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Improved Lower Bound on the Thermodynamic Pressure of the Spin 1/2 Heisenberg Ferromagnet★

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Abstract. We introduce a new stochastic representation of the partition function of the spin 1/2 Heisenberg ferromagnet. We express some of the relevant thermodynamic quantities in terms of expectations of functionals of so-called random stirrings on $\mathbb{Z}^d$. By use of this representation, we improve the lower bound on the pressure given by Conlon and Solovej in *Lett. Math. Phys.* **23**, 223–231 (1991).

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1. Introduction and Result

We consider the 1/2-spin isotropic quantum Heisenberg ferromagnet (QHF) on the d-dimensional hypercubic lattice. The Hamiltonian of the model is

$$H_\Lambda = \frac{1}{2} \sum_{|x-y|=1} [(\mathbf{S}(x) - \mathbf{S}(y))^2 - 1], \tag{1.1}$$

where

$$\mathbf{S}(x) = (S_X(x), S_Y(x), S_Z(x)), \quad x \in \Lambda,$$

are the local spin operators and the summation runs over nearest neighbour pairs of lattice sites in the rectangular box Λ, with periodic boundary conditions. The canonical commutation relations satisfied by the spin operators are

$$[S_\alpha(x), S_\beta(y)] = i\delta_{x,y}\varepsilon_{\alpha,\beta,\gamma}S_\gamma(x), \qquad S_X^2(x) + S_Y^2(x) + S_Z^2(x) = \tfrac{3}{4}. \tag{1.2}$$

The grand partition function and the thermodynamic pressure are defined in the usual way:

$$\Xi_\Lambda(\beta, h) = \mathrm{Tr}\left[\exp - \beta\left(H_\Lambda - h \sum_{x\in\Lambda} S_Z(x)\right)\right] \tag{1.3}$$

and

$$\beta p_\Lambda(\beta, h) = |\Lambda|^{-1} \log \Xi_\Lambda(\beta, h), \tag{1.4}$$

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$$p(\beta, h) = \lim_{\Lambda \nearrow \mathbb{Z}^d} p_\Lambda(\beta, h). \tag{1.5}$$

It is generally conjectured that in the thermodynamic limit, at zero external field ($h = 0$) and very low temperatures, the elementary excitations of the QHF (i.e. the so-called magnons) behave like noninteracting bosons on the lattice $\mathbb{Z}^d$, (see, e.g., [4], for historical origins of the magnon approximation see [2] and [5]). In particular, it is expected that

$$\lim_{\beta \to \infty} \beta^{(d+2)/2} p(\beta, 0) = C_d \stackrel{\text{def}}{=} \frac{-1}{(2\pi)^d} \int_{\mathbb{R}^d} \log(1 - \mathrm{e}^{-k^2})\,\mathrm{d}k. \tag{1.6}$$

Conlon and Solovej considered this asymptotics in [4] and proved the following bound:

$$\liminf_{\beta \to \infty} \beta^{(d+2)/2} p(\beta, 0) \geqslant \frac{1}{2} c_d \stackrel{\text{def}}{=} \frac{1}{2} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{e}^{-k^2}\,\mathrm{d}k. \tag{1.7}$$

Using a new stochastic representation of the partition function, which is of course closely related to that of [3] and [15], we improve this bound in the present Letter by proving the following theorem.

THEOREM 1. *In three and more dimensions,*

$$\liminf_{\beta \to \infty} \beta^{(d+2)/2} p(\beta, 0) \geqslant \log 2\; C_d. \tag{1.8}$$

Remark. In one and two dimensions, our proof yields the constant $\log 2\; c_d$ on the right-hand side of (1.8), which is still better than that of [4].

