1980 jan

Note on the "Magic Cube"

T
Tóth Bálint+

Since 1977 the Hungarian "Magic Cube" has conquered the world. This toy is a cube of liniar dimensions of approximately 6 cm. composed of 26 little plastic cubes. /Apparently there are 27 but in the center one little cube is missing./ The little cubes are so ingeniously assembled together, that all the six face-layers of the Cube can be rotated around their axes /see fig.1./. The really ingenious technical solution of this construction is not relevant from the mathematical point of view which I shall adopt in this note.

In the basic ordered state each of the six faces of the Cube has its own colour /fig.2./. But after a few random rotations of the faces the six different colours mix up on them and the Cube becomes motley. The point is to reestablish the initial basic order. This task seems practically hopeless without a wellformed strategy.

In 1979 Kéri Gerzson had published an algorithmic-programmable solution of the problem with the help of which one can rearrenge the Cube from any mixed state with a reduced number /max.186/ of rotations.

In this note I shall give a more formal description in the sense of matrix representations of the operations on the Cube. Kéri's results are formulable in this formal frame, but this representation gives a complete mathematical description of the Cube. In this sense my note can be considered as a completion to Kéri's work.

Terminology

The Cube contains three kinds of pieces:

--corner-pieces /8/

--edge-pieces /12/

--face-center-pieces /6/

The first two can change their situation on the Cube by switching the faces.The face-center-pieces are fixed.This is evident from the construction /fig.1,/.The situation of a mobile piece on the Cube is characterised by its place /site/ and its position on this place. Corner-pieces can have three,edge-pieces two different positions on one site.Thus for a full description of one state of the Cube,one has to specify the places and positions of the twenty mobile pieces. For the description of a transformation of the Cube one has to specify the changes suffered by these.

Further:corner-pieces can permute but on corner-sites and likewise can do edge-pieces.

The group of possible transformations of the Cube,or the proper group of the Cube is a subgroup of a group formed by the direct product of two groups,which themselves are direct products.The first of these is the following: the direct product of the group of permutations of the eight corner-pieces with the groups of position-changes of these on fixed sites (G_1). The second: the direct product of the group of permutations of the twelve edge-pieces with the groups of position changes of these on fixed sites /G_2/

$$\mathcal{G} \subset G = G_1 \times G_2$$

$\mathcal{G}$,the proper group is generated by six generating elements /elementary transformations/ :the rotations of the six faces around their axes in pozitive trigonometric sense with one quarter rotation. $(g_i;\ i=1,2,\dots 6)$

The representation

The group of corner-pieces, G_1,has a natural eight-dimensional complex matrix representation. For the fixation of the representation one has to choose a numeration of the corner-sites from 1 to 8 and a notation of the positions. /The choice is arbitrary, but once fixed the representation is also fixed./ I chose the one notated on fig. 3. The permutations are representable in the usual natural way. /In every line and column of the 8x8 matrices there is one 1 in the rest are 0s./ But this representation does not reflect the possible changes of positions. For that, one has to introduce other signs in the place of 1s. /Direct product of groups $\longrightarrow$ tensorial product of representations./

One can observe, that the group of changes of position of one corner-piece is simply the third-order cyclic group $C(3)$ /fig. 4./. Thus it can be represented in the following way:

$$a(=b^2) \longrightarrow e^{\frac{2i\pi}{3}} \; ; \; b(=a^2) \longrightarrow e^{\frac{4i\pi}{3}} \; ; \; \varepsilon(=a^3=b^3) \longrightarrow 1$$

One can now easily guess the faithful representation of the group G_1 : it will be the group of 8x8 matrices, which formally resemble the natural representations of 8th order permutations, but instead of 1s complex numbers represent the changes of positions of the corner-pieces. /This is exactly the tensorial product of the [redacted] representations mentioned above./

