Tied favourite edges for simple random walk

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Abstract

We show that there are almost surely only finitely many times at which there are at least 4 'tied' favourite edges for a simple random walk. This (partially) answers a question of P. Erdős and P. Révész.

Key Words: Random walks, favourite points, local time
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1 Introduction

In [2], [3], [4], [6], P. Erdős and P. Révész repeatedly raised the following problem (among other questions): Suppose $(X_n,\ n \geq 0)$ is an ordinary symmetric simple random walk on the set of integers $\mathbf{Z}$. Define, the 'local time' at x and time i by

$$\ell(x,i) = |\{j \in [0,i];\ X_j = x\}|$$

(here and throughout the paper $|D|$ denotes the cardinal of the set D). For any $i \geq 0$, the set of favourite points before i (the most visited points during the first i steps) is

$$\mathcal{A}_i = \{x \in \mathbf{Z};\ \ell(x,i) = \sup_{y \in \mathbf{Z}} \ell(y,i)\}.$$

Clearly, $|\mathcal{A}_i| \geq 1$ for all $i \geq 0$ and it is easy to see that for infinitely many i's, $|\mathcal{A}_i| \geq 2$. The question is: Does it happen that $|\mathcal{A}_i| \geq r$ infinitely many times (i.e. for infinitely many i's), for $r = 3, 4, \ldots$?

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The aim of this note is to show that

$$|\{i > 0;\ |\mathcal{A}_i| \geq 4\}| < \infty \quad \text{almost surely.}$$

In fact, we are not exactly going to show this result, but the analogous result for favourite edges (and not favourite points), which turns out to be more tractable. The case $r = 3$ remains open. The proof relies mainly on the fact that the process of 'local time on edges' associated to X at a suitable stopping time can be described explicitly in terms of certain Markov chains (as e.g. pointed out in Knight [5]). Let us just recall here the following related result of Bass and Griffin [1]: Almost surely,

$$\inf_{x\in\mathcal{A}_i} |x| \stackrel{i\to\infty}{\longrightarrow} \infty, \tag{1}$$

i.e. a fixed site cannot be favourite infinitely often.

Before stating our main result, we need some more notation: Define for all $i \geq 1$,

$$\widetilde{X}_i = \frac{X_i + X_{i-1} + 1}{2},$$

which characterizes the edge of the i-th jump (edge x is between points $x-1$ and x). We also define the 'local time on the edge x' as follows:

$$L(x,i) = |\{j \in [1,i];\ \widetilde{X}_j = x\}|,$$

and the set of favourite edges at step i:

$$\mathcal{K}_i = \{x \in \mathbf{Z};\ L(x,i) = \sup_{y\in\mathbf{Z}} L(y,i)\}.$$

For fixed $r \geq 1$, we define

$$f(r) = |\{i \geq 1;\ [\widetilde{X}_i \in \mathcal{K}_i] \ \wedge\ [|\mathcal{K}_i| \geq r]\}|$$

the number of times at which a new favourite edge appears, tied with at least $r-1$ other favourite edges. By definition $f(r) \geq f(r+1)$. The main result of the present note is the following:

Theorem 1 *Almost surely,* $f(4) < \infty$. *Moreover* $\mathbf{E}(f(4)) < \infty$.

This clearly implies also:

$$|\{i > 0;\ |\mathcal{K}_i| \geq 4\}| < \infty \quad \text{almost surely.}$$

Another direct consequence is that there almost surely exists (a random) r_0, such that $f(r_0) = 0$ (and consequently $f(r) = 0$, for all $r \geq r_0$).

The paper is structured as follows. We first introduce (section 2) and study some properties (section 3) of two relevant auxiliary Markov chains, before proving the theorem in section 4.

2 Preliminaries

We now introduce two auxiliary Markov chains and notation that will be needed in our proof.

