectively $(-\infty,\infty) \ni x \mapsto \widehat{\pi}(s,x) \in (0,\infty)$, are probability densities. That is:*

$$\int_{-\infty}^\infty \pi(t,x)dx = 1 = \int_{-\infty}^\infty \widehat{\pi}(s,x)dx$$

holds for any generalized Ray-Knight process.

In our forthcoming applications only self-similar generalized Ray-Knight processes will appear, i.e. generalized Ray-Knight processes patched together by triplets $(\widetilde{\mathrm{BES}}^q_{2-\delta}, \mathrm{BES}^q_\delta, \widetilde{\mathrm{BES}}^q_{2-\delta})$. We shall use the superscrtipt (δ,q) for emphasizing this fact. So, in these cases, the various objects defined above will be denoted as follows:

$$\Lambda^{(\delta,q)}_{x,h}(y), \quad T^{(\delta,q)}_{x,h}, \quad \varrho^{(\delta,q)}(t,x,h), \quad \pi^{(\delta,q)}(t,x), \quad \widehat{\varrho}^{(\delta,q)}(s,x,h), \quad \widehat{\pi}^{(\delta,q)}(s,x).$$

From the self-similarity

$$\alpha^{-q/2}\Lambda^{(\delta,q)}_{\alpha x,\alpha^{q/2}h}(\alpha y) \overset{\text{law}}{=} \Lambda^{(\delta,q)}_{x,h}(y), \qquad \alpha^{-(q+2)/2}T^{(\delta,q)}_{\alpha x,\alpha^{q/2}h} \overset{\text{law}}{=} T^{(\delta,q)}_{x,h},$$

directly follow the scaling identities

$$\begin{aligned}
\alpha\pi^{(\delta,q)}(\alpha^{(q+2)/2}t,\alpha x) &= \pi^{(\delta,q)}(t,x),\\
\alpha\widehat{\pi}^{(\delta,q)}(\alpha^{-(q+2)/2}s,\alpha x) &= \widehat{\pi}^{(\delta,q)}(s,x).
\end{aligned}$$

Later, in Chapter 2, when $\delta = 1$ and $q = 1$ throughout that chapter, and there is no danger of confusion, we shall drop the explicit notation of the superscript.

1.3. Limit Theorems for Self-Interacting Random Walks.

1.3.1. Proto-Theorems.

The limit theorems referring to the exponentially-, subexponentially-, polynomially self-repelling and the asymptotically free walks are *syntactically* very similar: there is first a Ray-Knight type functional limit theorem for the local time process (stopped at inverse local times), i.e. for $\Lambda^*_{z,m}(y)$, a corollary to this one is a limit theorem for the asymptotics of hitting times $T^*_{x,m}$, and eventually the full information contained in this corollary is converted to a limit theorem for the distribution of the late time position of the random walk. The most economic way of presenting these results is the following: First we formulate the common syntactical form of these theorems. In this general formulation there are some free parameters: the scaling exponents γ and ν, the dimensions δ, $2-\delta$ and the power q of the Bessel processes composing the generalized Ray-Knight processes arising as scaling limits of the local time processes, finally a time-scale factor τ arising in the same limit theorems. These parameters will be explicitly identified for the different concrete cases later in this section. This compact presentation of the results shouldn't mislead the reader: the asymptotic behaviour of the different self-interacting walks is rather varied and very sensitive to the concrete form of the weight function defining the self-interaction of the RW trajectories. Also, the techniques of proofs are rather different in the exponentially- and subexponentially self-repelling cases at one hand (cf [T1], [T2]), and the polynomially self-repelling and asymptotically free cases on the other (cf [T3]).

The weakly reinforced walk (with $w(n) \sim n^\alpha$, $\alpha \in (0,1)$) behaves even syntactically differently: the limit theorems for that case are formulated at the end of this section.

The following is a generalized Ray-Knight theorem for the local time process of exponentially-, subexponentially, polynomially self-repelling and asymptotically free random walks.

Theorem 1.3. *Let $x \in \mathbb{R}$, $h \in \mathbb{R}_+$ be fixed and the superscript $*$ stand for either U (upcrossing) or D (downcrossing). Then*

$$N^{-\gamma}\tau^{-1}\Lambda^*_{[Nx],[N^\gamma \tau h]}([Ny]) \Rightarrow \Lambda^{(\delta,q)}_{x,h}(y), \qquad \text{as } N\to\infty, \tag{1.18}$$

in the function space $D(-\infty,\infty)$ endowed with the Skorohod topology.

Using the identity (1.2) we get the following:

Corollary. *Under the same conditions*

$$N^{-(\gamma+1)}\tau^{-1}T^*_{[Nx],[N^\gamma\tau h]} \Rightarrow T^{(\delta,q)}_{x,h}, \qquad \text{as } N\to\infty.$$

Let $s>0$ be fixed and θ_N a geometrically distributed stopping time which is independent of the walk X_i,

$$\mathbf{P}\Big(\theta_N = n\Big) = (1-e^{-s/N})e^{-sn/N}. \tag{1.19}$$

The next statement is a limit theorem for the distribution of the location of the SIRW, stopped at the random stopping time θ_N with $N\to\infty$. It follows from the conversion of the full information contained in the previous Corollary.

Theorem 1.4. *Let $s>0$ and $x\in\mathbb{R}$ be fixed and θ_N a geometric stopping time, independent of the walk X_i, distributed according to (1.19). Then*

$$\mathbf{P}\Big(N^{-\nu}X_{\theta_N} < x\Big) \to \int_{-\infty}^{x} \widehat{\pi}^{(\delta,q)}(\tau s, y)dy, \qquad \text{as } N\to\infty, \tag{1.20}$$

where

$$\nu := \frac{1}{\gamma+1}$$

Remarks: (1) In the original papers a slightly stronger statement was proved: the local version of this limit theorem, i.e. pointwise convergence of the properly defined density functions, rather than convergence of the distribution functions.
(2) The statement in Theorem 1.4 is a little bit short of stating the limit theorems for *deterministic time*:

$$\mathbf{P}\Big(N^{-\nu}X_{[Nt]} < x\Big) \to \int_{-\infty}^{x} \pi^{(\delta,q)}(\tau^{-1}t, y)dy, \qquad \text{as } N\to\infty, \tag{1.21}$$

In order to convert (1.20) to (1.21) some refined Tauberian argument would be needed, which we were not able to push through. But, of course, we can conclude that, if $X_{[Nt]}$ obeys any limit law as $N\to\infty$, then (1.21) also must hold.