In three dimensions, the constants appearing in (1.6), (1.7), and (1.8) are, respectively,

$$C_3 = 0.0301, \qquad \tfrac{1}{2} c_3 = 0.0112, \qquad \log 2\, C_3 = 0.0209. \tag{1.9}$$

The proof is based on a Feynman–Kac formula applied to the partition function of lattice boson gas (Section 2), the so-called random stirring representation of the simple exclusion process (Section 3), a coupling argument, and a simple probabilistic estimate on the recurrence probability of the simple symmetric random walk on $\mathbb{Z}^d$ (Section 4). [Terminology: by random walk, throughout this paper we mean continuous time simple symmetric random walk of jump rate $2d$.] In Section 5, formulas are given for the spontaneous magnetization and the long-range order in terms of random stirring expectations.

2. Bose Gas Formulation and Feynman–Kac Formulas

The local raising, lowering and excitation (i.e. magnon) number operators are defined in the usual way:

$$S^{\pm}(x) = S_X(x) \pm iS_Y(x), \qquad m(x) = S_Z(x) + \tfrac{1}{2}. \tag{2.1}$$

They satisfy the commutation relations

$$[S^{\pm}(x), S^{\pm}(y)] = 0, \qquad [S^{-}(x), S^{+}(y)] = \delta_{x,y}(1 - 2m(x)), \tag{2.2}$$

$$m(x) = S^{+}(x)S^{-}(x) = m^2(x). \tag{2.3}$$

It is well known that these are exactly the commutation relations of the local creation, annihilation and occupation number operators of a *hard core* Bose lattice gas (see, e.g., [3] or [14]). On the other hand, the *canonical* bosonic creation, annihilation, and particle number operators satisfy the relations:

$$[a^{+}(x), a^{+}(y)] = [a(x), a(y)] = 0, \qquad [a(x), a^{+}(y)] = \delta_{x,y}, \tag{2.4}$$

$$n(x) = a^{+}(x)a(x). \tag{2.5}$$

Thus, the Heisenberg Hamiltonian written in terms of these operators,

$$H_{\Lambda} = -\sum_{x,y \in \Lambda} S^{+}(x)\Delta_{x,y}S^{-}(y) - \sum_{|x-y|=1} m(x)m(y), \tag{2.6}$$

is identified with a Bose lattice gas Hamiltonian

$$H_{\Lambda}^{\mathrm{BLG}} = -\sum_{x,y \in \Lambda} a^{+}(x)\Delta_{x,y}a(y) + \frac{1}{2}\sum_{x,y \in \Lambda} V(x, y)n(x)n(y), \tag{2.7}$$

with one-particle kinetic energy operator $-\Delta$ given by the discrete Laplacian on the lattice:

$$\Delta_{x,y} = \begin{cases} -2d, & \text{if } |x - y| = 0, \\ 1, & \text{if } |x - y| = 1, \\ 0, & \text{if } |x - y| > 1, \end{cases} \tag{2.8}$$

and pair interaction

$$V(x, y) = \begin{cases} \infty, & \text{if } |x - y| = 0, \\ -2, & \text{if } |x - y| = 1, \\ 0, & \text{if } |x - y| > 1. \end{cases} \tag{2.9}$$

By a standard Feynman–Kac argument, we express the N-particle canonical partition function of a Bose lattice gas with *arbitrary* pair interaction V, as follows:

$$Q_{\Lambda}(\beta, N) = \frac{1}{N!}\sum_{\pi \in \mathscr{P}_N} \sum_{x^1, \ldots, x^N \in \Lambda}$$

$$\mathbf{E}\left(\exp\left[-\int_0^{\beta} \sum_{i<j} V(X_s^i, X_s^j)\,\mathrm{d}s\right]\mathbb{I}[X_{\beta}^i = x^{\pi(i)}, i = 1, \ldots, N]\,\middle\|\, X_0^i = x^i, i = 1, \ldots, N\right), \tag{2.10}$$

where $X_s^1, \ldots, X_s^N$ are N independent continuous time simple symmetric random walks on the lattice Λ, each with generator Δ (i.e. with jump rate $2d$) and $\mathscr{P}_N$ is the group of permutations of the N indices $\{1, \ldots, N\}$. Here and throughout this Letter,