For the edge-pieces the situation is essentially similar. There are two differences: we shall have 12x12 matrices and real numbers /± 1/, because the group of position-changes of an edge-piece is the second order cyclic group $C(2)$ ./fig. 4._/

I shall denote the "corner-part" of the representation with D , the "edge-part" with Δ . The complete representation is:

$$\forall g \in \mathcal{G} \qquad \mathcal{D}(g) = \begin{pmatrix} D(g) & 0 \\ 0 & \Delta(g) \end{pmatrix}$$

In this representation the generating elements of the proper group $\mathcal{G}$ will be the following: (I shall adopt the notation $D(g_i) = D_i$; $\Delta(g_i) = \Delta_i$ for $i = 1, 2, \ldots, 6$), $\varepsilon = e^{\frac{2i\pi}{3}}$

$$D_1 = \begin{pmatrix} 0&0&0&1&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \end{pmatrix}; \quad D_2 = \begin{pmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \end{pmatrix}; \quad D_3 = \begin{pmatrix} 0&0&0&0&0&0&0&\varepsilon \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ \varepsilon^2&0&0&0&0&0&0&0 \\ 0&0&0&\varepsilon&0&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&\varepsilon^2&0&0&0 \end{pmatrix};$$

$$D_4 = \begin{pmatrix} 0&\varepsilon^2&0&0&0&0&0&0 \\ 0&0&0&0&0&0&\varepsilon&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&\varepsilon^2 \\ \varepsilon&0&0&0&0&0&0&0 \end{pmatrix}; \quad D_5 = \begin{pmatrix} 1&0&0&0&0&0&0&0 \\ 0&0&\varepsilon^2&0&0&0&0&0 \\ 0&0&0&0&0&\varepsilon&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&\varepsilon^2&0 \\ 0&\varepsilon&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1 \end{pmatrix}; \quad D_6 = \begin{pmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&\varepsilon^2&0&0&0&0 \\ 0&0&0&0&\varepsilon&0&0&0 \\ 0&0&0&0&0&\varepsilon^2&0&0 \\ 0&0&\varepsilon&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \end{pmatrix}$$

$$\Delta_1 = \begin{pmatrix} 0&0&0&1&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \end{pmatrix}; \quad \Delta_2 = \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \end{pmatrix}$$

$$\Delta_3 = \begin{pmatrix} 0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0 \\ 1&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix}; \quad \Delta_4 = \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&-1&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&-1&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&-1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&-1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \end{pmatrix}$$

$$\Delta_5 = \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \end{pmatrix}; \quad \Delta_6 = \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&-1 \\ 0&0&0&0&0&0&0&0&0&0&-1&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&-1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&-1&0&0&0&0&0&0&0 \end{pmatrix}$$

where the indices refere to the faces as follows: 1---front-face, 2---back-face, 3---upper-face, 4---left-side-face, 5---lower-face, 6---right-side-face.

Theorems concerning the Cube

Theorem 1: There exists no [redacted] transformation in the proper group of the Cube, which can change the position of one corner-piece /at fixed site/, leaving unchanged the situation of the other seven. /No matter ~~of~~ what changes in the "edge-piece-space" occur./

~~Proof~~

Proof : $i=1,2,\dots,6$. $\det D(g_i)=-1 \Longrightarrow \forall g\in\mathcal{G}$ $\det D(g)=\pm 1$,

while for such a transformation g' $\det D(g')=e^{\frac{2i\pi}{3}}$ or $e^{\frac{4i\pi}{3}}$

$\Longrightarrow g'\notin\mathcal{G}$ Q.E.D.