1.3.2. Simple symmetric random walk.

For sake of comparison we recall that simple symmetric random walk is just a very special case of our general notion of self-interacting random walks, with weight function

$$w(n) = \text{const.}$$

In this case Theorem 1.3 is a compact formulation of the invariance principle form of the classical Ray-Knight theorems, see e.g. [K], [R], [KW], etc. The scaling exponents are

$$\gamma = 1, \qquad \nu = \frac{1}{2}$$

The dimension and power of the Bessel processes composing the generalized Ray-Knight process on the right hand side of (1.18) are

$$\delta = 2, \qquad 2-\delta = 0, \qquad q = 2.$$

That is: the generalized Ray-Knight process is patched together by the triplet $(\widetilde{\mathrm{BES}}_0^2, \mathrm{BES}_2^2, \widetilde{\mathrm{BES}}_0^2)$ on the intervals $(-\infty, x\wedge 0]$, $[x\wedge 0, x\vee 0]$, $[x\vee 0, \infty)$. The time-scale factor τ is

$$\tau = 1.$$

It is worth comparing these data with the corresponding ones for the different truely interacting cases.

1.3.3. Exponentially self-repelling (or myopic) walk.

This model has some notoriety in the probabilistic and mathematical physics literature. For a historical account see [Lw] and [MS]. Theorems 1.3 and 1.4 and for this model were proved in [T1], for detailed proofs see that paper. The interaction is defined by the exponential weight function

$$w(n) = \exp(-\beta n)$$

where $\beta > 0$ is a fixed parameter.

Theorems 1.3 and 1.4 hold with the scaling exponents

$$\gamma = \frac{1}{2}, \qquad \nu = \frac{2}{3}.$$

The dimension and power of the Bessel processes composing the generalized Ray-Knight process on the right hand side of (1.18) are

$$\delta = 1, \qquad 2-\delta = 1, \qquad q = 1.$$

The generalized Ray-Knight process is patched together by $(\widetilde{\mathrm{BES}}_1^1, \mathrm{BES}_1^1, \widetilde{\mathrm{BES}}_1^1)$ on the intervals $(-\infty, x\wedge 0]$, $[x\wedge 0, x\vee 0]$, $[x\vee 0, \infty)$. That is: absorbed Brownian motions on the two wings and reflecting Brownian motion in the central interval. The time-scale factor τ is

$$\tau = 2\sqrt{\frac{\sum_{z\in\mathbb{Z}} z^2 \exp(-\beta z^2)}{\sum_{z\in\mathbb{Z}} \exp(-\beta z^2)}}. \tag{1.22}$$

1.3.4. Sub-exponentially self-repelling (or generalized myopic) walk.

This family of models was investigated in [T2]. The interaction is defined now by the stretched exponential weight function

$$w(n) = \exp(-\beta n^{\kappa})$$

where $\beta > 0$ and $\kappa \in (0,1)$ are fixed parameters. So, this family of models interpolates between the simple symmetric random walk (with $\kappa = 0$) and the exponentially (or myopic) self-repelling walk (with $\kappa = 1$).

Theorems 1.3 and 1.4 hold with the scaling exponents

$$\gamma = \frac{1}{\kappa+1}, \qquad \nu = \frac{\kappa+1}{\kappa+2}.$$

The dimension and power of the Bessel processes composing the generalized Ray-Knight process on the right hand side of (1.18) are

$$\delta = 1, \qquad 2-\delta = 1, \qquad q = \frac{2}{\kappa+1}.$$

The generalized Ray-Knight process is patched together by ($\widetilde{\mathrm{BES}}_1^{2/(\kappa+1)}$,$\mathrm{BES}_1^{2/(\kappa+1)}$, $\widetilde{\mathrm{BES}}_1^{2/(\kappa+1)}$) on the intervals $(-\infty, x\wedge 0]$, $[x\wedge 0, x\vee 0]$, $[x\vee 0,\infty)$. That is: absorbed Brownian motions raised to power $2/(\kappa+1)$ on the two wings and reflecting Brownian motion raised to the same power in the central interval. The time-scale factor τ is

$$\tau = 2\left(\frac{(\kappa+1)^2}{2^{\kappa+2}\kappa\beta}\right)^{1/(\kappa+1)}.$$

Remarks: (1) It is worth noting that in the limit $\kappa \to 1$ the exponents γ, ν, the dimension δ and the power q behave continuously but the time-scale-factor τ doesn't:

$$\lim_{\kappa\to 1}\tau(\kappa) = \sqrt{\frac{2}{\beta}} \neq \tau(1)$$

The discrepancy is caused by interchanging the order of two limiting procedures. For details see [T2].

(2) In the limit $\kappa \to 0$ the behaviour is even stranger. Now only the scaling exponents γ, ν and the power q behave continuously, the dimension δ and the time-scale-factor τ do not:

$$\begin{aligned}
&\lim_{\kappa\downarrow 0}\delta(\kappa) = 1 \neq 2 = \delta(0),\\
&\lim_{\kappa\downarrow 0}\tau(\kappa) = \infty \neq 1 = \tau(0).
\end{aligned}$$

The proper meaningful limit to be taken, is

$$\kappa \to 0, \qquad \beta \to \infty, \qquad \kappa\beta \to \alpha \in (0,\infty).$$

In this limit the weight function becomes

$$\frac{w(n)}{w(m)} = \exp\{-\beta(n^\kappa - m^\kappa)\} \to \frac{n^{-\alpha}}{m^{-\alpha}}.$$

That is: in this limit we get *polynomial self-repelling mechanism*, to be treated in the next paragraph.

1.3.5. Polynomially self-repelling walk.

This model was treated in detail in [T3]. The weight function decays now as a power of n:

$$w(n) = (1+\alpha)^{-1}(n/2)^{-\alpha} - B(1+\alpha)^{-2}(n/2)^{-\alpha-1} + \mathcal{O}(n^{-\alpha-2}),$$

where $\alpha \in (0,\infty)$ and $B \in \mathbb{R}$ are fixed parameters. It is also assumed that $w(n)$ is monotone non-increasing function of $n \in \mathbb{N}$.