$\mathbb{1}[\ldots]$ denotes the indicator function of the event appearing as argument, $\mathbf{E}(\ldots)$, $\mathbf{E}(\ldots \| \ldots)$ and $\mathbf{P}(\ldots)$ denote expectation, conditional expectation, and probability. The grand partition function is

$$\Xi_\Lambda(\beta, \mu) = \sum_{N=0}^{\infty} e^{\beta\mu N} Q_\Lambda(\beta, N), \tag{2.11}$$

and the pressure is defined by (1.4), (1.5). (The chemical potential μ and the magnetic field h are identical, the choice of the wording depends on the interpretation only.) We conclude this section by giving a formula in terms of random walk recurrence probabilities for the pressure of the *free* Bose lattice gas: in the $V = 0$ case, one can get from (2.10), (2.11) by straightforward combinatorial manipulations, the following expression for the pressure:

$$\beta p_0(\beta, \mu) = \sum_{n=1}^{\infty} \frac{e^{\beta\mu n}}{n} \mathbf{P}(X_{n\beta} = 0) = \sum_{n=1}^{\infty} \frac{e^{\beta\mu n}}{n} (e^{-2n\beta} I_0(2n\beta))^d. \tag{2.12}$$

Here, and in the sequel, $X_.$ (with no superscript) denotes a continuous time random walk on $\mathbb{Z}^d$, with generator Δ, starting from the origin. I_0 is the modified Bessel function of order 0. Using the asymptotics

$$\lim_{t\to\infty} e^{-t}\sqrt{t} I_0(t) = (2\pi)^{-1/2} \tag{2.13}$$

(see, e.g., [7]), we indeed find

$$\lim_{\beta\to\infty} \beta^{(d+2)/2} p_0(\beta, 0) = (4\pi)^{-d/2} \sum_{n=1}^{\infty} n^{-(d+2)/2} = C_d \tag{2.14}$$

in the case of noninteracting bosons. We gave this expression for the free Bose gas because of its suitability for comparison with a similar asymptotic expression of the pressure of the QHF. (See (4.3) below.)

3. The Random Stirring Representation

This section is devoted to further development of the stochastic representations (2.10), (2.11) in the special case of QHF, i.e. the interaction potential V given in (2.9).

For the N independent random walks $X_s^1, \ldots X_s^N$, denote by τ the first collision time:

$$\tau = \inf\{s\colon \text{for some } (i \neq j) X_s^i = X_s^j\}. \tag{3.1}$$

Then the form (2.10) of the canonical partition function becomes

$$\begin{aligned} &Q_\Lambda(\beta, N) \\ &= \sum_{\substack{A\subset\Lambda \\ |A|=N}} \mathbf{E}\Bigg(\left[\exp \int_0^\beta \mathscr{B}(\{X_s^1, \ldots, X_s^N\})\, ds\right] \mathbb{1}[\tau > \beta] \\ &\quad \times \mathbb{1}[\{X_\beta^1, \ldots, X_\beta^N\} = A] \Bigg\| \{X_0^1, \ldots, X_0^N\} = A\Bigg), \end{aligned} \tag{3.2}$$

where for $A \subset \Lambda$

$$\mathscr{B}(A) = |\{(x, y) \in A \times A : |x - y| = 1\}| \tag{3.3}$$

(Each neighbouring pair is counted twice!)

At this stage, we have to introduce the *simple symmetric exclusion process* (SSEP) defined in Spitzer's classic [12] and extensively studied since then. (For a survey, see also [10].) The SSEP of N particles on Λ is a Markov process η_t on $\{A : A \subset \Lambda, |A| = N\}$. It is an evolution of configurations of N indistinguishable particles on Λ with at most one particle per site. A particle at $x \in \Lambda$ waits an exponentially distributed time with mean $(2d)^{-1}$, then chooses a neighbouring site y with probability $(2d)^{-1}$. If y is vacant at that time, the particle jumps from x to y, otherwise it stays at x. All the waiting times and choices of neighbouring sites are independent.

