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# Perturbation of Singular Equilibria of Hyperbolic Two-Component Systems: A Universal Hydrodynamic Limit

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**Abstract:** We consider one-dimensional, locally finite interacting particle systems with two conservation laws which under the Eulerian hydrodynamic limit lead to two-by-two systems of conservation laws:

$$\begin{cases} \partial_t \rho + \partial_x \Psi(\rho, u) = 0 \\ \partial_t u + \partial_x \Phi(\rho, u) = 0, \end{cases}$$

with $(\rho, u) \in \mathcal{D} \subset \mathbb{R}^2$, where $\mathcal{D}$ is a convex compact polygon in $\mathbb{R}^2$. The system is *typically* strictly hyperbolic in the interior of $\mathcal{D}$ with possible non-hyperbolic degeneracies on the boundary $\partial\mathcal{D}$. We consider the case of an isolated singular (i.e. non-hyperbolic) point on the interior of one of the edges of $\mathcal{D}$, call it $(\rho_0, u_0)$. We investigate the propagation of *small nonequilibrium perturbations* of the steady state of the microscopic interacting particle system, corresponding to the densities $(\rho_0, u_0)$ of the conserved quantities. We prove that for a very rich class of systems, under a proper hydrodynamic limit the propagation of these small perturbations are *universally* driven by the two-by-two system

$$\begin{cases} \partial_t \rho + \partial_x \big(\rho u\big) = 0 \\ \partial_t u + \partial_x \big(\rho + \gamma u^2\big) = 0, \end{cases}$$

where the parameter $\gamma$ is the only trace of the microscopic structure.

The proof relies on the relative entropy method and thus, it is valid only in the regime of smooth solutions of the pde. But there are essential new elements: in order to control the fluctuations of the terms with Poissonian (rather than Gaussian) decay coming from the low density approximations we have to apply refined pde estimates. In particular Lax entropies of these pde systems play a *not merely technical* key role in the main part of the proof.

## 1. Introduction

*1.1. The PDE to be derived and some facts about it.* We consider the pde

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t u + \partial_x(\rho + \gamma u^2) = 0 \end{cases} \tag{1.1}$$

for $(t, x) \in [0, \infty) \times (-\infty, \infty)$, where $\rho = \rho(t, x) \in \mathbb{R}_+$, $u = u(t, x) \in \mathbb{R}$ are density, respectively, velocity field and $\gamma \in \mathbb{R}$ is a fixed parameter. For any fixed $\gamma$ this is a *hyperbolic system of conservation laws* in the domain $(\rho, u) \in \mathbb{R}_+ \times \mathbb{R}$.

Phenomenologically, the pde describes a deposition/domain growth – or, in biological terms: chemotaxis – mechanism: $\rho(t, x)$ is the density of population performing the deposition and $h(t, x)$ is the height of the deposition. Let

$$u(t, x) := -\partial_x h(t, x).$$

The physics of the phenomenon is contained in the following two rules:

(a) The velocity field of the population is proportional to the *negative gradient of the height* of the deposition. That is, the population is pushed towards the local decrease of the deposition height. This rule, together with the conservation of total mass of the population leads to the continuity equation (the first equation in our system).
(b) The deposition rate is

$$\partial_t h = \rho + \gamma\left(\partial_x h\right)^2.$$

The first term on the right-hand side is just saying that deposition is done additively by the population. The second term is a self-generating deposition, introduced and phenomenologically motivated by Kardar-Parisi-Zhang [9] and commonly accepted in the literature. Differentiating this last equation with respect to the space variable $x$ results in the second equation of our system.

The pde (1.1) is invariant under the following scaling, if $\rho(t, x)$, $u(t, x)$ is a solution then

$$\widetilde{\rho}(t, x) := A^{2\beta}\rho(A^{1+\beta}t, Ax), \qquad \widetilde{u}(t, x) := A^{\beta}u(A^{1+\beta}t, Ax),$$

is also a solution, where $A > 0$ and $\beta \in \mathbb{R}$ are arbitrarily fixed. The choice $\beta = 0$ gives the straightforward hyperbolic scale invariance, valid for any system of conservation laws. More interesting is the $\beta = 1/2$ case. This is the natural scale invariance of the system, since the physical variables (density and velocity fields) change *covariantly* under this scaling. This is the (presumed, but never rigorously proved) asymptotic scale invariance of the Kardar-Parisi-Zhang deposition phenomena. The nontrivial scale invariance of the pde (1.1) suggests its *universality* in some sense. Our main result indeed states its validity in a very wide context.

It is also clear that the pde is invariant under the left-right reflection symmetry $x \mapsto -x$:

$$\widetilde{\rho}(t, x) := \rho(t, -x), \qquad \widetilde{u}(t, x) := -u(t, -x)$$

also satisfies (1.1).

The parameter $\gamma$ of the pde (1.1) is of crucial importance: different values of $\gamma$ lead to completely different behavior. Some particular cases which arose in the past in various contexts are listed:

- The pde (1.1) with $\gamma = 0$ arose in the context of the 'true self-repelling motion' constructed by Tóth and Werner in [23]. For a survey of this case see also [24]. The same equation, with viscosity terms added, appear in mathematical biology under the name of (negative) chemotaxis equations (see e.g. [17, 15, 14]).
- Taking $\gamma = 1/2$ we get the 'shallow water equation'. See [3, 13]. This is the only value of the parameter $\gamma$ when $m = \rho u$ is conserved and as a consequence the pde (1.1) can be interpreted as gas dynamics equation.
- With $\gamma = 1$ the pde is called 'Leroux's equation' which is of Temple class and for this reason much investigated. For many details about this equation see [19]. In the recent paper [6] Leroux's system has been derived as a hydrodynamic limit under Eulerian scaling for a two-component lattice gas, going even beyond the appearance of shocks.