Theorem 2: There exists no [redacted] transformation in the proper group $\mathcal{G}$, which can change the position of one edge-piece /at fixed site/, leaving unchanged the situation of the other eleven. /No matter of what changed occur in the "corner-piece-space"./

Proof : $i=1,2,\dots,6$ $\frac{\det\Delta(g_i)}{\det\hat{\Delta}(g_i)}=1$ where $\hat{\Delta}_{ij}=|\Delta_{ij}| \Longrightarrow (\forall g\in\mathcal{G})\ \frac{\det\Delta(g)}{\det\hat{\Delta}(g)}=1$

while for such a transformation g' $\frac{\det\Delta(g')}{\det\hat{\Delta}(g')}=-1 \Longrightarrow$

$\Longrightarrow g'\notin\mathcal{G}$ Q.E.D.

Theorem 3: There exists no [redacted] transformation in the proper group $\mathcal{G}$, which can interchange the place of two mobile pieces and leaves the other eighteen at their initial sites. /No matter of position-changes./

Proof : $i=1,2,\dots,6$ $\det\mathcal{D}(g_i)=\det D(g_i)\cdot\det\Delta(g_i)=+1 \Longrightarrow (\forall g\in\mathcal{G})\ \det\mathcal{D}(g)=1$

while for such a transformation g' $\det\mathcal{D}(g')=-1 \Longrightarrow$

$\Longrightarrow g'\notin\mathcal{G}$ Q.E.D.

These theorems are also stated in Kéri's work. But there their proofs are lengthy and more verbal. Shortness, simplicity and transparence are adventages of this formal description.

One can conclude from these theorems, that there exist at least twelve classes of configurations of the Cube, which are not connectable by means of proper transformations /corner-piece-pozitions x edge-piece-pozitions x place-interchanges = 3x2x2 =12 /. Thus the Cube may be assembled in twelve different ways of which one alone is correct.

Of course in its mixed-up state one can not find if the Cube is correctly assembled or not.

The solution of the problem

IN this last section I shall give five simple transformation series belonging to the proper group ζ ,which are sufficient for the reordering of the Cube from any mixed-up state,and their matrix representations.These transformation series are those indicated by Kéri in his work.I do not know of any simpler than those.

$$\gamma_1 = g_3^3 g_6 g_3 g_6 g_1 g_6^3 g_1^3$$

$$\gamma_2 = g_1 g_5^3 g_1^3 g_5 g_6^3 g_5 g_6 g_5^3 g_6 g_1^3 g_6^3 g_1$$

$$\gamma_3 = g_1^2 g_6^3 g_5^3 g_3 g_1^2 g_3^3 g_5 g_6^3 g_1^2$$

$$\gamma_4 = g_3 g_1 g_6 g_2 g_4 g_3^3 g_4^3 g_2^3 g_6^3 g_1^3$$

$$\gamma_5 = g_6 g_1 g_6 g_4^3 g_3^2 g_6^2 g_4^2 g_5 g_6^3 g_5^3 g_4^2 g_6^2 g_3^2 g_4 g_6^3 g_1^3$$

$\Delta(\Gamma_1)$ and $\Delta(\Gamma_2)$ are unimportant.

From the matrix represenations of Γs one can find the followings:

--By means of Γ_1 /applying it a few times consecutively in different positions of the Cube/ one can rearrange the corner-pieces on their right sites.

--After this, by means of Γ_2 and Γ_2^2 /applying them at most seven times/ one can spin these to their correct positions on their places./The position of the last is determined!/

--Applying Γ_3 and Γ_4 a few times one can rearrange the edge-pieces to their right sites, leaving unchanged the corners, which are already ordered./The places of the last two are determined!/

--Finally, by means of Γ_5 /applying it at most eleven times/ it is possible to arrange the positions of the edge-pieces. /The position of the last is determined!/

From this way of ordering we can conclude, that there are exactly twelve classes of configurations on the Cube.

From a simple calculation it results that there are

$$8! \cdot 12! \cdot 3^8 \cdot 2^{12} = 519\ 024\ 039\ 293\ 878\ 272\ 000$$

transformations in the total group G, from which the 12-th part /43 252 003 274 489 856 000 / are elements of the proper group .