Let $(\xi_j^t,\ j \geq 0,\ t \geq 0)$ be a family of i.i.d. geometric random variables such that for all $n \geq 0$, $\mathbf{P}(\xi_j^t = n) = 2^{-n-1}$. We define the Markov chain $(Z_t,\ t \geq 0)$ on the state space $\mathbf{N}$ as follows: $Z_0 \in \mathbf{N}$, and for all $t \geq 0$,

$$Z_{t+1} = \sum_{j=0}^{Z_t} \xi_j^t.$$

The transition probabilities of the Markov chain Z_t are:

$$\pi(i,j) := \mathbf{P}\Big(Z_{t+1} = j \,\Big|\, Z_t = i \Big) = 2^{-i-j-1} \frac{(i+j)!}{i!j!}, \qquad i,j \geq 0.$$

Similarly, we define another Markov chain $(Y_t,\ t \geq 0)$ on $\mathbf{N}$ such that $Y_0 \in \mathbf{N}$ and for all $t \geq 0$,

$$Y_{t+1} = \sum_{j=1}^{Y_t} \xi_j^t$$

(we set the empty sum equal to zero). The transition probabilities of the Markov chain Y_t are (for $i,j \geq 0$):

$$\mu(i,j) := \mathbf{P}\Big(Y_{t+1} = j \,\Big|\, Y_t = i \Big) = \begin{cases} \delta_{0,j} & \text{if} \quad i = 0, \\ \dfrac{(i+j-1)!}{(i-1)!j!} 2^{-i-j} & \text{if} \quad i > 0. \end{cases}$$

In other words: Y_t is a critical branching process with geometric offspring distribution, and Z_t is the same with 'one intruder at each generation'.

In the sequel, Y^1 and Y^2 will denote two independent copies of Y, which are also independent from Z (only the starting points Y_0^1 and Y_0^2 may depend on Z).

These processes are useful to describe the local time process of $\widetilde{X}$ in the space variable, taken at certain stopping times. We now recall some results from Knight [5]: Define the inverse local time, for $x \in \mathbf{Z}$ and $s > 0$,

$$T(x,s) = \inf\{i \geq 1;\ L(x,i) \geq s\}.$$

Fix $x \geq 1$ and $s \geq 1$; remark that $L(y, T(x,s))$ is odd for $y \in [1, x-1]$ (we set $[1, x-1] = \emptyset$ for $x = 1$) and even if $y \notin [1, x]$ (of course, one also has $L(x, T(x,s)) = s$). The following statement follows easily from the results in [5]:

Proposition 1 *The processes*

$$\left\{ \left(\frac{L(x-y,T(x,s))-1}{2}, y \in [1, x-1] \right), \left(\frac{L(1-y,T(x,s))}{2}, y \geq 1 \right), \right.$$
$$\left. \left(\frac{L(x+y,T(x,s))}{2}, y \geq 1 \right) \right\}$$

have the same joint law as

$$\left\{ \quad (Z_y, y \in [1, x-1]) \quad , \quad (Y^1_y, y \geq 1) \quad , \quad (Y^2_y, y \geq 1) \quad \right\},$$

where

- $Z_0 = h$, $Y^1_0 = Z_{x-1}$ *and* $Y^2_0 = h$ *if* $s = 2h+1$ *is odd.*
- $Z_0 = h-1$, $Y^1_0 = Z_{x-1}$ *and* $Y^2_0 = h$ *if* $s = 2h$ *is even.*

3 Some properties of two relevant Markov chains

In this section, which is the longest of this note, we derive some technical lemmas concerning Z, Y, their hitting times and overshoots.