The main facts about the pde (1.1) are presented in Subsect. 10.1 in the Appendix. Here we only mention that

1. For any $\gamma \in \mathbb{R}$ the system (1.1) is strictly *hyperbolic* in $(\rho, u) \in (0, \infty) \times \mathbb{R}$, with hyperbolicity marginally lost at $(\rho, u) = (0, 0)$ for $\gamma \neq 1/2$ and at $\rho = 0$ for $\gamma = 1/2$.
2. The *Riemann invariants* (or characteristic coordinates) are explicitly computed in Sect. 10.1, for a first impression see Fig. 1 of the Appendix where the level lines of the Riemann invariants are shown. It turns out that the picture changes qualitatively at the critical values $\gamma = 1/2$, $\gamma = 3/4$ and $\gamma = 1$. It is of crucial importance for our later problem that the level curves, expressed as $u \mapsto \rho(u)$ are convex for $\gamma < 1$, linear for $\gamma = 1$ and concave for $\gamma > 1$.
3. For any $\gamma \geq 0$ the system (1.1) is *genuinely nonlinear* in $(\rho, u) \in (0, \infty) \times \mathbb{R}$, with genuine nonlinearity marginally lost at $(\rho, u) = (0, 0)$ for $\gamma \neq 0, 1/2$ and at $\rho = 0$ for $\gamma = 0, 1/2$.
4. The system is sufficiently rich in *Lax entropies*.
5. For $\gamma \geq 0$ the system (1.1) satisfies the conditions of the Lax-Chuey-Conley-Smoller *Maximum Principle* (see [11, 12, 19]). However, this maximum principle yields a priori bounds for entropy solutions with bounded initial data *only for* $\gamma \geq 1$.

The goal of the present paper is to derive the two-by-two hyperbolic system of conservation laws (1.1) as a decent hydrodynamic limit of some systems of interacting particles with two conserved quantities.

We consider one-dimensional, locally finite interacting particle systems with two conservation laws with periodic boundary conditions which under the *Eulerian* hydrodynamic limit lead to two-by-two systems of conservation laws

$$\begin{cases} \partial_t \rho + \partial_x \Psi(\rho, u) = 0 \\ \partial_t u + \partial_x \Phi(\rho, u) = 0, \end{cases}$$

with $(t, x) \in [0, \infty) \times \mathbb{T}$, $(\rho, u) \in \mathcal{D} \subset \mathbb{R}^2$. Here $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the unit torus and $\mathcal{D}$ is a convex compact polygon in $\mathbb{R}^2$. The system is *typically* strictly hyperbolic in the interior of $\mathcal{D}$ with possible non-hyperbolic degeneracies on the boundary $\partial\mathcal{D}$. We consider the case of an isolated singular (i.e. non-hyperbolic) point on the interior of one of the edges of $\mathcal{D}$, call it $(\rho_0, u_0) = (0, 0)$ and assume $\mathcal{D} \subset \{\rho \geq 0\}$ (otherwise we apply an appropriate linear transformation on the conserved quantities). We investigate the propagation of *small nonequilibrium perturbations* of the steady state of the microscopic interacting particle system, corresponding to the densities $(\rho_0, u_0)$ of the conserved quantities. We prove that for a very rich class of systems, under a proper hydrodynamic limit the propagation of these small perturbations is *universally* driven by the system (1.1) on the unit torus, where the parameter $\gamma := \frac{1}{2}\Phi_{uu}(\rho_0, u_0)$ (with a proper choice of space and time scale) is the only trace of the microscopic structure. The proof is valid for the cases with $\gamma > 1$.

The proof essentially relies on H-T. Yau's relative entropy method and thus, it is valid only in the regime of smooth solutions of the pde (1.1).

We should emphasize here the essential new ideas of the proof. Since we consider a *low density* limit, the distribution of particle numbers in blocks of mesoscopic size will have a *Poissonian* rather than Gaussian tail. The fluctuations of the other conserved quantity will be Gaussian, as usual. It follows that when controlling the fluctuations of the empirical block averages the usual large deviation approach would lead us to the disastrous estimate $\mathbf{E}\big(\exp\{\varepsilon\ GAU \cdot POI\}\big) = \infty$. It turns out that some very special cutoff must be applied. Since the large fluctuations which are cut off cannot be estimated by robust methods (i.e. by applying entropy inequality), only some cancellation due to martingales can help. This is the reason why the cutoff function must be chosen in a very special way, in terms of a particular Lax entropy of the Euler equation. In this way the proof becomes an interesting mixture of probabilistic and pde arguments. The fine properties of the limiting pde, in particular the global behavior of Riemann invariants and some particular Lax entropies, play an essential role in the proof. The radical difference between the $\gamma \geq 1$ vs. $\gamma < 1$ cases, in particular applicability vs. non-applicability of the Lax-Chuey-Conley-Smoller maximum principle, manifests itself on the microscopic, probabilistic level.

*1.2. The structure of the paper.* In Sect. 2 we define the class of models to which our main theorem applies: we formulate the conditions to be satisfied by the interacting particle systems to be considered, we compute the steady state measures and the fluxes corresponding to the conserved quantities. At the end of this section we formulate the Eulerian hydrodynamic limit, for later reference.