The results of this note are not different in content from those published by Kéri, but I consider that this formal, mathematical language is more adequate. Further this formal frame makes possible the complete description of the Cube facing Kéris algorithmical approach.

1. Kéri Gerzson: About the Magic Cube; MTA SZTAKI preprint, Budapest, 28.11.1979. (in Hungarian)

2. Morton Hamermesh: Group Theory and its Application to Physical Problems; Pergamon Press, 1962.

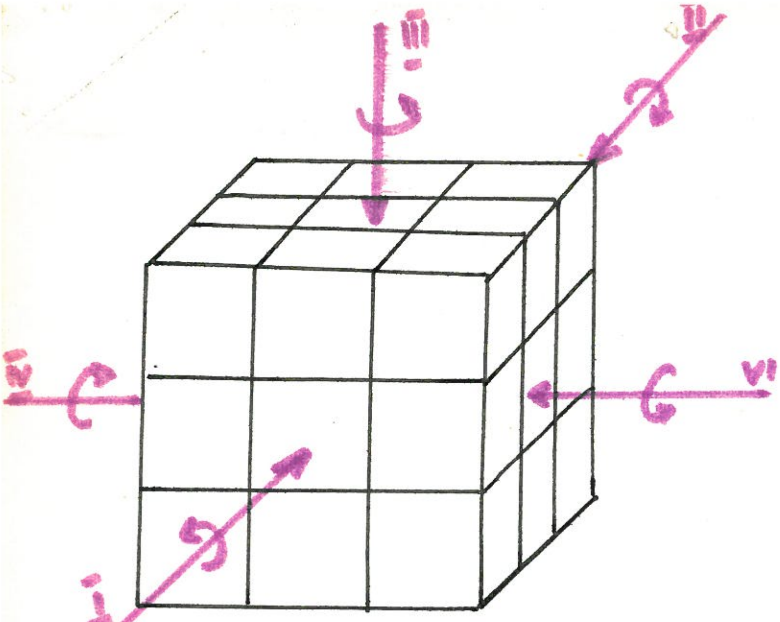


fig1: Rotation axes of the face-layers

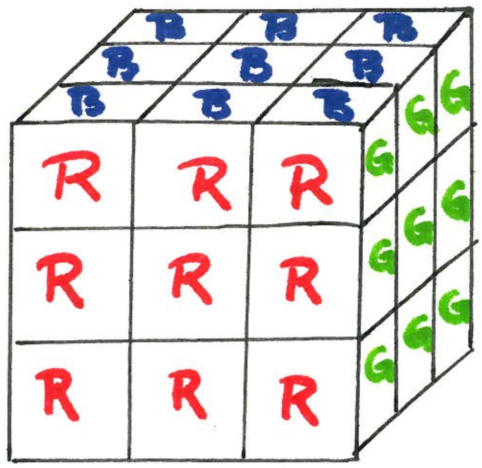
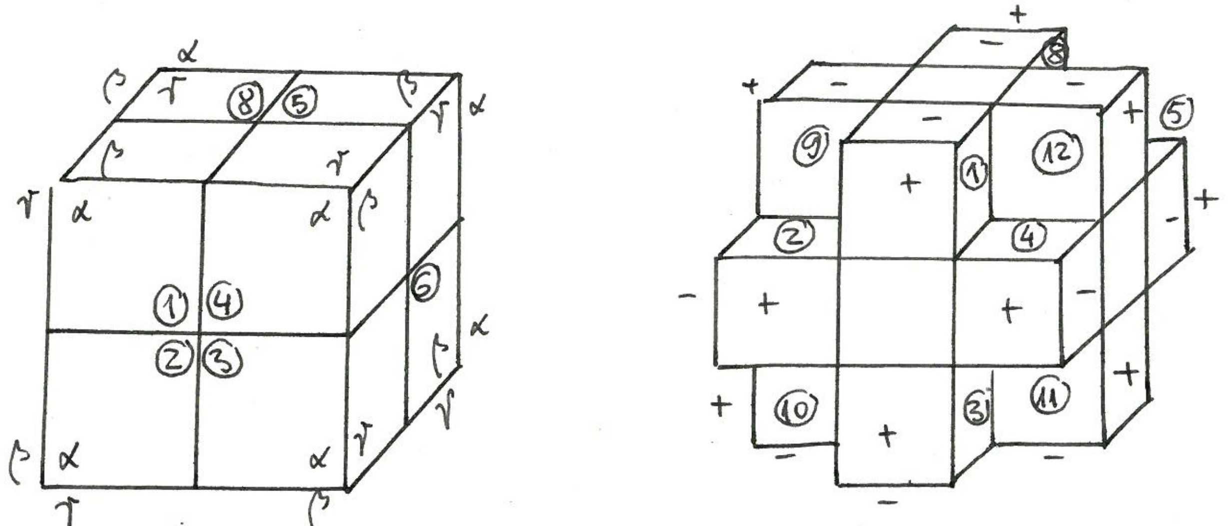