3.1 Results for Z

For $h \in \mathbf{N}$ we define $\tau_h = \inf\{t > 0,\ Z_t \geq h\}$. More generally, we put $\tau_{h,0} = 0$, and for all $i \geq 1$,

$$\tau_{h,i} = \inf\{t > \tau_{h,i-1},\ Z_t \geq h\}.$$

In the last section of this note we shall also need the following hitting times:

$$\widehat{\tau}_{h+1,i} = \inf\{t > \tau_{h,i-1},\ Z_t \geq h+1\}.$$

Lemma 1 *(i) There exists a constant* $c < \infty$ *such that for all* $h > 0$

$$\mathbf{E}\Big(\tau_h \,\Big|\, Z_0 = h \Big) \leq \mathbf{E}\Big(\tau_{h+1} \,\Big|\, Z_0 = h \Big) \leq c\sqrt{h}. \tag{2}$$

(ii) For any $\varepsilon > 0$*, there exists a constant* $c(\varepsilon) < \infty$ *such that for all* $h > 0$*,*

$$\mathbf{P}\Big(Z_{\tau_h} = h \,\Big|\, Z_0 = h \Big) \leq c(\varepsilon) h^{-1/2+\varepsilon}. \tag{3}$$

In the sequel, $c(\varepsilon)$ and c will denote finite positive constants, which may vary from line to line. Before proving this, we need another lemma:

Lemma 2 *For all $x \le h$,*

$$\mathbf{E}\Big(Z_1 \Big| [Z_0 = x] \wedge [Z_1 > h] \Big) \le \mathbf{E}\Big(Z_1 \Big| [Z_0 = h] \wedge [Z_1 > h] \Big). \tag{4}$$

Proof of Lemma 2- Define for all $i \le j$ and $k \ge 0$,

$$p_{i,j}(k) = \mathbf{P}\Big(Z_1 = j + k \Big| [Z_0 = i] \wedge [Z_1 \ge j] \Big) = \frac{\pi(i, j+k)}{\sum_{l=0}^{\infty} \pi(i, j+l)}. \tag{5}$$

Then, for any $0 \le i < i+1 \le j$ the distribution $p_{i+1,j}(\cdot)$ stochastically dominates $p_{i,j}(\cdot)$, i.e. for any $l > 0$

$$\sum_{k \ge l} p_{i+1,j}(k) > \sum_{k \ge l} p_{i,j}(k). \tag{6}$$

Indeed, using (5), we only have to prove that for any $l \ge 1$,

$$\sum_{k=0}^{l-1} \pi(i, j+k) \sum_{k=l}^{\infty} \pi(i+1, j+k) > \sum_{k=0}^{l-1} \pi(i+1, j+k) \sum_{k=l}^{\infty} \pi(i, j+k). \tag{7}$$

But,

$$\frac{\pi(i+1, j+k)}{\pi(i, j+k)} = \frac{i+j+k+1}{2(i+1)} < \frac{i+j+k+2}{2(i+1)} = \frac{\pi(i+1, j+k+1)}{\pi(i, j+k+1)} \tag{8}$$

and hence, for any $k' > k$,

$$\pi(i+1, j+k)\pi(i, j+k') < \pi(i+1, j+k')\pi(i, j+k)$$

which implies (7) and (6). Finally, (6) yields readily that for any increasing $f : \mathbf{N} \to \mathbf{R}$

$$\max_{0 \le i \le h} \mathbf{E}\Big(f(Z_1) \Big| [Z_0 = i] \wedge [Z_1 > h] \Big) = \mathbf{E}\Big(f(Z_1) \Big| [Z_0 = h] \wedge [Z_1 > h] \Big)$$

and in particular, (4) follows.

Proof of Lemma 1-(i) It is clear that $Z_t - t$ is a martingale and that $\mathbf{E}(\tau_{h+1}) < \infty$ (for instance because, for any $x \le h$, $\mathbf{P}(\tau_{h+1} = 1 \,|\, Z_0 = x) \ge 2^{-h-1}$). Hence,

$$\mathbf{E}\Big(\tau_{h+1} \Big| Z_0 = h \Big) = \mathbf{E}\Big(Z_{\tau_{h+1}} - h \Big| Z_0 = h \Big).$$

But

$$\mathbf{E}\Big(Z_{\tau_{h+1}} - h \Big| Z_0 = h \Big) = \sum_{x=0}^{h} \mathbf{E}\Big(Z_1 - h \Big| [Z_0 = x] \wedge [Z_1 > h] \Big) \mathbf{P}\Big(Z_{\tau_{h+1}-1} = x \Big| Z_0 = h \Big).$$

Hence, using Lemma 2,

$$\mathbf{E}\Big(\tau_{h+1} \Big| Z_0 = h \Big) \le \mathbf{E}\Big(Z_1 - h \Big| [Z_0 = h] \wedge [Z_1 > h] \Big) \le c\sqrt{h},$$

where the last inequality follows from the central limit theorem. This concludes the proof of (2).