In Sect. 3 first we perform asymptotic analysis of the Euler equations close to the singular point considered, then we formulate our main result, Theorem 1, and its immediate consequences.

In Sect. 4 we perform the necessary preliminary computations for the proof. After introducing the minimum necessary notation we apply some standard procedures in the context of the relative entropy method. Empirical block averages are introduced, numerical error terms are separated and estimated. In these first estimates only straightforward numerical approximations (Taylor expansion bounds) and the most direct entropy inequality are applied.

Section 5 is of crucial importance: here it is shown why the traditional approach of the relative entropy method fails to apply. Here it becomes apparent that in the fluctuation bound (usually referred to as *large deviation estimate*) instead of the tame $\mathbf{E}\big(\exp\{\varepsilon\ GAU^2\}\big)$ we would run into the wild $\mathbf{E}\big(\exp\{\varepsilon\ GAU \cdot POI\}\big)$, which is, of course, infinite. Here we describe our special cutoff function and we state its main properties in Lemma 2. The construction of the cutoff is outlined in the Appendix. The proof, that the constructed functions indeed possess the properties described in Lemma 2, is pure classical pde theory. It is a straightforward, although quite lengthy (and not entirely trivial) calculation. Since the detailed proof would lengthen our paper considerably and also because it would stick out a bit from the framework of the paper, it is omitted completely. The interested reader may look up the detailed proof in [25]. At the end of the section the outline of the further steps is presented.

In Sect. 6 all the necessary probabilistic ingredients of the forthcoming steps are gathered. These are: fixed time large deviation bounds and fixed time fluctuation bounds, the time averaged block replacement bounds (one block estimates) and the time averaged gradient bounds (closely related to the so-called two block estimates). The proof of these last two rely on Varadhan's large deviation bound cited in that section and on some probability lemmas stated and proved in Sect. 9. We should mention here that these proofs, in particular the probability lemmas involved also contain some new and instructive elements.

Sections 7 and 8 conclude the proof: the various terms arising in Sect. 5 are estimated using all the tools (probabilistic and pde) described in earlier sections. One can see that these estimates rely heavily on the fine properties of the Lax entropy used in the cutoff procedure.

As we already mentioned Sect. 9 is devoted to proofs of various lemmas stated in earlier parts.

In the first subsection of the Appendix we give some details about the pde (1.1). This is included for the sake of completeness and in order to let the reader have some more information about these, certainly interesting, pde-s. Strictly technically speaking this is not used in the proof. In the second subsection we outline the construction of the cutoff function.

## 2. Microscopic Models

Our interacting particle systems to be defined in the present section model on a microscopic level the same deposition phenomena as the pde (1.1). There will be two conserved physical quantities: the particle number $\eta_j \in \mathbb{N}$ and the (discrete) negative gradient of the deposition height $\zeta_j \in \mathbb{Z}$.

The dynamical driving mechanism is of such nature that

(i) The deposition height growth is influenced by the local particle density. Typically: growth is enhanced by higher particle densities.
(ii) The particle motion is itself influenced by the deposition profile. Typically: particles are pushed in the direction of the negative gradient of the deposition height.

*2.1. State space, conserved quantities.* Throughout this paper we denote by $\mathbb{T}^n$ the discrete tori $\mathbb{Z}/n\mathbb{Z}$, $n \in \mathbb{N}$, and by $\mathbb{T}$ the continuous torus $\mathbb{R}/\mathbb{Z}$. We will denote the local spin state by $\Omega$; we only consider the case when $\Omega$ is finite. The state space of the interacting particle system of size $n$ is

$$\Omega^n := \Omega^{\mathbb{T}^n}.$$

Configurations will be denoted

$$\underline{\omega} := (\omega_j)_{j\in\mathbb{T}^n} \in \Omega^n.$$

For sake of simplicity we consider discrete (integer valued) conserved quantities only. The two conserved quantities are

$$\eta : \Omega \to \mathbb{N}, \qquad \zeta : \Omega \to v_0\mathbb{Z} \text{ or } v_0\big(\mathbb{Z}+1/2\big). \tag{2.1}$$

The trivial scaling factor $v_0$ will be conveniently chosen later (see (2.4)). We also use the notations $\eta_j = \eta(\omega_j)$, $\zeta_j = \zeta(\omega_j)$. This means that the sums $\sum_j \eta_j$ and $\sum_j \zeta_j$ are conserved by the dynamics. We assume that the conserved quantities are different and non-trivial, i.e. the functions $\zeta, \eta$ and the constant function 1 on $\Omega$ are linearly independent.