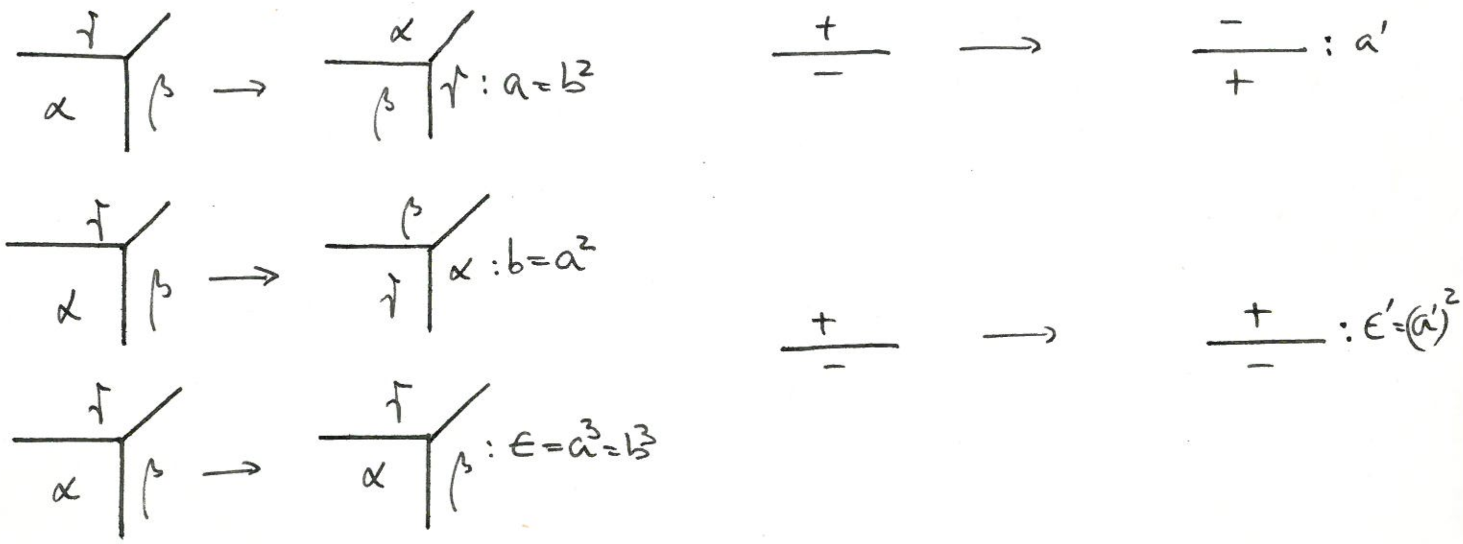
fig2: The ordered state of the Cube



corners

edges

fig3: Numeration and notation of sites respectively pozitions



corners

edges

fig4: The groups of position-changes