Theorems 1.3 and 1.4 hold with the scaling exponents

$$\gamma = 1, \qquad \nu = \frac{1}{2}.$$

The dimension and power of the Bessel processes composing the generalized Ray-Knight process on the right hand side of (1.18) are

$$\delta = 1, \qquad 2-\delta = 1, \qquad q = 2.$$

That is: the generalized Ray-Knight process is patched together by ($\widetilde{\mathrm{BES}}^2_1$, BES^2_1, $\widetilde{\mathrm{BES}}^2_1$) on the intervals $(-\infty, x\wedge 0]$, $[x\wedge 0, x\vee 0]$, $[x\vee 0,\infty)$. In plain words: squared absorbed Brownian motions on the two wings and squared reflecting Brownian motion in the central interval. The time-scale factor τ is

$$\tau = \frac{1}{2\alpha+1}$$

Remark: It is rather surprizing (at least for the author) that the limiting Ray-Knight process does not depend on the power α of the polynomial self-repellence.

1.3.6. Asymptotically free walk.

The weight function is:

$$w(n) = 1 - 2Bn^{-1} + \mathcal{O}(n^{-2}),$$

where $B \in \mathbb{R}$ is fixed parameter. It is also assumed that $w(n)$ is either monotone non-increasing or monotone non-decreasing function of $n \in \mathbb{N}$. For details see [T3]

Theorems 1.3 and 1.4 hold with the scaling exponents

$$\gamma = 1, \qquad \nu = \frac{1}{2}.$$

The dimension and power of the Bessel processes composing the generalized Ray-Knight process on the right hand side of (1.18) are

$$\begin{aligned}\delta &= 2w(0)^{-1} + 2\sum_{j=1}^{\infty}\Big(w(2j)^{-1} - w(2j-1)^{-1}\Big)\\ &= 2\sum_{j=0}^{\infty}\Big(w(2j)^{-1} - w(2j+1)^{-1}\Big),\\ q &= 2\end{aligned}$$

The generalized Ray-Knight process is patched together by $(\widetilde{\text{BES}}^2_{2-\delta}, \text{BES}^2_\delta, \widetilde{\text{BES}}^2_{2-\delta})$ on the intervals $(-\infty, x\wedge 0]$, $[x\wedge 0, x\vee 0]$, $[x\vee 0,\infty)$. That is: squared Bessel processes of dimension $2-\delta$ stopped at first hiting of 0, at the two wings and a squared Bessel process of dimension δ, in the central interval. Note that if the weight function $w(n)$ is monotone non-increasing, i.e. the SIRW is self-repelling, then $\delta\in(0,2]$. On the other hand, if the weight function $w(n)$ is monotone non-decreasing, i.e. the SIRW is self-attracting, then $\delta\in[2,\infty)$. The time-scale factor τ is

$$\tau = 1.$$

1.3.7. Weakly reinforced walk.

The model was studied in [T4]. The monotone non-decreasing weight function

$$w(n) = (1-\alpha)^{-1}(n/2)^{\alpha} - B(1-\alpha)^{-2}(n/2)^{\alpha-1} + \mathcal{O}(n^{\alpha-2}),$$

with $\alpha\in(0,1)$ and $B\in\mathbb{R}$ fixed, defines a self-attracting (reinforced) random walk. It follows from results of [D1], that $w(n)\sim n^{\alpha}$ with $\alpha>1$ would define an excessively strong self-attraction, causing that a.s. the random walker eventually sticks to one single edge of the lattice $\mathbb{Z}$, jumping forth-and-back indefinitely. The weight functions $w(n) = Cn$, with $C>0$ fixed define the (linearly) reinforced random walks. Results of [CD] and [P] imply that this walk has an asymptotic (random) distribution on $\mathbb{Z}$, *without any scaling.* This is why we restrict our investigations to $\alpha\in(0,1)$.

In this case our asymptotic results are structuraly different from the previous ones.

Due to our assumptions, the infinite sum

$$D := \sum_{j=0}^{\infty}\Big(w(2j)^{-1} - w(2j+1)^{-1}\Big)$$

is convergent and $D\in(0,\infty)$.

Theorem 1.5. *Let $x \in [0,\infty)$ and $h \geq 0$ be fixed. Then*

$$\sup_y \left| N^{-1/(1-\alpha)} \Lambda_{[Nx],[N^{1/(1-\alpha)}h]}([Ny]) - \left\{ h^{1-\alpha} + D(x - |y-x|) \right\}_+^{1/(1-\alpha)} \right| \xrightarrow{\mathbf{P}} 0$$

as $N \to \infty$.

Remark: Note that the non-trivial scaling of the local time process provides *convergence in probability to a deterministic function rather than convergence in distribution to a genuinely stochastic process.* In this non-trivial scaling the fluctuations of the local time process become negligeable.

Corollary. *Under the same conditions*

$$N^{-(2-\alpha)/(1-\alpha)} T_{[Nx],[N^{1/(1-\alpha)}h]} \xrightarrow{\mathbf{P}} \frac{2-2\alpha}{2-\alpha} D\Big(x + \big(D^{-1}h\big)^{1-\alpha}\Big)^{(2-\alpha)/(1-\alpha)}$$

as $N \to \infty$.

Again, we can convert these information into a limit theorem for the distribution of the late-time position of the random walker.

Theorem 1.6. *Let $s > 0$ and $x \in \mathbb{R}$ be fixed and θ_N a geometric stopping time independent of the walk X_i, distributed according to (1.19). Then*

$$\mathbf{P}\Big(N^{-(1-\alpha)/(2-\alpha)} X_{\theta_N} < x \Big) \to \int_{-\infty}^{x} \widehat{p}_D^{(\alpha)}(s,y)dy$$

as $N \to \infty$, where

$$\widehat{p}_D^{(\alpha)}(s,x) := s \int_0^\infty e^{-st} p_D^{(\alpha)}(t,x) dt$$

and

$$p_D^{(\alpha)}(t,x) := \frac{1}{2-2\alpha} \left(\frac{2-2\alpha}{2-\alpha} \cdot \frac{D}{t} \right)^{\frac{1}{2-\alpha}} \left\{ \left(\frac{2-\alpha}{2-2\alpha} \cdot \frac{t}{D} \right)^{\frac{1-\alpha}{2-\alpha}} - |x| \right\}_+^{\frac{\alpha}{1-\alpha}}$$

2. The True Self-Repelling Motion

2.1. Setup.

In [T1] a limit theorem was proven for the *one dimensional marginal distributions* of $X_t^{(n)} := n^{-2/3}X_{[nt]}$, where X_j was the myopic self-avoiding random walk on $\mathbb{Z}$, defined by exponentially decaying weight function $w(n) = \exp(-\beta n)$. The problem of *invariance principle*, i.e. that of the weak convergence of the *process* $X_t^{(n)}$, as $n \to \infty$, was completely left open. So much so, that even the slightest guess about the possible limiting process was not formulated. The present section of this survey is based on [TW] and devoted to the construction of the presumed limit-process: a robust, self-similar stochastic process $\mathbb{R}_+ \ni t \mapsto X_t \in \mathbb{R}$, with all the natural properties requested from a locally self-repelling continuous motion.