The left-right reflection symmetry of the model is implemented by an involution

$$R : \Omega \to \Omega, \qquad R \circ R = Id$$

which acts on the conserved quantities as follows:

$$\eta(R\omega) = \eta(\omega), \qquad \zeta(R\omega) = -\zeta(\omega). \tag{2.2}$$

*2.2. Rate functions, infinitesimal generators, stationary measures.* Consider a (fixed) probability measure $\pi$ on $\Omega$, which is invariant under the action of the involution $R$, i.e. $\pi(R\omega) = \pi(\omega)$ and puts positive measure on every $\omega \in \Omega$. Since eventually we consider *low densities* of $\eta$, in order to exclude trivial cases we assume that

$$\pi\big(\, \zeta = 0 \,\big|\, \eta = 0 \,\big) < 1. \tag{2.3}$$

The scaling factor $v_0$ in (2.1) is chosen so that

$$\mathbf{Var}\big(\zeta \mid \eta = 0) = 1. \tag{2.4}$$

This choice simplifies some formulas (fixing a recurring constant to be equal to 1, see (3.4)) but does not restrict generality. For later use we introduce the notations

$$\rho^* := \max_{\omega\in\Omega} \eta(\omega), \qquad u^* := \max_{\omega\in\Omega} \zeta(\omega), \qquad u_* := \max_{\substack{\omega\in\Omega,\\ \eta(\omega)=0}} \zeta(\omega).$$

For $\tau, \theta \in \mathbb{R}$ let $G(\tau, \theta)$ be the moment generating function defined below:

$$G(\tau, \theta) := \log \sum_{\omega\in\Omega} e^{\tau\eta(\omega)+\theta\zeta(\omega)}\pi(\omega).$$

In thermodynamic terms $G(\tau, \theta)$ corresponds to the Gibbs free energy. We define the probability measures

$$\pi_{\tau,\theta}(\omega) := \pi(\omega) \exp(\tau\eta(\omega) + \theta\zeta(\omega) - G(\tau, \theta)) \tag{2.5}$$

on $\Omega$. We are going to define dynamics which conserve the quantities $\sum_j \eta_j$ and $\sum_j \zeta_j$, possess no other (hidden) conserved quantities and for which the product measures

$$\pi^n_{\tau,\theta} := \prod_{j\in\mathbb{T}^n} \pi_{\tau,\theta}$$

are stationary.

We need to separate a symmetric (reversible) part of the dynamics which will be speeded up sufficiently in order to enhance convergence to local equilibrium and thus helps to estimate some error terms in the hydrodynamic limiting procedure. So we consider two *rate functions* $r : \Omega \times \Omega \times \Omega \times \Omega \to \mathbb{R}_+$ and $s : \Omega \times \Omega \times \Omega \times \Omega \to \mathbb{R}_+$, $r$ will define the *asymmetric* component of the dynamics, while $s$ will define the *reversible* component. The dynamics of the system consists of elementary jumps affecting nearest neighbor spins, $(\omega_j, \omega_{j+1}) \longrightarrow (\omega'_j, \omega'_{j+1})$ performed with rate

$$\lambda\, r(\omega_j, \omega_{j+1}; \omega'_j, \omega'_{j+1}) + \kappa\, s(\omega_j, \omega_{j+1}; \omega'_j, \omega'_{j+1}),$$

where $\lambda, \kappa > 0$ are speed-up factors, depending on the size of the system in the limiting procedure.

We require that the rate functions $r$ and $s$ satisfy the following conditions:

(A) *Conservation laws:* If $r(\omega_1, \omega_2; \omega'_1, \omega'_2) > 0$ *or* $s(\omega_1, \omega_2; \omega'_1, \omega'_2) > 0$ then

$$\eta(\omega_1) + \eta(\omega_2) = \eta(\omega'_1) + \eta(\omega'_2),$$
$$\zeta(\omega_1) + \zeta(\omega_2) = \zeta(\omega'_1) + \zeta(\omega'_2).$$

(B) *Irreducibility:* For every $N \in [0, n\rho^*]$, $Z \in [-nu^*, nu^*]$ the set

$$\Omega^n_{N,Z} := \left\{ \underline{\omega} \in \Omega^n : \sum_{j\in\mathbb{T}^n} \eta_j = N,\ \sum_{j\in\mathbb{T}^n} \zeta_j = Z \right\}$$

is an irreducible component of $\Omega^n$, i.e. if $\underline{\omega}, \underline{\omega}' \in \Omega^n_{N,Z}$ then there exists a series of elementary jumps with positive rates transforming $\underline{\omega}$ into $\underline{\omega}'$.

(C) *Left-right symmetry:* The jump rates are invariant under left-right reflection *and* the action of the involution $R$ (jointly):

$$r(R\omega_2, R\omega_1; R\omega'_2, R\omega'_1) = r(\omega_1, \omega_2; \omega'_1, \omega'_2),$$
$$s(R\omega_2, R\omega_1; R\omega'_2, R\omega'_1) = s(\omega_1, \omega_2; \omega'_1, \omega'_2).$$

(D) *Stationarity of the asymmetric part:* For any $\omega_1, \omega_2, \omega_3 \in \Omega$,

$$Q(\omega_1, \omega_2) + Q(\omega_2, \omega_3) + Q(\omega_3, \omega_1) = 0,$$

where

$$Q(\omega_1, \omega_2) := \sum_{\omega'_1, \omega'_2 \in \Omega} \left\{ \frac{\pi(\omega'_1)\pi(\omega'_2)}{\pi(\omega_1)\pi(\omega_2)} r(\omega'_1, \omega'_2; \omega_1, \omega_2) - r(\omega_1, \omega_2; \omega'_1, \omega'_2) \right\}.$$

(E) *Reversibility of the symmetric part:* For any $\omega_1, \omega_2, \omega'_1, \omega'_2 \in \Omega$,

$$\pi(\omega_1)\pi(\omega_2)s(\omega_1, \omega_2; \omega'_1, \omega'_2) = \pi(\omega'_1)\pi(\omega'_2)s(\omega'_1, \omega'_2; \omega_1, \omega_2).$$