A SSEP on Λ can be expressed in terms of the so-called *random stirring process* (RSP) on Λ, defined originally by Harris [9] (see also [8]). The RSP is a Markov process σ_t on the set of permutations of Λ,

$$\{\pi : \Lambda \to \Lambda : [x \neq y] \Rightarrow [\pi(x) \neq \pi(y)]\}.$$

The labels $\sigma_t(x)$ and $\sigma_t(y)$ are interchanged at rate 1, with independent Poisson flow of event times for each pair of neighbouring sites (x, y) in Λ. In other words, the random permutation σ_t starts from the identity and a transposition (x, y) is appended to it after exponentially distributed times with mean 1, independently for each pair of nearest neighbours.

Given a RSP σ_t on Λ and a subset $A \subset \Lambda$, $|A| = N$

$$\eta_t = \{\sigma_t(x) : x \in A\} \tag{3.4}$$

is clearly a SSEP of N particles on Λ, starting initially from the set A. (The particles of the SSEP are indistinguishable!)

We are now going to express Q_Λ in terms of expectations over SSEP trajectories and eventually Ξ_Λ in terms of expectations over RSP trajectories. The point is that until the first collision time the trajectories of N independent random walkers at one hand and N particles performing SSE random walks on the other hand, can be identified. More precisely: given a SSEP η_t, enlarge the probability space by an extra random event ('killing') which occurs at a random time $\tilde{\tau}$ with instantaneous rate $\mathscr{B}(\eta_t)$. Clearly, $(\tau, \{X_s^1, \ldots, X_s^N\} : s < \tau)$ and $(\tilde{\tau}, \eta_s : s < \tilde{\tau})$ have the same joint distribution, given $\{X_0^1, \ldots, X_0^N\} = \eta_0$. Consequently,

$$\begin{aligned}
&\mathbf{E}\Bigg(\left[\exp \int_0^\beta \mathscr{B}(\{X_s^1, \ldots, X_s^N\})\, \mathrm{d}s\right] \mathbb{I}[\tau > \beta] \\
&\qquad \times \mathbb{I}[\{X_\beta^1, \ldots, X_\beta^N\} = A] \,\Bigg\|\, \{X_0^1, \ldots, X_0^N\} = A\Bigg) \\
&= \mathbf{E}\Bigg(\left[\exp \int_0^\beta \mathscr{B}(\eta_s)\, \mathrm{d}s\right] \mathbb{I}[\tilde{\tau} > \beta] \mathbb{I}[\eta_\beta = A] \,\Bigg\|\, \eta_0 = A\Bigg)
\end{aligned}$$

$$= \mathbf{E}\left(\mathbf{P}(\tilde{\tau} > \beta \,\|\, \eta_s \colon 0 \leqslant s \leqslant \beta) \left[\exp \int_0^\beta \mathscr{B}(\eta_s)\,\mathrm{d}s \right] \mathbb{I}[\eta_\beta = A] \,\middle\|\, \eta_0 = A \right)$$

$$= \mathbf{P}(\eta_\beta = A \,\|\, \eta_0 = A). \tag{3.5}$$

In the last step, we used the equality

$$\mathbf{P}(\tilde{\tau} > \beta \,\|\, \eta_s \colon 0 \leqslant s \leqslant \beta) = \exp - \int_0^\beta \mathscr{B}(\eta_s)\,\mathrm{d}s \tag{3.6}$$

which holds by definition of $\tilde{\tau}$. Inserting (3.5) into (2.10) we get the following form of the grand canonical partition function

$$\Xi_\Lambda(\beta, \mu) = \sum_{A \subset \Lambda} \mathrm{e}^{\beta\mu|A|} \mathbf{P}(\eta_\beta = A \,\|\, \eta_0 = A). \tag{3.7}$$

This form is, of course, equivalent to the one given in [3], where similar arguments are applied to the QHF of any spin. Our forthcoming arguments (the random stirring representation and its consequences) work in the $S = 1/2$ case only.