I should emphasize that our problem is not an isolated one raised for its own sake. For a very closely related question see e.g. [DR] where the authors ask about the possible scaling limit of processes satisfying the stochastic differential equation

$$dY_t = \left(\int_0^t f(Y_t - Y_u)du \right) dt + dB_t,$$

where $f : \mathbb{R} \to \mathbb{R}$ is a smooth function of compact support which is either
(1) even: $f(-x) = f(x)$, and nonnegative: $f(x) \geq 0$, or
(2) odd: $f(-x) = -f(x)$, and satisfies $\text{sign}(f(x)) = \text{sign}(x)$.
These two possibilities define two different kinds of self-repellence mechanisms of the trajectories. Durrett and Rogers study in detail the asymptotics in the first case and they get ballistic behaviour, i.e. a weak law for $t^{-1}Y_t$, as $t \to \infty$. About the second choice, which is the more interesting and natural one, they say that it would be interesting even to guess its asymptotics. After short inspection one could convince her(him)self, that this second case is a very close relative of the myopic self-avoiding walk and it should exhibit the same scaling limit. In the physicists' terminology: they belong to the same class of universality. The phenomenological derivation presented in the next subsection can be easily transposed to the self-repelling polymer proposed by Durrett and Rogers.

Next, I give a summary of the fundamental properties of the true self-repelling motion X_t constructed in [TW]:

Continuity, recurrence. *Almost surely, $X_0 = 0$, the process $t \mapsto X_t$ is continuous on $[0, \infty)$ and for any $x \in \mathbb{R}$, $\{t \geq 0 : X_t = x\}$ is unbounded.*

Scaling. *For all $\alpha > 0$, $(X_{\alpha t}, t \geq 0)$ and $(\alpha^{2/3}X_t, t \geq 0)$ are identical in law.*

Given the limit law for $n^{-2/3}X_n$, this self-similar scale invariance of the limit process is very natural to expect. The scaling property shows that X_t is super-diffusive. The local counterpart of this property is the finite variation of order $3/2$:

Local variation. *For all* $\varepsilon > 0$*, define by induction* $\theta_0^\varepsilon := 0$ *and for all* $n \geq 1$*,*

$$\theta_n^\varepsilon := \inf\{t > \theta_{n-1}^\varepsilon \;:\; |X_t - X_{\theta_{n-1}^\varepsilon}| = \varepsilon\}. \tag{2.1}$$

Then, for all $t \geq 0$*,*

$$\text{P-}\lim_{\varepsilon\downarrow 0} \varepsilon^{3/2} \sup\{n \geq 0 \;:\; \theta_n^\varepsilon \leq t\} = \frac{2}{\sqrt{\pi}} t.$$

Here and in the sequel P- lim stands for limit in probability.

Occupation-time density. *Almost surely, for all* $t \geq 0$*, the occupation-time measure* μ_t *of* X_s *on the time-interval* $[0,t]$*, defined for all Borel subset* A *of* $\mathbb{R}$ *by*

$$\mu_t(A) := \int_0^t 1\!\!1_{\{X_s \in A\}} ds$$

has a bounded density with respect to the Lebesgue measure and this density has a continuous version that we denote by $L_t(\cdot)$*. We call* $L_t(x)$ *the local time of* X *at time* t *and position* x*.*

Markov property of (X_t, μ_t)**.** *The process* $(X_t, L_t(\cdot))_{t\geq 0}$ *,or equivalently the process* $(X_t, \mu_t)_{t\geq 0}$*, is a Markov process.*

In other words, the future of X after t depends only on the occupation-time measure at time t (i.e. μ_t), and on the position of X at time t.

Locality. *The self-interaction is local in the following sense: For all* $t \geq 0$*, the law of* X *just after* t *depends only on* L_t *restricted to the immediate neighbourhood of the point* X_t*. In other words, the process* X_t *is 'feeling' only the self-interaction due to* the germ *of its own past occupation-time measure at the points it is currently visiting.*

For a more precise formal description of this property, see the original publication [TW].

The following property is of crucial importance: it describes in proper mathematical terms the phenomenon of local self-repellence.

Dynamical driving mechanism. *There exists a random set* $I \subset \mathbb{R}_+$*, which is a.s. of full Lebesgue measure, such that for any* $T \in I$

$$\text{P-}\lim_{\varepsilon\downarrow 0} \int_0^T \frac{L_s(X_s + \varepsilon) - L_s(X_s - \varepsilon)}{2\varepsilon} ds = -X_T + \frac{1}{4}\left(\sup_{0\leq s\leq T} X_s + \inf_{0\leq s\leq T} X_s\right). \tag{2.2}$$

Unfortunately, we could prove (2.2) only for the random set of (stopping) times $I \subset \mathbb{R}_+$. Actually, this property should hold for all $T \in \mathbb{R}_+$. Phenomenologically, this equation states that the motion is driven by the negative gradient of the local time at the actual position, as long as the moving point is in the interior of the range swept in the past. This behaviour entitles us to call this process 'truely self-repelling'. In addition, at the edges of this range an instantaneous partial reflection (moving boundary condition) is felt. Indeed: writing (2.2) *formally* in differential form we find:

$$dX_t = -\frac{\partial L_t(X_t)}{\partial x} dt + \left(\text{boundary effects at } \sup_{0\le s\le t} X_s \text{ and } \inf_{0\le s\le t} X_s \right). \tag{2.3}$$

Strictly speaking, (2.3) (or (0.1)) does not make sense mathematically: the local time process is so singular that a 'differential equation' involving its gradient can not be rigorously defined (we shall see that $L_t(\cdot)$ has the same regularity properties as Brownian motion). Nevertheless, this formal way of writing may help the intuition about the dynamics of the process. Note, that there is no 'external noice', or 'external source of randomness' in the driving mechanism. One could think naively that such a mechanism would give rise to a deterministic motion. This is not the case: due to the extremely high singularity of this 'differential equation', (2.3) (or (0.1)) has only truely stochastic solutions.

One of the main novelties of the process X_t is exactly the fact that it is in striking contrast with our traditional intuition about a random motion being driven by local drift and external noice.

In the next section we present a phenomenological derivation of the driving mechanism. Finally, in the last section the main lines of the construction are presented.