For a precise formulation of the infinitesimal generator on $\Omega^n$ we first define the map $\Theta_j^{\omega'\omega''} : \Omega^n \to \Omega^n$ for every $\omega', \omega'' \in \Omega$, $j \in \mathbb{T}^n$:

$$\left(\Theta_j^{\omega'\omega''}\underline{\omega}\right)_i = \begin{cases} \omega' & \text{if } i = j \\ \omega'' & \text{if } i = j+1 \\ \omega_i & \text{if } i \neq j, j+1. \end{cases}$$

The infinitesimal generators defined by these rates will be denoted:

$$L^n f(\underline{\omega}) = \sum_{j\in\mathbb{T}^n} \sum_{\omega',\omega''\in\Omega} r(\omega_j,\omega_{j+1};\omega',\omega'')\big(f(\Theta_j^{\omega'\omega''}\underline{\omega}) - f(\underline{\omega})\big),$$

$$K^n f(\underline{\omega}) = \sum_{j\in\mathbb{T}^n} \sum_{\omega',\omega''\in\Omega} s(\omega_j,\omega_{j+1};\omega',\omega'')\big(f(\Theta_j^{\omega'\omega''}\underline{\omega}) - f(\underline{\omega})\big).$$

We denote by $\mathcal{X}_t^n$ the Markov process on the state space $\Omega^n$ with infinitesimal generator

$$G^n := \lambda(n)L^n + \kappa(n)K^n \tag{2.6}$$

with speed-up factors $\lambda(n)$ and $\kappa(n)$ to be specified later. Let $\mu_0^n$ be a probability distribution on $\Omega^n$ which is the initial distribution of the microscopic system of size $n$, and

$$\mu_t^n := \mu_0^n e^{tG^n} \tag{2.7}$$

the distribution of the system at (macroscopic) time $t$.

*Remarks.*

(1) Conditions (A) and (B) together imply that $\sum_j \eta_j$ and $\sum_j \zeta_j$ are indeed the only conserved quantities of the dynamics.
(2) Condition (C) together with (2.2) is the implementation of the left-right symmetry of the pde (1.1) on a microscopic level. Actually, our main result, Theorem 1, is valid without this assumption but some of the arguments would be more technical.
(3) Condition (D) implies that the product measures $\pi_{\tau,\theta}^n$ are indeed stationary for the dynamics defined by the asymmetric rates $r$. This is seen by applying similar computations to those of [1, 2, 18] or [22]. Mind that this is *not* a detailed balance condition for the rates $r$.
(4) Condition (E) is a straightforward detailed balance condition. It implies that the product measures $\pi_{\tau,\theta}^n$ are reversible for the dynamics defined by the symmetric rates $s$.

We will refer to the measures $\pi_{\tau,\theta}^n$ as the *canonical* measures. Since $\sum_j \zeta_j$ and $\sum_j \eta_j$ are conserved the canonical measures on $\Omega^n$ are *not* ergodic. The conditioned measures defined on $\Omega_{N,Z}^n$ by:

$$\pi_{N,Z}^n(\underline{\omega}) := \pi_{\tau,\theta}^n\Big(\underline{\omega}\Big| \sum_{j\in\mathbb{T}^n} \eta_j = N,\ \sum_{j\in\mathbb{T}^n} \zeta_j = Z\Big) = \frac{\pi_{\tau,\theta}^n(\underline{\omega})\,1\!\!1\{\underline{\omega}\in\Omega_{N,Z}^n\}}{\pi_{\tau,\theta}^n(\Omega_{N,Z}^n)}$$

are also stationary and, due to condition (B) satisfied by the rate functions, they are ergodic. We shall call these measures the *microcanonical measures* of our system. (It is easy to see that the measure $\pi_{N,Z}^n$ does not depend on the choice of the values of $\tau$ and $\theta$ in the previous definition.)

The assumptions are by no means excessively restrictive. Here follow some concrete examples of interacting particle systems which belong to the class specified by conditions (A)–(E) and also satisfy the further conditions (F), (G), (H), (I) to be formulated later.

<u>$\{-1, 0, +1\}$-*model.*</u> The model is described and analyzed in full detail in [22] and [6]. The one spin state space is $\Omega = \{-1, 0, +1\}$ . The left-right reflection symmetry is implemented by $R : \Omega \to \Omega$, $R\omega = -\omega$. The dynamics consists of nearest neighbor spin exchanges and the two conserved quantities are $\eta(\omega) = 1 - |\omega|$ and $\zeta(\omega) = \omega$. The jump rates are

$$\begin{aligned} r(1, -1; -1, 1) &= 0, & r(-1, 1; 1, -1) &= 2,\\ r(0, -1; -1, 0) &= 0, & r(-1, 0; 0, -1) &= 1,\\ r(1, 0; 0, 1) &= 0, & r(0, 1; 1, 0) &= 1, \end{aligned}$$

and

$$s(\omega_1, \omega_2; \omega_1', \omega_2') = \begin{cases} 1 & \text{if } (\omega_1, \omega_2) = (\omega_2', \omega_1') \text{ and } \omega_1 \neq \omega_2 \\ 0 & \text{otherwise.} \end{cases}$$

The one dimensional marginals of the stationary measures are

$$\pi_{\rho,u}(0) = \rho, \quad \pi_{\rho,u}(\pm 1) = \frac{1 - \rho \pm u}{2}$$

with the domain of variables $\mathcal{D} = \{(\rho, u) \in \mathbb{R}_+ \times \mathbb{R} : \rho + |u| \leq 1\}$.