We are going to exploit now the random stirring representation of the simple exclusion process. Let us denote by $\mathfrak{m}_s(l)$, $l \geqslant 1$ the number of cycles of length l in the random permutation σ_s.

THEOREM 2.

$$\Xi_\Lambda(\beta, \mu) = \mathbf{E}\left(\prod_{l \geqslant 1} (1 + \mathrm{e}^{\beta\mu l})^{\mathfrak{m}_\beta(l)} \right). \tag{3.8}$$

Proof. According to (3.7) and (3.4)

$$\Xi_\Lambda(\beta, \mu) = \sum_{A \subset \Lambda} \mathrm{e}^{\beta\mu|A|} \mathbf{P}(\sigma_\beta(A) = A). \tag{3.9}$$

Denote

$$p = \frac{\mathrm{e}^{\beta\mu}}{1 + \mathrm{e}^{\beta\mu}}. \tag{3.10}$$

Then

$$\Xi_\Lambda(\beta, \mu) = (1 + \mathrm{e}^{\beta\mu})^{|\Lambda|} \sum_{A \subset \Lambda} p^{|A|} (1 - p)^{|\Lambda \setminus A|} \mathbf{P}(\sigma_\beta(A) = A). \tag{3.11}$$

But the sum on the right-hand side has a straightforward probabilistic meaning: imagine that at time zero we put a red or a blue marble with probability p, respectively $1 - p$, on each site of Λ, independently of one another. After this we perform the random stirring of the marbles up to time β, independently of the initial choice of colours. The sum on the right-hand side of (3.11) is equal to the probability of the event 'the initial and final configuration of coloured marbles are the same'. Which is the same as 'all cycles of the permutation σ_β are monocolour'.

That is

$$\Xi_\Lambda(\beta,\mu) = (1 + e^{\beta\mu})^{|\Lambda|}\mathbf{E}\left(\prod_{l\geqslant 1}(p^l + (1-p)^l)^{\mathfrak{m}_\beta(l)}\right). \tag{3.12}$$

Let us denote by $\lambda_s(x)$, $x \in \Lambda$, $s \geqslant 0$, the length of the cycle containing $x \in \Lambda$ in the random permutation σ_s. The following identity is straightforward:

$$\sum_{l\geqslant 1} F(l)l\mathfrak{m}_s(l) = \sum_{x\in\Lambda} F(\lambda_s(x)) \tag{3.13}$$

for any function $F\colon \mathbb{N} \to \mathbb{R}$. Using (3.13) with $F \equiv 1$, from (3.12) we get exactly (3.8).

Remark. Random permutations have been used recently in different contexts of one-dimensional quantum spin chains [1] and of interacting Bose gas [13].

4. Proof of Theorem 1

By Jensen's inequality, we have

$$\beta p_\Lambda(\beta, 0) = |\Lambda|^{-1}\log \mathbf{E}(2^{\Sigma_{l\geqslant 1}\mathfrak{m}_\beta(l)}) \geqslant |\Lambda|^{-1}\log 2\ \mathbf{E}\left(\sum_{l\geqslant 1}\mathfrak{m}_\beta(l)\right) \tag{4.1}$$

We apply (3.13) with the choice $F(l) = 1/l$ and use translation invariance of the distribution of $\lambda_\beta(x)$ to get

$$\beta p_\Lambda(\beta, 0) \geqslant \log 2\ \mathbf{E}\left(\frac{1}{\lambda_\beta(0)}\right). \tag{4.2}$$

Taking the thermodynamic limit, we have

$$\beta p(\beta, 0) \geqslant \log 2 \sum_{n\geqslant 1}\frac{1}{n}\mathbf{P}(\lambda_\beta = n), \tag{4.3}$$

where now λ_β is the length of the (possibly infinite) cycle containing $0 \in \mathbb{Z}^d$, under the random permutation σ_β on the infinitely extended lattice $\mathbb{Z}^d$. (There is no difficulty in extending the RSP to the whole hypercubic lattice, see [9].)