2.2. Phenomenological Derivation of the Dynamical Driving Mechanism.

In [T1] a limit theorem was proved, essentially for the distribution of $n^{-2/3}X_n$ as $n \uparrow \infty$, but the natural question of the asymptotics of the *process*

$$X_t^{(N)} := N^{-2/3} X_{[Nt]}, \qquad t \in \mathbb{R}_+ \tag{2.4}$$

in the limit $N \uparrow \infty$ remained open. In the following paragraphs we argue that, if the sequence of processes $t \mapsto X_t^{(N)}$ converges in distribution to a process $t \mapsto X_t^{(\infty)}$, as $N \uparrow \infty$, then the limit process is driven by the gradient of its local time, as claimed in (2.3 (or (0.1)). We warn the reader that the forthcoming argument is based on a somewhat *formal computation* and it is by no means mathematically rigorous, but we hope, it provides a convincing motivation for the construction of a process with the prescribed properties.

Beside the scaled position process $t \mapsto X_t^{(N)}$ defined in (2.4) we define the properly scaled local time process of the exponentially self-repelling walk

$$L_t^{(N)}(x) := N^{-1/3} L_{[Nt]}([N^{2/3}x]), \qquad t \in \mathbb{R}_+, \quad x \in \mathbb{R} \tag{2.5}$$

and we assume that the sequences of processes $X_t^{(N)}$ and $L_t^{(N)}(x)$ converge jointly weakly (in some vague topological space):

$$\left(X_t^{(N)}, L_t^{(N)}(x)\right) \Rightarrow \left(X_t^{(\infty)}, L_t^{(\infty)}(x)\right)$$

where $(t,x) \mapsto L_t^{(\infty)}(x)$ is assumed to be the local time of the process $t \mapsto X_t^{(\infty)}$. Let $\mathcal{F}_n$ be the σ-algebra generated by $(X_0, \dots, X_n)$, then

$$\mathbf{E}\Big(X_{n+1} - X_n \,\Big|\, \mathcal{F}_n\Big) = -\tanh\Big(\beta(L_n(X_n) - L_n(X_n - 1))\Big)$$
$$\mathbf{Var}\Big(X_{n+1} - X_n \,\Big|\, \mathcal{F}_n\Big) = \cosh^{-2}\Big(\beta(L_n(X_n) - L_n(X_n - 1))\Big)$$

So:

$$X_n + \sum_{k=0}^{n-1} \tanh\Big(\beta(L_k(X_k) - L_k(X_k - 1))\Big) =: M_n \tag{2.6}$$

is a martingale with quadratic variation process

$$\langle M \rangle_n = \sum_{k=0}^{n-1} \cosh^{-2}\Big(\beta(L_k(X_k) - L_k(X_k - 1))\Big) < n \tag{2.7}$$

Our object of study is the scaled form of (2.6):

$$N^{-2/3} X_{[Nt]} + N^{-2/3} \sum_{k=0}^{[Nt]-1} \tanh\left(\beta(L_k(X_k) - L_k(X_k - 1))\right) = N^{-2/3} M_{[Nt]} \tag{2.8}$$

The first term on the left-hand side of (2.8) is just $X_t^{(N)}$. From (2.7) in particular it follows that for any $T < \infty$

$$\text{P-}\lim_{N\uparrow\infty} \left(\sup_{0\le t\le T} \left|N^{-2/3} M_{[Nt]}\right|\right) = 0 \tag{2.9}$$

so that the right hand-side of (2.8) is asymptotically negligible. A formal computation of the second term on the left-hand side of (2.8) follows: the first two steps are straightforward transformations using the definitions (2.4) and (2.5) of the scaled process and scaled local time:

$$\begin{aligned}
&N^{-2/3} \sum_{k=0}^{[Nt]-1} \tanh\Big(\beta(L_k(X_k) - L_k(X_k - 1))\Big)\\
=&N^{-1} \sum_{k=0}^{[Nt]-1} N^{1/3} \tanh\Big(\beta(L_{Nk/N}(N^{2/3} X_{k/N}^{(N)}) - L_{Nk/N}(N^{2/3} X_{k/N}^{(N)} - 1))\Big)\\
=&N^{-1} \sum_{k=0}^{[Nt]-1} N^{1/3} \tanh\Big(\beta N^{1/3}(L_{k/N}^{(N)}(X_{k/N}^{(N)}) - L_{k/N}^{(N)}(X_{k/N}^{(N)} - N^{-2/3}))\Big)
\end{aligned}$$

The next step is the formal, non-rigorous one: we treat formally $L_t^{(N)}(x)$ as a smooth function and replace

$$L_t^{(N)}(x) - L_t^{(N)}(x-\delta x) \qquad \text{by} \qquad \frac{\partial L_t^{(N)}(x)}{\partial x}\delta x$$

to get

$$\begin{aligned}
&N^{-2/3}\sum_{k=0}^{[Nt]-1} \tanh\big(\beta(L_k(X_k) - L_k(X_k-1))\big)\\
\text{“}{=}\text{”}\ &N^{-1}\sum_{k=0}^{[Nt]-1} N^{1/3}\tanh\big(\beta N^{1/3}N^{-2/3}\frac{\partial L_{k/N}^{(N)}(X_{k/N}^{(N)})}{\partial x}\big)\\
\text{“}{=}\text{”}\ &\beta N^{-1}\sum_{k=0}^{[Nt]-1} \frac{\partial L_{k/N}^{(N)}(X_{k/N}^{(N)})}{\partial x} + \mathcal{O}(N^{-1/3})\\
\text{“}{\Rightarrow}\text{”}\ &\beta\int_0^t \frac{\partial L_s^{(\infty)}(X_s^{(\infty)})}{\partial x}ds
\end{aligned} \tag{2.10}$$

With the quotation marks “. . ” we intend to emphasize that these last equalities and convergence should not be taken too seriously. From (2.8), (2.9) and (2.10) we get

$$X_t^{(\infty)} + \text{const.}\int_0^t \frac{\partial L_s^{(\infty)}(X_s^{(\infty)})}{\partial x}ds = 0$$

which is indeed somewhat reminiscent of (2.3) (or (0.1)). The effect of ‘pushing the boundaries of the range’ and the right constant in front of the gradient term can not be recovered on this level of formal computations. We repeat again: this computation is nothing like rigorous, but on the phenomenological level it is convincing.