<u>*Two-lane models.*</u> The following family of examples are finite state space versions of the bricklayers models introduced in [24]. Let $\Omega = \{0, 1, \ldots, \bar{n}\} \times\{-\bar{z}, -\bar{z} + 1, \ldots, \bar{z} - 1, \bar{z}\}$, where $\bar{n} \in \mathbb{N}$ and $\bar{z} \in \{\frac{1}{2}, 1, \frac{3}{2}, 2 \ldots\}$. The elements of $\Omega$ will be denoted $\omega := \binom{\eta}{\zeta}$. Naturally enough, $\sum_j \eta_j$ and $\sum_j \zeta_j$ will be the conserved quantities of the dynamics. Left-right reflection symmetry is implemented as $R : \Omega \to \Omega$, $R\binom{\eta}{\zeta} = \binom{\eta}{zeta}$. We allow only the following elementary changes to occur at neighboring sites $j$, $j + 1$:

$$\binom{\eta_j\ \ \eta_{j+1}}{\zeta_j, \zeta_{j+1}} \to \binom{\eta_j \qquad \eta_{j+1}}{\zeta_j \mp 1, \zeta_{j+1} \pm 1}, \qquad \binom{\eta_j\ \ \eta_{j+1}}{\zeta_j, \zeta_{j+1}} \to \binom{\eta_j \mp 1\ \ \eta_{j+1} \pm 1}{\zeta_j \quad , \quad \zeta_{j+1}}$$

with appropriate rates. Beside the conditions already imposed we also assume that the one dimensional marginals of the steady state measures factorize as follows:

$$\pi(\omega) = \pi\binom{\eta}{\zeta} = p(\eta) q(\zeta).$$

The simplest case, with $\bar{n} = 1$ and $\bar{z} = 1/2$, that is with $\Omega = \{0, 1\} \times \{-1/2, +1/2\}$, was introduced and fully analyzed in [22] and [16]. For a full description (i.e. identification of the rates which satisfy the imposed conditions, Eulerian hydrodynamic limit, etc., see those papers.) It turns out that conditions (A)–(E) impose some nontrivial combinatorial constraints on the rates which are satisfied by a finite parameter family of models. The number of free parameters increases with $\bar{n}$ and $\bar{z}$. Since the concrete expressions of the rates are not relevant for our further presentation we omit the lengthy computations.

*2.3. Expectations.* Expectation, variance, covariance with respect to the measures $\pi^n_{\tau,\theta}$ will be denoted by $\mathbf{E}_{\tau,\theta}(.)$, $\mathbf{Var}_{\tau,\theta}(.)$, $\mathbf{Cov}_{\tau,\theta}(.)$.

We compute the expectations of the conserved quantities with respect to the canonical measures, as functions of the parameters $\tau$ and $\theta$:

$$\rho(\tau,\theta) := \mathbf{E}_{\tau,\theta}(\eta) = \sum_{\omega\in\Omega} \eta(\omega)\pi_{\tau,\theta}(\omega) = G_\tau(\tau,\theta),$$

$$u(\tau,\theta) := \mathbf{E}_{\tau,\theta}(\zeta) = \sum_{\omega\in\Omega} \zeta(\omega)\pi_{\tau,\theta}(\omega) = G_\theta(\tau,\theta).$$

Elementary calculations show that the matrix-valued function

$$\begin{pmatrix} \rho_\tau & \rho_\theta \\ u_\tau & u_\theta \end{pmatrix} = \begin{pmatrix} G_{\tau\tau} & G_{\tau\theta} \\ G_{\theta\tau} & G_{\theta\theta} \end{pmatrix} =: G''(\tau,\theta)$$

is equal to the covariance matrix $\mathbf{Cov}_{\tau,\theta}(\eta,\zeta)$, and therefore it is strictly positive definite. It follows that the function $(\tau,\theta) \mapsto (\rho(\tau,\theta), u(\tau,\theta))$ is invertible. We denote the inverse function by $(\rho,u) \mapsto (\tau(\rho,u), \theta(\rho,u))$. Denote by $(\rho,u) \mapsto S(\rho,u)$ the convex conjugate (Legendre transform) of the strictly convex function $(\tau,\theta) \mapsto G(\tau,\theta)$:

$$S(\rho,u) := \sup_{\tau,\theta} \big(\rho\tau + u\theta - G(\tau,\theta)\big), \tag{2.8}$$

and

$$\begin{aligned} \mathcal{D} &:= \overline{\{(\rho,u) \in \mathbb{R}_+ \times \mathbb{R} : S(\rho,u) < \infty\}} \\ &= \mathrm{co}\{(\eta,\zeta) : \pi(\omega) > 0\}, \end{aligned} \tag{2.9}$$

where co stands for convex hull and $\overline{A}$ is the closure of $A$. The nondegeneracy condition (2.3) implies that $\partial\mathcal{D} \cap \{\rho = 0\} = \{(0,u) : |u| \le u_*\}$. For $(\rho,u) \in \mathcal{D}$ we have

$$\tau(\rho,u) = S_\rho(\rho,u), \qquad \theta(\rho,u) = S_u(\rho,u).$$

In probabilistic terms: $S(\rho,u)$ is the rate function of joint large deviations of $\big(\sum_j \eta_j, \sum_j \zeta_j\big)$. In thermodynamic terms: $S(\rho,u)$ corresponds to the equilibrium thermodynamic entropy. Let

$$\begin{pmatrix} \tau_\rho & \tau_u \\ \theta_\rho & \theta_u \end{pmatrix} = \begin{pmatrix} S_{\rho\rho} & S_{\rho u} \\ S_{u\rho} & S_{uu} \end{pmatrix} =: S''(\rho,u).$$