We define now a random process $Z_t^{(\beta)}$, $t \geqslant 0$ on $\mathbb{Z}^d$, which is loosely speaking the trajectory of a particle starting from the origin, induced by the random stirring σ_s, $s \in [0, \beta)$ periodically continued in time. More precisely, we periodically continue the random permutation process

$$\bar{\sigma}_t^{(\beta)} = \sigma_{t-\beta[t/\beta]} \circ (\sigma_\beta)^{\circ[t/\beta]}, \quad t \geqslant 0 \tag{4.4}$$

and define $Z_t^{(\beta)}$ as the trajectory of a particle starting from the origin, under this permutation process:

$$Z_t^{(\beta)} = \bar{\sigma}_t^{(\beta)}(0), \quad t \geqslant 0. \tag{4.5}$$

In terms of this process the cycle length λ_β is

$$\lambda_\beta = \min\{k \geqslant 1 : Z^{(\beta)}_{k\beta} = 0\}. \tag{4.6}$$

The process $Z^{(\beta)}_\cdot$ can be easily realized as a function of the trajectory of a random walk $X_\cdot$ in the following way. Until $t = \beta\lambda_\beta$, $Z^{(\beta)}$ performs the same jumps as X except for two deterministic modifications: Let $t > \beta$, assume $Z^{(\beta)}_{t-0} = x$ and y is a neighbouring site.

(a) *Forced jumps:* If at some time $t - k\beta$, $k = 1, 2, \ldots, [t/\beta]$, a jump $y \to x$ of $Z^{(\beta)}$ occurred, then, at time t, $Z^{(\beta)}$ is forced to jump backwards from x to y.
(b) *Erased jumps:* If at time t, a jump of X occurs, which would force $Z^{(\beta)}$ to jump from x to y, but y was occupied by $Z^{(\beta)}$ at some time $t - k\beta$, $k = 1,2,\ldots,[t/\beta]$, then this jump is not performed by $Z^{(\beta)}$.

After $t = \beta\lambda_\beta$, $Z^\beta_\cdot$ is continued periodically, with period $\beta\lambda_\beta$.

Denote by θ_β the moment when the trajectories $Z^{(\beta)}_\cdot$ and $X_\cdot$ come apart for the first time, if this happens before $\beta\lambda_\beta$, or $\beta\lambda_\beta$ otherwise, i.e. the time when the first deterministic modification (forced jump, erased jump or periodic continuation) of the random walk trajectory occurs. This is clearly the following stopping time of X.

$$\begin{aligned}\theta_\beta &= \inf\{t > 0 : Z^{(\beta)}_t \neq X_t\} \wedge \beta\lambda_\beta \\ &= \inf\{t > \beta : X_t = X_{t-k\beta} \text{ for some } k = 1,2,\ldots,[t/\beta]\}.\end{aligned} \tag{4.7}$$

Consequently,

$$\begin{aligned}\mathbf{P}(\lambda_\beta = n) &\geqslant \mathbf{P}(\theta_\beta = \beta\lambda_\beta = n\beta) \\ &= \mathbf{P}(X_{n\beta} = 0) - \mathbf{P}([X_{n\beta} = 0] \wedge [\theta_\beta < n\beta]).\end{aligned} \tag{4.8}$$

The subtracted rightmost term (4.8) is the probability of the event that the random walk $X_\cdot$ returns to the origin at time $n\beta$ after somewhere making a loop of time span $m\beta$, $1 \leqslant m < n$. This probability is easily estimated:

$$\begin{aligned}&\mathbf{P}([X_{n\beta} = 0] \wedge [(\exists t \in [0,(n-m)\beta]) : X_{t+m\beta+0} = X_t]) \\ &\quad\leqslant 2\mathbf{P}([X_{n\beta} = 0] \wedge ([\exists t \in [0,(n-m)\beta]) : [|X_{t+m\beta} - X_t| = 1] \\ &\quad\wedge [X_{t+m\beta+\mathrm{d}t} = X_t]])\end{aligned} \tag{4.9}$$