Proof of Lemma 1-(ii) We divide this proof into several short steps:

STEP 1- Suppose $1/2 \le \alpha \le 1$ and $h - h^\alpha \le z \le h$. For $k \le \sqrt{h}$

$$\begin{aligned}\frac{\pi(z,h+k)}{\pi(z,h)} &= 2^{-k}\frac{(h+z+1)(h+z+2)\dots(h+z+k)}{(h+1)(h+2)\dots(h+k)} \\ &\ge 2^{-k}\left(\frac{2h-h^\alpha}{h+h^{1/2}}\right)^k \\ &\ge (1-2h^{\alpha-1})^k.\end{aligned}$$

Hence,

$$\frac{\mathbf{P}\Big(Z_1 \ge h \,\Big|\, Z_0 = z\Big)}{\mathbf{P}\Big(Z_1 = h \,\Big|\, Z_0 = z\Big)} \ge \sum_{k=0}^{[\sqrt{h}]}(1-2h^{\alpha-1})^k \ge \frac{1-(1-2h^{\alpha-1})^{\sqrt{h}-1}}{2h^{\alpha-1}}.$$

But,

$$(1-2h^{\alpha-1})^{\sqrt{h}-1} \le (1-2h^{-1/2})^{\sqrt{h}-1} \stackrel{h\to\infty}{\longrightarrow} e^{-2} < 1/2.$$

Hence, for some $h_0 < \infty$ and for all $h > h_0$,

$$\mathbf{P}\Big(Z_1 = h \,\Big|\, [Z_0 = z] \wedge [Z_1 \ge h]\Big) \le 4h^{\alpha-1} \tag{9}$$

STEP 2- We now define for $u \ge 0$,

$$\theta_u = \inf\{t > 0;\ Z_t \le u\}.$$

As $Z_t - t$ is a martingale, for $0 \le u \le h$ we have

$$\mathbf{E}\Big(Z_{\tau_h} \,\Big|\, Z_0 = h\Big) - \mathbf{E}\Big(\tau_h \,\Big|\, Z_0 = h\Big) = \mathbf{E}\Big(Z_{\tau_h \wedge \theta_u} \,\Big|\, Z_0 = h\Big) - \mathbf{E}\Big(\tau_h \wedge \theta_u \,\Big|\, Z_0 = h\Big)$$

Hence

$$\mathbf{E}\Big((Z_{\tau_h} - Z_{\theta_u})\mathbb{1}\{\theta_u < \tau_h\} \,\Big|\, Z_0 = h\Big) \le \mathbf{E}\Big(\tau_h \,\Big|\, Z_0 = h\Big)$$

($\mathbb{1}\{\dots\}$ denotes the indicator function). As $Z_{\tau_h} \ge h$, $Z_{\theta_u} \le u$, and using (2), one has

$$\mathbf{P}\Big(\theta_u < \tau_h \,\Big|\, Z_0 = h\Big) \le \frac{ch^{1/2}}{h-u}. \tag{10}$$

and in particular, for $\beta \in [1/2, 1]$

$$\mathbf{P}\Big(\theta_{h-h^\beta} < \tau_h \,\Big|\, Z_0 = h\Big) \le ch^{1/2-\beta}. \tag{11}$$