The same reasoning (on the same level of ‘rigour’) can be applied to the ‘polymer model’ proposed by Durrett and Rogers in [DR]:

$$X_t = B_t + \int_0^t\left\{\int_0^s f(X_s - X_u)du\right\}ds$$

where $f:\mathbb{R}\to\mathbb{R}$ is a smooth function of compact support and satisfies $f(-x) = -f(x)$ and $\operatorname{sgn}(f(x)) = \operatorname{sgn}(x)$. Defining $X_t^{(N)} = N^{-2/3}X_{Nt}$, in the limit $N\to\infty$ f transforms into δ' and the same dynamical driving mechanism is found.

2.3. Construction and Primary Properties of the True Self-Repelling Motion.

2.3.1. Main Idea of Construction — Outline

Assume for the moment that there is a continuous process $\mathbb{R}_+ \ni t \mapsto X_t \in \mathbb{R}$ which does have local time (occupation time density) $L_t(x)$ *with due regularity properties.* Define the inverse local time $T(x,h)$ and the Ray-Knight process in the usual way:

$$\begin{aligned} T(x,h) &:= \inf\{t \geq 0 : L_t(x) \geq h\}, \\ \Lambda_{x,h}(y) &:= L_{T(x,h)}(y). \end{aligned} \tag{2.11}$$

The relation (2.11) can be inverted to recover $(x,t) \mapsto L_t(x)$ from $(x,h) \mapsto T(x,h)$:

$$L_t(x) = \inf\{h > 0 : T(x,h) \geq t\}. \tag{2.12}$$

It is a fact that the full Ray-Knight process

$$\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R} \ni (x,h,y) \mapsto \Lambda_{x,h}(y) \in \mathbb{R}_+ \tag{2.13}$$

contains all information about the process $t \mapsto X_t$, i.e. the process $X.$ can be fully reconstructed from its Ray-Knight process. Moreover, only 'half' of the data in (2.13) are actually needed: using the assumed regularity properties of the local time process $L_t(x)$ one can easily prove that for any $x, y \in \mathbb{R}$

$$\Lambda_{y,h}(x) = \sup\{\, h' \,:\, \Lambda_{x,h'}(y) < h \,\} \tag{2.14}$$

and thus full information about

$$\Lambda^+ := \{\, \Lambda_{x,h}(y) \,:\, x \in \mathbb{R},\ h \in \mathbb{R}_+,\ y \geq x \,\}$$

siffices for the reconstruction of the proces $X.$, since, as a first step

$$\Lambda^- := \{\, \Lambda_{x,h}(y) \,:\, x \in \mathbb{R},\ h \in \mathbb{R}_+,\ y \leq x \,\}$$

can be obtained from Λ^+ by applying (2.14).

The reconstruction procedure goes as follows: $T(x,h)$ is recovered from the straightforward identity

$$T(x,h) = \int_{-\infty}^{\infty} \Lambda_{x,h}(y)dy, \tag{2.15}$$

where the domain of integration is actually compact. Given the assumed properties of $X.$ one can see that the random mapping $\mathbb{R} \times \mathbb{R}_+ \ni (x,h) \mapsto T(x,h) \in \mathbb{R}$ is injective, lower semicontinuous (hence, Borel measurable) and maps the Lebesgue measure of its domain $\mathbb{R}\times\mathbb{R}_+$ into the Lebesgue measure of $\mathbb{R}_+$ (hence, in particular, its range is of full Lebesgue measure in $\mathbb{R}_+$). For $t \in \mathrm{Ran}(T)$, using injectivity of the map T, define $(\widetilde{X}_t, H_t)$ as the unique $(x,h) \in \mathbb{R} \times \mathbb{R}_+$, for which $T(x,h) = t$.

$(\widetilde{X}_t, H_t)$ so defined can be extended by continuity to all $t \in \mathbb{R}_+$ and it is not at all surprizing that actually, almost surely, for all $t \in \mathbb{R}_+$

$$(\widetilde{X}_t, H_t) \equiv (X_t, L_t(X_t)).$$

The procedure is very clear, though technically not completely trivial. In this philosophy it is essential that the process $t \mapsto X_t$ is assumed to exist and to have regular local times.

In our case, of course, we do not have the process $t \mapsto X_t$ a priori given, we just want to construct it. Our morale of the reconstruction procedure outlined above is the following: we first *guess* the forward Ray-Knight process Λ^+, then we turn the identities (2.14), (2.15) and (1.12) (in this order) into the definitions of Λ^-, $T(x,h)$ and $L_t(x)$. Next we (try to) apply the reconstruction procedure outlined in the last paragraph into the construction of the process X_t. The main difficulty in this step is that we do not have a priori the necessary regularity properties of Λ, T and L, which were previously guaranteed by the fact that L, T and Λ were the local time-, inverse local time- and Ray-Knight processes of an a priori existing, sufficiently regular process. The process X_t being constructed we have to prove that $L_t(x)$ is *indeed* its local time process. Consequently $T(x,h)$ and $\Lambda_{x,h}(y)$ are indeed its inverse local time, respectively, its Ray-Knight processes. Note, that in this construction the Ray-Knight type theorems come for free. Finally, there are two more things to be done: prove rigorously the dynamical driving mechanism (2.2), and the local variation of order 3/2. These proofs are technically considerably involved, we refer the reader to the original paper [TW].

2.3.2. Forward and Backward lines: the Λ-Process.

Let us guess first what the Λ-process should be. Recall the main result of [T1], that is Theorem 3 of this survey, specified for exponentially (or myopic) self-repelling random walk.

Let $x \in \mathbb{R}$ and $h \in [0, \infty)$ be fixed. Denote by $\Lambda_{x,h}(y)$ a Brownian motion in $\mathbb{R}_+$ defined for $y \in [x, \infty)$ starting at 'time' x from level h obeying the following boundary conditions at 0: in the time interval $[x, x \vee 0]$, $\Lambda_{x,h}(\cdot)$ is instantaneously reflected at 0 and in the time interval $[x \vee 0, \infty)$, $\Lambda_{x,h}(\cdot)$ is absorbed at the first hitting of 0. We call such a process *reflected/absorbed Brownian motion*, abbreviated RAB. Note that this is just $\Lambda^{(1,1)}_{x,h}(y)$ of the first chapter, restricted to $y \geq x$. The compound Ray-Knight type theorem (Theorem 3 of the first chapter) specified for myopic self-repelling random walk states the following weak convergence: for any $x \in \mathbb{R}$ and $h > 0$ fixed,

$$\frac{\Lambda^*_{[Nx],[\sqrt{N}\tau h]}([Ny])}{\tau\sqrt{N}} \Rightarrow \Lambda_{x,h}(y), \qquad y \in [x, \infty)$$

when $N \to \infty$, as *process* in the time parameter y (the constant τ is explicitly given in (84). Recall that $\Lambda^*_{k,m}(l)$ denotes the edge-local-time of the myopic self-repelling walk on $\mathbb{Z}$, stopped at the hiting time $T_{k,m}$. (See section 1.1 for the definitions of the discrete objects.)