by invariance with respect to time reversal. Let $t \in [0,(n-m)\beta]$ be fixed

$$\begin{aligned}&\mathbf{P}([X_{n\beta} = 0] \wedge [|X_{t+m\beta} - X_t| = 1] \wedge [X_{t+m\beta+\mathrm{d}t} = X_t]) \\ &\quad= \sum_{x\in\mathbb{Z}^d} \mathbf{P}(X_t = x)\,\mathbf{P}(|X_{t+m\beta} - x| = 1 \,\|\, X_t = x)\,\mathrm{d}t\,\mathbf{P}(X_{n\beta=0} \,\|\, X_{t+m\beta} = x) \\ &\quad= \left\{\sum_{x\in\mathbb{Z}^d} \mathbf{P}(X_t = x)\,\mathbf{P}(X_{n\beta=0} \,\|\, X_{t+m\beta} = x)\right\}\mathbf{P}(|X_{m\beta}| = 1)\,\mathrm{d}t \\ &\quad= \mathbf{P}(X_{(n-m)\beta} = 0)\,\mathbf{P}(|X_{m\beta}| = 1)\,\mathrm{d}t \leqslant \text{const}\,\beta^{-d}(m(n-m))^{-d/2}\,\mathrm{d}t\end{aligned} \tag{4.10}$$

As t may occur anytime in the interval $[0,(n-m)\beta]$, we find

$$\mathbf{P}([X_{n\beta}=0]\wedge[(\exists t\in[0,(n-m)\beta]):[|X_{t+m\beta}-X_t|=1]\wedge[X_{t+m\beta+\mathrm{d}t}=X_t]]) \leqslant \mathrm{const}\,\beta^{1-d}m^{-d/2}(n-m)^{(2-d)/2} \tag{4.11}$$

and summation over $m\in\{1,2,\ldots,n-1\}$ yields

$$\mathbf{P}([X_{n\beta}=0]\wedge[\theta_\beta<n\beta])=(n-1)\mathcal{O}(\beta^{1-d}) \tag{4.12}$$

From (4.8) and (4.12), we get

$$\liminf_{\beta\to\infty}\beta^{d/2}\mathbf{P}(\lambda_\beta=n)\geqslant\lim_{\beta\to\infty}\beta^{d/2}\mathbf{P}(X_{n\beta}=0) \tag{4.13}$$

for any $n\in\mathbb{N}$ in three and more dimensions, and for $n=1$ in one and two dimensions. Inserting (4.13) into (4.3), a comparison with (2.12) yields the proof of Theorem 1.

5. Further Consequences of Theorem 2

In this section, we give formulas for the spontaneous magnetization and the long-range order parameter of the QHF in terms of random stirring expectations. These formulas will show that the expected phase transition of the model is closely related to the appearance of an infinite cycle in the random stirring σ_β of $\mathbb{Z}^d$, for β sufficiently large.

After straightforward manipulations, from (3.8) we get the following expression of the magnetization (i.e. density of the Bose gas minus $\frac{1}{2}$)

$$m_\Lambda(\beta,h)=\frac{1}{2}\,\frac{\mathbf{E}\left(\tanh((h/2)\lambda_\beta(0))\prod_{l\geqslant 1}(1+\mathrm{e}^{\beta hl})^{\mathrm{m}_\beta(l)}\right)}{\mathbf{E}\left(\prod_{l\geqslant 1}(1+\mathrm{e}^{\beta hl})^{\mathrm{m}_\beta(l)}\right)}. \tag{5.1}$$