It is obvious that the matrices $G''(\tau,\theta)$ and $S''(\rho,u)$ are strictly positive definite and are inverse of each other:

$$G''(\tau,\theta)S''(\rho,u) = I = S''(\rho,u)G''(\tau,\theta), \tag{2.10}$$

where either $(\tau,\theta) = (\tau(\rho,u), \theta(\rho,u))$ or $(\rho,u) = (\rho(\tau,\theta), u(\tau,\theta))$. With slight abuse of notation we shall denote:

$$\pi_{\tau(\rho,u),\theta(\rho,u)} =: \pi_{\rho,u}, \qquad \pi^n_{\tau(\rho,u),\theta(\rho,u)} =: \pi^n_{\rho,u}, \qquad \mathbf{E}_{\tau(\rho,u),\theta(\rho,u)} =: \mathbf{E}_{\rho,u}, \text{ etc.}$$

As a general convention, if $\xi : \Omega^m \to \mathbb{R}$ is a local function then its expectation with respect to the canonical measure $\pi^m_{\rho,u}$ is denoted by

$$\Xi(\rho,u) := \mathbf{E}_{\rho,u}(\xi) = \sum_{\omega_1,\dots,\omega_m\in\Omega^m} \xi(\omega_1,\dots,\omega_m)\pi_{\rho,u}(\omega_1)\cdots\pi_{\rho,u}(\omega_m).$$

*2.4. Fluxes.* We introduce the fluxes of the conserved quantities. The infinitesimal generators $L^n$ and $K^n$ act on the conserved quantities as follows (recall condition (A) on the rates):

$$\begin{aligned} L^n\eta_i &= -\psi(\omega_i,\omega_{i+1})+\psi(\omega_{i-1},\omega_i) &&=: -\psi_i+\psi_{i-1},\\ L^n\zeta_i &= -\phi(\omega_i,\omega_{i+1})+\phi(\omega_{i-1},\omega_i) &&=: -\phi_i+\phi_{i-1},\\ K^n\eta_i &= -\psi^s(\omega_i,\omega_{i+1})+\psi^s(\omega_{i-1},\omega_i) &&=: -\psi^s_i+\psi^s_{i-1},\\ K^n\zeta_i &= -\phi^s(\omega_i,\omega_{i+1})+\phi^s(\omega_{i-1},\omega_i) &&=: -\phi^s_i+\phi^s_{i-1}, \end{aligned}$$

where

$$\begin{aligned} \psi(\omega_1,\omega_2) &:= \sum_{\omega'_1,\omega'_2\in\Omega} r(\omega_1,\omega_2;\omega'_1,\omega'_2)\big(\eta(\omega'_2)-\eta(\omega_2)\big),\\ \phi(\omega_1,\omega_2) &:= \sum_{\omega'_1,\omega'_2\in\Omega} r(\omega_1,\omega_2;\omega'_1,\omega'_2)\big(\zeta(\omega'_2)-\zeta(\omega_2)\big), \end{aligned} \tag{2.11}$$

$$\begin{aligned} \psi^s(\omega_1,\omega_2) &:= \sum_{\omega'_1,\omega'_2\in\Omega} s(\omega_1,\omega_2;\omega'_1,\omega'_2)\big(\eta(\omega'_2)-\eta(\omega_2)\big),\\ \phi^s(\omega_1,\omega_2) &:= \sum_{\omega'_1,\omega'_2\in\Omega} s(\omega_1,\omega_2;\omega'_1,\omega'_2)\big(\zeta(\omega'_2)-\zeta(\omega_2)\big). \end{aligned} \tag{2.12}$$

Note that due to the left-right symmetry and conservations, i.e. (2.2) and conditions (A) and (C), the microscopic fluxes have the following symmetries:

$$\phi(\omega_1,\omega_2) = \phi(R\omega_2,R\omega_1), \qquad \psi(\omega_1,\omega_2) = -\psi(R\omega_2,R\omega_1).$$

In order to simplify some of our further arguments we impose one more microscopic condition

(F) *Gradient condition on symmetric fluxes:* The microscopic fluxes of the symmetric part, defined in (2.12) satisfy the following gradient conditions:

$$\begin{aligned} \psi^s(\omega_1,\omega_2) &= \kappa(\omega_1)-\kappa(\omega_2) =: \kappa_1-\kappa_2,\\ \phi^s(\omega_1,\omega_2) &= \chi(\omega_1)-\chi(\omega_2) =: \chi_1-\chi_2. \end{aligned} \tag{2.13}$$

*Remarks.*

(1) This is a technical assumption (referring actually to the measure $\pi$) which simplifies considerably the arguments of Sect. 7. The symmetric part $K^n$ has the role of enhancing convergence to local equilibrium. Its effect is *not seen* in the limit, so in principle we can choose it conveniently. Without this assumption we would be forced to use all the non-gradient technology developed in [26] (see also [10]), which would make the paper even longer.