It is not explicitly stated in [T1], but the methods of the cited paper allow for the proof of a more general, *joint* weak convergence: let finitely many pairs of coordinates $(x_1, h_1), \ldots, (x_p, h_p) \in \mathbb{R} \times (0, \infty)$ be fixed, then

$$\left(\frac{\Lambda^*_{[Nx_1],[\sqrt{N}\tau h_1]}([Ny_1])}{\tau\sqrt{N}}, \ldots, \frac{\Lambda^*_{[Nx_p],[\sqrt{N}\tau h_p]}([Ny_p])}{\tau\sqrt{N}}\right) \Rightarrow \left(\Lambda_{x_1,h_1}(y_1), \ldots, \Lambda_{x_p,h_p}(y_p)\right)$$
$$y_1 \in [x_1, \infty), \ldots, y_p \in [x_p, \infty)$$

when $N \to \infty$, where $(\Lambda_{x_1,h_1}(\cdot), \ldots, \Lambda_{x_p,h_p}(\cdot))$ are *independent coalescing Brownian motions* starting at times x_k from level h_k, reflected from 0 in the time intervals $[x_k, x_k \vee 0]$, $k = 1, \ldots, p$ and absorbed instanteneously at the first hitting of 0 in the time-intervals $[x_k \vee 0, \infty)$, $k = 1, \ldots, p$, respectively. In other words, independent *coalescing* RABs, or shortly CRABs.

It is clear now what should be the proposed forward Ray-Knight process Λ^+ to be used in the construction outlined in the previous subsection: for any finite collection of starting points $(x_1, h_1), \ldots, (x_p, h_p) \in \mathbb{R} \times (0, \infty)$ the processes $(\Lambda_{x_1,h_1}(\cdot), \ldots, \Lambda_{x_p,h_p}(\cdot))$ should be independent coalescing RAB-s and the full process Λ^+ must have due regularity properties requested by the forthcoming construction. The existence and uniqueness of such a process is stated in the next theorem:

Theorem 2.1.
There exists a random process $(x, h, y) \mapsto \Lambda_{x,h}(y) \in \mathbb{R}_+$ defined on $\{(x, h, y) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R} : y \geq x\}$ such that:
(i) for any finite collection of starting points $(x_1, h_1), \ldots, (x_p, h_p) \in \mathbb{R} \times (0, \infty)$ the processes $(\Lambda_{x_1,h_1}(\cdot), \ldots, \Lambda_{x_p,h_p}(\cdot))$ are independent coalescing RAB-s;
(ii) almost surely, for all $(x, h) \in \mathbb{R} \times \mathbb{R}_+$, $\Lambda_{x,h}(x) = h$;
(iii) almost surely, for all (x_1, h_1), (x_2, h_2) in $\mathbb{R} \times \mathbb{R}_+$ and $z \geq y \geq \max\{x_1, x_2\}$:

$$[\Lambda_{x_1,h_1}(y) < \Lambda_{x_2,h_2}(y)] \quad \Longrightarrow \quad [\Lambda_{x_1,h_1}(z) \leq \Lambda_{x_2,h_2}(z)];$$

(iv) almost surely, for all $x \leq y$, the mapping $h \mapsto \Lambda_{x,h}(y)$ is left-continuous on $(0, \infty)$.

Furthermore: the law of the process $(x, h, y) \mapsto \Lambda_{x,h}(y)$ is uniquely determined by the above properties.

Remark: This theorem is very close to similar results stated by R. Arratia in the unpublished work [A1, A2]. For comments on the details of these similarities see our original paper [TW].

The proof of this theorem is rather technical. Essentially the following happens: there is no difficulty in constructing a countable family of independent CRABs, starting from all points $(x, h) \in \mathbb{R} \times \mathbb{R}_+$ with rational coordinates. These CRAB trajectories will be dense everywhere in $\mathbb{R} \times \mathbb{R}_+$. The full process Λ^+ will be obtained from this countable family, by the unique extension which is left-continuous in the h variable, as required in (iv) of the Theorem. We mention here that R. Arratia requires a different regularity property (instead of left continuity in h) which guarantees a certain flow-property of the process Λ. For details see [A1, A2] or [TW].

The system of forward-lines $\{\, \Lambda^+_{x,h}(y) \,:\, x \in \mathbb{R},\, h \in \mathbb{R}_+,\, y \in [x, \infty) \,\}$ being constructed, next we define the system of backward lines $\{\, \Lambda^-_{x,h}(y) \,:\, x \in \mathbb{R},\, h \in \mathbb{R}_+,\, y \in (-\infty, x] \,\}$ by formula (2.14). The following theorem states a subtle duality between forward- and backward lines:

Theorem 2.2. *The two processes* $(x, h, y) \mapsto \Lambda_{(x,h)}(y)$ *and* $(x, h, y) \mapsto \Lambda_{(-x,h)}(-y)$ *are identical in law.*

The proof of this theorem is far non trivial: it relies on fine topological properties of the process Λ.

2.3.3. Construction and First Properties of the Process $t \mapsto X_t$.

Given the full process $\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R} \ni (x, h, y) \mapsto \Lambda_{x,h}(y) \in \mathbb{R}_+$ we now *define* the random mapping $\mathbb{R} \times \mathbb{R}_+ \ni (x, h) \mapsto T(x, h) \in \mathbb{R}_+$ by the formula (2.15). The domain of integration in (2.15) is actually compact, almost surely for all $(x, h) \in \mathbb{R} \times \mathbb{R}_+$. From the properties of the process $\Lambda_{x,h}(y)$ we can derive the following important facts about the random mapping T:

Theorem 2.3. *The map* $T : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+$ *almost surely has the following properties:*
(i) T *is injective; for all* $x \in \mathbb{R}$, $h \mapsto T(x, h)$ *is strictly increasing and continuous from the left.*
(ii) T *is lower semicontinuous; in particular, it is Borel measurable.*
(iii) T *maps the Lebesgue measure on* $\mathbb{R} \times \mathbb{R}_+$ *into the Lebesgue measure on* $\mathbb{R}_+$ *and the range of* T *is of full Lebesgue measure in* $\mathbb{R}_+$.