STEP 3- Fix $\varepsilon > 0$, and choose N such that $N > 1/2\varepsilon$. For $i \in \{0,1,2,\dots,N\}$, we define

$$u_i = u_i(h) = h - h^{\frac{N+i}{2N}}$$

and

$$\Delta_0 = [u_0, h], \qquad \Delta_i = [u_i, u_{i-1}), \quad i = 1,2,\dots,N.$$

Then,

$$\mathbf{P}\Big(Z_{\tau_h} = h \,\Big|\, Z_0 = h\Big) = \sum_{i=0}^{N}\mathbf{P}\Big([Z_{\tau_h} = h] \wedge [Z_{(\tau_h - 1)} \in \Delta_i] \,\Big|\, Z_0 = h\Big) \tag{12}$$

- For $i \in \{1, \ldots, N\}$, using the strong Markov property and the first two steps we get

$$\begin{aligned}\mathbf{P}\Big([Z_{\tau_h} = h] \ &\wedge \ [Z_{(\tau_h-1)} \in \Delta_i] \Big| Z_0 = h\Big) \\ &= \mathbf{P}\Big([Z_{\tau_h} = h] \wedge [Z_{\tau_h-1} \in \Delta_i] \wedge [\theta_{u_{i-1}} \leq \tau_h] \Big| Z_0 = h\Big) \\ &\leq \mathbf{P}\Big(\theta_{u_{i-1}} \leq \tau_h \Big| Z_0 = h\Big) \sup_{z \in \Delta_i} \mathbf{P}\Big(Z_1 = h \Big| [Z_0 = z] \wedge [Z_1 \geq h]\Big) \\ &\leq c_1 h^{-\frac{i-1}{2N}} c_2 h^{-\frac{N-i}{2N}} = c h^{-1/2+1/(2N)} \leq c h^{-1/2+\varepsilon} \qquad (13)\end{aligned}$$

- For $i = 0$, $\mathbf{P}([Z_{\tau_h} = h] \wedge [Z_{(\tau_h-1)} \geq u_0] \,|\, Z_0 = h)$ can be bounded directly, as in step 1. Take $z \in [h - \sqrt{h}, h]$; then for all $k \in [0, \sqrt{h}]$,

$$\frac{\pi(z, h+k)}{\pi(z,h)} \geq (1 - 2h^{-1/2})^k \geq (1 - 2h^{-1/2})^{\sqrt{h}} \geq c.$$

  Hence, summing over k,

$$\frac{\mathbf{P}\Big(Z_1 \geq h \Big| Z_0 = z\Big)}{\mathbf{P}\Big(Z_1 = h \Big| Z_0 = z\Big)} \geq c\sqrt{h}$$

  and finally

$$\mathbf{P}\Big([Z_{\tau_h} = h] \wedge [Z_{(\tau_h-1)} \in \Delta_0] \Big| Z_0 = h\Big) \leq c h^{-1/2}. \qquad (14)$$

Eventually, putting the pieces together using (12), (13) and (14), we get exactly (3).

3.2 Results for Y

We now derive analogous results for Y. This subsection is very much similar to the previous one.

For $h \in \mathbf{N}$, let us now put $\sigma_h = \inf\{t > 0;\ Y_t \geq h\}$ (with the convention $\inf \emptyset = \infty$). We also define the following hitting times that we will need in the last section of this paper: $\sigma_{h,0} = 0$ and for $i \geq 1$,

$$\sigma_{h,i} = \inf\{t > \sigma_{h,i-1};\ Y_t \geq h\},$$

and

$$\widehat{\sigma}_{h+1,i} = \inf\{t > \sigma_{h,i-1};\ Y_t \geq h+1\}.$$

The following lemma is the analogue of Lemma 1:

Lemma 3 *(i) There exists a constant $c < \infty$ such that for all $h \geq 0$*

$$\mathbf{P}\Big(\sigma_h = \infty \Big| Y_0 = h\Big) \leq \mathbf{P}\Big(\sigma_{h+1} = \infty \Big| Y_0 = h\Big) \leq c h^{-1/2}. \qquad (15)$$

(ii) For any $\varepsilon > 0$, there exists a constant $c(\varepsilon) < \infty$ such that for all $h > 0$,

$$\mathbf{P}\Big([\sigma_h < \infty] \wedge [Y_{\sigma_h} = h] \Big| Y_0 = h \Big) \leq c(\varepsilon) h^{-1/2+\varepsilon}. \tag{16}$$

We also state and prove the following analogue to Lemma 2.