Taking the thermodynamic limit, we find the following formula for the spontaneous magnetization

$$\lim_{h\to 0}m(\beta,h)=m(\beta)=\frac{1}{2}\lim_{n\to\infty}\lim_{\Lambda\nearrow\mathbb{Z}^d}\frac{\mathbf{E}(\mathbb{I}[\lambda_\beta(0)>n]2^{\Sigma_{l\geqslant 1}\mathrm{m}_\beta(l)})}{\mathbf{E}(2^{\Sigma_{l\geqslant 1}\mathrm{m}_\beta(l)})}. \tag{5.2}$$

The long-range order parameter (as defined, e.g., in [6]) is $r(\beta)=\lim_{\Lambda\nearrow\mathbb{Z}^d}r_\Lambda(\beta)$, where

$$r_\Lambda(\beta)=\frac{1}{|\Lambda|^2}\sum_{x,y\in\Lambda}\langle\mathbf{S}(x)\cdot\mathbf{S}(y)\rangle_\Lambda. \tag{5.3}$$

(Notice that in the Bose gas formulation, $\frac{4}{3}r(\beta)$ is exactly the density of the condensate, as defined by [11].) Applying similar considerations as in the derivation of Theorem 2, the following expression of the long-range order parameter is found

$$r(\beta) = \frac{3}{4} \lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \frac{\mathbf{E}(\lambda_\beta(0) 2^{\sum_{l \geq 1} m_\beta(l)})}{\mathbf{E}(2^{\sum_{l \geq 1} m_\beta(l)})}. \tag{5.4}$$

Formulas (5.2) and (5.4) show striking similarity to percolation theoretical objects with 'cluster size' replaced by 'cycle length'. They could allow nice random-geometrical speculations about the still open problem of existence of phase transition in QHF. On a technical level their value is less clear for two reasons: (1) the random cycles seem to be less transparent geometric objects than the random clusters and (2) calculating the averages in (5.2), (5.4) is further complicated by the weights $2^{\sum_{l \geq 1} m_\beta(l)}$ assigned to the permutations, over the bare random stirring measure.

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References

1. Aizenman, M. and Nachtergaele, B., Geometric aspects of quantum spin states., in preparation (1993).
2. Bloch, F., *Z. Physik* **61**, 206 (1930), **74**, 295 (1932).
3. Conlon, J. and Solovej, J. P., Random walk representations of the Heisenberg model, *J. Stat. Phys.* **64**, 251–270 (1991).
4. Conlon, J. and Solovej, J. P., Upper bound on the free energy of the spin 1/2 Heisenberg ferromagnet, *Lett. Math. Phys.* **23**, 223–231 (1991).
5. Dyson, F., General theory of spin-wave interactions, *Phys. Rev.* **102**, 1217–1244 (1956).
6. Dyson, F., Lieb, E. H., and Simon, B., Phase transitions in quantum spin systems with isotropic and nonisotropic interactions, *J. Stat. Phys.* **18**, 335–383 (1978).
7. Gradshteyn, I. S. and Ryzhik, I. M., *Table of Integrals, Series and Products*, Academic Press, New York, 1980.
8. Griffeath, D., *Additive and Cancellative Interactive Particle Systems*, Lecture Notes in Mathematics 724, Springer, New York, 1979.
9. Harris, T. E., Nearest neighbour Markov interaction processes on multidimensional lattices. *Adv. Math.* **9**, 66–89 (1972).
10. Liggett, T., *Interacting Particle Systems*, Springer, New York, 1985.
11. Penrose, O. and Onsager, L., Bose–Einstein condensation and liquid Helium, *Phys. Rev.* **104**, 576–584 (1956).
12. Spitzer, F., Interaction of Markov processes. *Adv. Math.* **5**, 256–290 (1970).
13. Sütő, A., Percolation transition in the Bose gas, submitted to *J. Phys. A.* (1993).
14. Tóth, B., Upper bound on the condensate in hard core Bose lattice gas, *J. Stat. Phys.* **65**, 373–378 (1991).
15. Tóth, B., Probabilistic representation of the quantum Heisenberg ferromagnet, in A. Truman and I. M. Davies (eds), *Stochastics and Quantum Mechanics, Swansea 1990*, World Scientific, Singapore, 1991.