We are ready now to define the process $t \mapsto X_t$. For any $t \geq 0$ define the set

$$P_t := \bigcap_{\varepsilon > 0} \overline{\{(x, h) \in \mathbb{R} \times \mathbb{R}_+ \;:\; T(x, h) \in (t - \varepsilon, t + \varepsilon)\}}.$$

As limit of nested compacta, P_t is clearly nonempty. From the properties of $T(x,h)$ (with some additional technical details) it follows that actually P_t consists of exactly one single point of $\mathbb{R}\times\mathbb{R}_+$ and the following definition makes sense:

Definition. *For all $t\geq 0$, we denote*

$$P_t =: \{(X_t, H_t)\}$$

and call $\mathbb{R}_+ \ni t\mapsto X_t\in\mathbb{R}$ the true self-repelling motion.

Remark: As we shall soon see

$$H_t = L_t(X_t)$$

where $L_t(x)$ is the occupation time density of $X_\cdot$.

The first properties of the process $t\mapsto X_t$ readily follow:

Theorem 2.4.
(i) (Continuity.)
Almost surely, $t\mapsto (X_t,H_t)$ is continuous on $[0,\infty)$ and $(X_0,H_0)=(0,0)$.
(ii) (Recurrence.)
Almost surely, the set $\{t\in\mathbb{R}_+ : X_t = x\}$ is unbounded for any $x\in\mathbb{R}$.
(iii) (Symmetry.)
The processes $t\mapsto (X_t,H_t)$ and $t\mapsto(-X_t,H_t)$ are identical in law.
(iv) (Scaling.)
For any $\alpha>0$, the processes $t\mapsto(X_{\alpha t},H_{\alpha t})$ and $t\mapsto(\alpha^{2/3}X_t,\alpha^{1/3}H_t)$ are identical in law.

Next we turn to the occupation time measure and local time of the process constructed. For all $t\geq 0$ and $y\in\mathbb{R}$, we now *define* $L_t(y)$ by formula (2.12). Since almost surely, for all $y\in\mathbb{R}$, $h\mapsto T(y,h)$ is *strictly* increasing, $t\mapsto L_t(y)$ is necessarily continuous, and of course, also monotone non-decreasing. The following theorem states that (as expected) $L_t(x)$ is actually the occupation time density (i.e. local time) of our process $t\mapsto X_t$.

Theorem 2.5. *Almost surely, for any measurable, bounded, real-valued function $g:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ and for any time $t\geq 0$,*

$$\int_0^t g(s,X_s)ds = \int_{-\infty}^{\infty}\left\{\int_0^t g(s,y)d_sL_s(y)\right\}dy.$$

Furthermore: The local time process is regular in the following sense: The mapping $t\mapsto L_t(\cdot)$ is non-decreasing continuous from $[0,\infty)$ into the space of continuous real-valued functions with compact support, with topology induced by uniform convergence on compact intervals.

2.3.4. Local Variation and Dynamics.

The two most striking characteristic features of the newly constructed process are its local behaviour and its dynamical driving mechanism. The first one shows its *analytic*, while the second one its *phenomenological/physical* peculiarity.

Consider the stopping times θ_n^ε explicitly defined in (2.1) and let

$$N_t^\varepsilon := \sup\{n \ : \ \theta_n^\varepsilon \leq t\}.$$

The following theorem states that the process $t \mapsto X_t$ has nontrivial variation of order 3/2, as opposed to the nontrivial quadratic variation of semimartingales:

Theorem 2.6. *For all* $t \geq 0$*,*

$$\text{P-}\lim_{\varepsilon \downarrow 0} \sum_{n=1}^{N_t^\varepsilon} \left| X_{\theta_n^\varepsilon} - X_{\theta_{n-1}^\varepsilon} \right|^{3/2} \equiv \text{P-}\lim_{\varepsilon \downarrow 0} \varepsilon^{3/2} N_t^\varepsilon = \frac{2t}{\sqrt{\pi}}.$$

Finally, the next theorem gives rigorous sense to the dynamical driving mechanism of our process, hinted at in the introduction. In plain words: the process is driven by the negative gradient of its own local time, at the current time and position. The phenomenological, non-rigorous derivation of this behaviour was presented in subsection 2.2. of this survey.

Theorem 2.7. *For any* $(x_0, h_0) \in \mathbb{R} \times \mathbb{R}_+$ *fixed*

(2.16)
$$\text{P-}\lim_{\varepsilon \downarrow 0} \int_0^{T(x_0,h_0)} \frac{L_s(X_s + \varepsilon) - L_s(X_s - \varepsilon)}{2\varepsilon} ds = -X_{T(x_0,h_0)} + \frac{1}{4}\left(\sup_{0 \leq s \leq T(x_0,h_0)} X_s + \inf_{0 \leq s \leq T(x_0,h_0)} X_s \right).$$

Remarks: (1) It would be more satisfactory to state this statement for all $t \in \mathbb{R}_+$, i.e. for deterministic times:

$$\text{(2.17)} \quad \text{P-}\lim_{\varepsilon \downarrow 0} \int_0^t \frac{L_s(X_s + \varepsilon) - L_s(X_s - \varepsilon)}{2\varepsilon} ds = -X_t + \frac{1}{4}\left(\sup_{0 \leq s \leq t} X_s + \inf_{0 \leq s \leq t} X_s \right).$$

This, stronger statement is very likely to be true, but we were not able to prove it. Note, however, that according to Theorem 2.3 the range of the random map $\mathbb{R} \times \mathbb{R}_+ \ni (x, h) \mapsto T(x, h) \in \mathbb{R}_+$ is almost surely of full Lebesgue measure.

(2) The $\frac{1}{4}(\cdots)$ term on the right hand side of (2.16) is a 'moving boundary term': it is due to the fact that at the two edges of the domain swept in the past, the process gets some instantaneous extra outwards kick, a similar effect as the 'Brownian motion perurbed at its extrema' (see e.g. [CPY, D2, D3, PW]). This term influences the process only when it is at its previous minimum or maximum, i.e. when the

process pushes the boundaries of its range swept. It is possible to define a version of our process with stationary increments:

$$\widehat{X}_t := \operatorname*{w-lim}_{t_0\to\infty} \left(X_{t_0+t} - X_{t_0}\right).$$

This variant will have exactly the same proparties as X_t, just the moving boundary terms on the right hand side of (2.16) will be missing, due to the fact that in this limit the boundaries of the swept range will be already pushed away to $\pm\infty$.

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