Lemma 4 *For all $x \leq h$,*

$$\mathbf{E}\Big(Y_1 \Big| [Y_0 = x] \wedge [Y_1 > h] \Big) \leq \mathbf{E}\Big(Y_1 \Big| [Y_0 = h] \wedge [Y_1 > h] \Big). \tag{17}$$

Proof of Lemma 4- The proof of this lemma is identical to that of Lemma 2. One just has to change equation (8) into:

$$\frac{\mu(i+1, j+k)}{\mu(i, j+k)} = \frac{i+j+k}{2i} < \frac{i+j+k+1}{2i} = \frac{\mu(i+1, j+k+1)}{\mu(i, j+k+1)}.$$

Proof of Lemma 3-(i) As Y is a martingale,

$$h = \Big(1 - \mathbf{P}\Big(\sigma_{h+1} = \infty \Big| Y_0 = h \Big)\Big) \mathbf{E}\Big(Y_{\sigma_{h+1}} \Big| [Y_0 = h] \wedge [\sigma_{h+1} < \infty] \Big).$$

Lemma 4, combined with the central limit theorem implies that for some constant c,

$$\mathbf{E}\Big(Y_{\sigma_{h+1}} - h \Big| [Y_0 = h] \wedge [\sigma_{h+1} < \infty] \Big) \leq c\sqrt{h}.$$

Hence,

$$\Big(1 - \mathbf{P}\Big(\sigma_{h+1} = \infty \Big| Y_0 = h \Big)\Big) (h + c\sqrt{h}) \geq h$$

and (15) follows.

Proof of Lemma 3-(ii) Again, this is almost a word by word copy of the proof of (3) in the previous section. We leave it safely to the reader.

4 Proof of the main result

Fix $r \geq 2$ and rewrite $f(r)$ in the following way:

$$\begin{aligned}
f(r) &= \sum_{i=1}^{\infty} \mathbb{1}\Big\{ [\widetilde{X}_i \in \mathcal{K}_i] \wedge [|\mathcal{K}_i| \geq r] \Big\} \\
&= \sum_{i=1}^{\infty} \sum_{x \in \mathbf{Z}} \sum_{s=1}^{\infty} \mathbb{1}\Big\{ [\widetilde{X}_i = x] \wedge [L(x,i) = s] \wedge [x \in \mathcal{K}_i] \wedge [|\mathcal{K}_i| \geq r] \Big\} \\
&= \sum_{i=1}^{\infty} \sum_{x \in \mathbf{Z}} \sum_{s=1}^{\infty} \mathbb{1}\Big\{ [T(x,s) = i] \wedge [x \in \mathcal{K}_{T(x,s)}] \wedge [|\mathcal{K}_{T(x,s)}| \geq r] \Big\} \\
&= \sum_{x \in \mathbf{Z}} \sum_{s=1}^{\infty} \mathbb{1}\Big\{ [x \in \mathcal{K}_{T(x,s)}] \wedge [|\mathcal{K}_{T(x,s)}| \geq r] \Big\}
\end{aligned}$$

Put

$$g(r,s) = \sum_{x\in\mathbf{Z}} \mathbb{1}\Big\{[x \in \mathcal{K}_{T(x,s)}] \ \wedge\ [|\mathcal{K}_{T(x,s)}| \geq r]\Big\}$$

so that

$$f(r) = \sum_{s=1}^{\infty} g(r,s). \tag{18}$$

By symmetry,

$$\mathbf{E}\Big(\, g(r,s)\Big) = 2\sum_{x=1}^{\infty} \mathbf{P}\Big(\, [x \in \mathcal{K}_{T(x,s)}] \ \wedge\ [|\mathcal{K}_{T(x,s)}| \geq r]\Big).$$

Let us first consider the case where $s = 2h+1$ is odd. Note that in this case, $\mathcal{K}_{T(x,s)} \subset [1,x]$. It is immediate to check that

$$\mathbf{E}\Big(\, g(r,1)\Big) < \infty. \tag{19}$$

Suppose now that $h \geq 1$. Using Proposition 1, we get

$$\begin{aligned}
\mathbf{E}\Big(\, g(r,2h+1)\Big) \ &= \ 2\sum_{x=1}^{\infty} \mathbf{E}\Big(\, \mathbb{1}\{\, [Z_{\tau_{h,1}} = \ldots = Z_{\tau_{h,r-1}} = h] \ \wedge\ [\tau_{h,r-1} \leq x < \widehat{\tau}_{h+1,r}]\,\} \\
&\qquad \times \mathbf{P}(\, \sigma_{h+1} = \infty \,|\, Y_0 = Z_x \,) \,\Big|\, Z_0 = h\Big) \mathbf{P}\Big(\, \sigma_{h+1} = \infty \,\Big|\, Y_0 = h\Big) \\
&\leq \ 2\mathbf{P}\Big(\, Z_{\tau_h} = h \,\Big|\, Z_0 = h\Big)^{r-1} \mathbf{E}\Big(\, \tau_{h+1} \,\Big|\, Z_0 = h\Big) \mathbf{P}\Big(\, \sigma_{h+1} = \infty \,\Big|\, Y_0 = h\Big)
\end{aligned}$$

Hence, using lemmas 1 and 3 we find that for all $\varepsilon > 0$, there exists $c(\varepsilon)$ such that for all $h \geq 1$,

$$\mathbf{E}\Big(\, g(r,2h+1)\Big) \leq c(\varepsilon) h^{-(r-1)(1/2-\varepsilon)}. \tag{20}$$

Suppose now that $s = 2h$ is even. This time $\mathcal{K}_{T(x,s)} \cap [1, x-1] = \emptyset$. We define for $r' \geq 1$,

$$a(r,r',h) = \sum_{x=1}^{\infty} \mathbf{P}\Big(\, [x \in \mathcal{K}_{T(x,s)}] \ \wedge\ [|\mathcal{K}_{T(x,s)}| \geq r] \ \wedge\ [|\mathcal{K}_{T(x,s)} \cap [x,\infty)| = r']\Big),$$

so that

$$\mathbf{E}\Big(\, g(r,2h)\Big) = 2\sum_{r'=1}^{\infty} a(r,r',h). \tag{21}$$

For $r' \in [1, r-1]$, Proposition 1 yields

$$\begin{aligned}
a(r,r',h) \ &\leq \ \mathbf{P}\Big(\, [\sigma_{h,r'-1} < \infty = \sigma_{h,r'}] \ \wedge\ [Y_{\sigma_{h,0}} = \ldots = Y_{\sigma_{h,r'-1}} = h] \,\Big|\, Y_0 = h\Big) \\
&\quad \times \sum_{x=1}^{\infty} \mathbf{E}\Big(\, \mathbf{P}(\, [\sigma_{h,r-r'} < \infty = \widehat{\sigma}_{h+1,r-r'+1}] \ \wedge\ [Y_{\sigma_{h,1}} = \ldots = Y_{\sigma_{h,r-r'}} = h] \,|\, Y_0 = Z_x \,) \\
&\qquad\qquad \times \mathbb{1}\{\, 0 \leq x < \tau_h \,\} \,\Big|\, Z_0 = h-1\Big)
\end{aligned}$$

Using again the strong Markov property of the processes Z and Y, and Lemmas 1 and 3, we get for $1 \le r' < r$:

$$\begin{aligned} a(r,r',h) &\le \mathbf{E}\Big(\tau_h \,\Big|\, Z_0 = h-1\Big)\mathbf{P}\Big([\sigma_h < \infty] \wedge [Y_{\sigma_h} = h] \,\Big|\, Y_0 = h\Big)^{r-2} \\ &\qquad \times \mathbf{P}\Big(\sigma_{h+1} = \infty \,\Big|\, Y_0 = h\Big)^2 \\ &\le ch^{-(r-1)(1/2-\varepsilon)}. \end{aligned} \tag{22}$$

Similarly, Proposition 1 implies that

$$\begin{aligned} \sum_{r'=r}^{\infty} a(r,r',h) &\le \mathbf{P}\Big([\sigma_{h,r-1} < \infty = \widehat{\sigma}_{h+1,r}] \wedge [Y_{\sigma_{h,1}} = \ldots = Y_{\sigma_{h,r-1}} = h] \,\Big|\, Y_0 = h\Big) \\ &\qquad \times \sum_{x=1}^{\infty} \mathbf{E}\Big(\mathbb{1}\{0 \le x < \tau_h\} \,\Big|\, Z_0 = h-1\Big). \end{aligned}$$

Lemmas 1 and 3 then also imply that

$$\sum_{r'=r}^{\infty} a(r,r',h) \le c(\varepsilon) h^{-(r-1)(1/2-\varepsilon)}. \tag{23}$$

Putting the pieces together, (18), (19), (20), (21), (22) and (23) show that for all $\varepsilon > 0$, there exists $c(\varepsilon)$ such that

$$\mathbf{E}\Big(f(4)\Big) = \sum_{s=1}^{\infty} \mathbf{E}\Big(g(4,s)\Big) \le \mathbf{E}\Big(g(4,1)\Big) + c(\varepsilon)\sum_{h\ge 1} h^{-3/2+3\varepsilon}$$

and the theorem is proved (take e.g $\varepsilon = 1/9$).

Remarks- (1) The upper bounds given in the lemmas seem to be sharp; it is therefore unlikely that this proof can be directly improved to cover the case $r = 3$. We think that $f(3) < \infty$ almost surely, but presumably $\mathbf{E}(f(3)) = \infty$.

(2) It is worthwhile noticing that, using exactly the same technique, one can actually prove a slightly stronger result: Suppose $M > 0$ is a fixed integer, and consider for all $i > 0$, the set of 'almost favourite edges'

$$\mathcal{K}_i^M = \{x \in \mathbf{Z};\ L(x,i) \ge \sup_{y \in Z} L(y,i) - M\}.$$

Then, again, defining

$$f^M(r) = |\{i \ge 1;\ [\widetilde{X}_i \in \mathcal{K}_i^M] \wedge [|\mathcal{K}_i^M| \ge r]\}|$$

we have $\mathbf{E}(f^M(4))$ finite.

(3) Our approach does not seem to be well-suited to derive result (1) of Bass and Griffin. It seems that the expected number of times at which a fixed edge is favourite, is infinite; this does of course not imply (1).

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References

[1] Bass, R.F., Griffin, P.S. (1985): The most visited site of Brownian motion and random walk, *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **70**: 417-436

[2] Erdős, P., Révész, P. (1987): Problems and results on random walks. In: *Mathematical Statistics and Probability Theory* (eds.: P. Bauer, F. Koneczny, W. Wertz) pp. 59-65. D. Reidel, Dordrecht.

[3] Erdős, P., Révész, P. (1991): Three problems on the random walk in $\mathbf{Z}^d$. *Studia Sci. Math. Hung.* **26**: 309-320.

[4] Erdős, P. (1994): My work with Pál Révész. *Lecture delivered at the conference dedicated to the 60th birthday of Pál Révész.* Budapest, 8 June 1994.

[5] Knight, F.B. (1963): Random walks and a sojourn density process of Brownian motion. *Transactions of AMS* **109**: 56-86.

[6] Révész, P. (1990): *Random Walk in Random and Non-Random Environment.* World Scientific, Singapore.

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