(2) It is easy to see that $\eta(\omega_1)=\eta(\omega_2)=0$ implies $\psi^s(\omega_1,\omega_2)=0$ and thus (by choosing a suitable additive constant) $\omega\mapsto\kappa(\omega)$ can be chosen so that

$$\eta(\omega)=0 \;\Rightarrow\; \kappa(\omega)=0. \tag{2.14}$$

The *macroscopic fluxes* are:

$$\begin{aligned}\Psi(\rho,u) &:= \mathbf{E}_{\rho,u}(\psi) = \sum_{\omega_1,\omega_2} \psi(\omega_1,\omega_2)\pi_{\rho,u}(\omega_1)\pi_{\rho,u}(\omega_2),\\ \Phi(\rho,u) &:= \mathbf{E}_{\rho,u}(\phi) = \sum_{\omega_1,\omega_2} \phi(\omega_1,\omega_2)\pi_{\rho,u}(\omega_1)\pi_{\rho,u}(\omega_2).\end{aligned} \tag{2.15}$$

These are smooth regular functions of the variables $(\rho, u) \in \mathcal{D}$. Note that due to reversibility of $K^n$ (condition (E)), for any value of $\rho$ and $u$,

$$\mathbf{E}_{\rho,u}(\psi^s) = 0 = \mathbf{E}_{\rho,u}(\phi^s).$$

The following lemma is proved in [22].

**Lemma 1** (Onsager reciprocity relation)**.** *Suppose we have a particle system with two conserved quantities and rates satisfying Conditions (A) and (D). Then the following relation holds:*

$$\partial_\theta \Psi(\rho(\tau,\theta), u(\tau,\theta)) = \partial_\tau \Phi(\rho(\tau,\theta), u(\tau,\theta)).$$

We will use the lemma in the following equivalent form:

$$\begin{aligned}&\Psi_u(\rho,u)\mathbf{Var}_{\rho,u}(\zeta) - \Phi_u(\rho,u)\mathbf{Cov}_{\rho,u}(\eta,\zeta)\\ &\qquad = \Phi_\rho(\rho,u)\mathbf{Var}_{\rho,u}(\eta) - \Psi_\rho(\rho,u)\mathbf{Cov}_{\rho,u}(\eta,\zeta).\end{aligned} \tag{2.16}$$

For the concrete examples presented at the end of Subsect. 2.2 the following domains $\mathcal{D}$ and macroscopic fluxes are obtained:

$\{-1,0,+1\}$-*model.*:

$$\begin{aligned}&\mathcal{D} = \{(\rho,u) \in \mathbb{R}_+ \times \mathbb{R} \,:\, \rho + |u| \le 1\},\\ &\Psi(\rho,u) = \rho u, \quad \Phi(\rho,u) = \rho + u^2.\end{aligned}$$

*Two lane models* with $\bar{n} = 1$.

$$\begin{aligned}&\mathcal{D} = \{(\rho,u) \in \mathbb{R}_+ \times \mathbb{R} \,:\, \rho \le 1, \ \ |u| \le \bar{z}\},\\ &\Psi(\rho,u) = \rho(1-\rho)\psi(u), \quad \Phi(\rho,u) = \varphi_0(u) + \rho\varphi_1(u),\end{aligned}$$

where $\psi(u)$ is odd, while $\varphi_0(u)$ and $\varphi_1(u)$ are even functions of $u$, determined by the jump rates of the model. In the simplest particular case with $\bar{z} = 1/2$ ,

$$\Psi(\rho,u) = \rho(1-\rho)u, \qquad \Phi(\rho,u) = (\rho-\gamma)(1-u^2),$$

where $\gamma \in \mathbb{R}$ is the only model dependent parameter which appears in the macroscopic fluxes. For details see [22].

*2.5. The hydrodynamic limit under Eulerian scaling.* Given a system of interacting particles as defined in the previous subsections, under Eulerian scaling the local densities of the conserved quantities $\rho(t,x)$, $u(t,x)$ evolve according to the system of partial differential equations:

$$\begin{cases} \partial_t \rho + \partial_x \Psi(\rho, u) = 0 \\ \partial_t u + \partial_x \Phi(\rho, u) = 0, \end{cases} \tag{2.17}$$

where $\Psi(\rho, u)$ and $\Phi(\rho, u)$ are the macroscopic fluxes defined in (2.15).

The precise statement of the hydrodynamical limit is as follows: Consider a microscopic system which satisfies Conditions (A)–(E) of Subsect. 2.2. Let $\Psi(\rho, u)$ and $\Phi(\rho, u)$ be the macroscopic fluxes computed for this system and $\rho(t,x)$, $u(t,x)$ $x \in \mathbb{T}$, $t \in [0, T]$ be the *smooth* solution of the pde (2.17). Let the microscopic system of size $n$ be driven by the infinitesimal generator $G^n$ defined in (2.6) with $\lambda(n) = n$ and $\kappa(n) = n^{1+\delta}$ where $\delta \in [0, 1)$, is fixed. This means that the main, asymmetric part of the generator is speeded up by $n$ and the additional symmetric part by $n^{1+\delta}$. Let $\mu_t^n$ be the distribution of the system on $\Omega^n$ at (macroscopic) time $t$ defined by (2.7). The *local equilibrium* measure $\nu_t^n$ (itself a probability measure on $\Omega^n$) is defined by

$$\nu_t^n := \prod_{j \in \mathbb{T}^n} \pi_{\rho(t, \frac{j}{n}), u(t, \frac{j}{n})}.$$

This measure *mimics on a microscopic scale* the macroscopic evolution driven by the pde (2.17).

We denote by $H(\mu_t^n | \pi^n)$, respectively, by $H(\mu_t^n | \nu_t^n)$ the relative entropy of the measure $\mu_t^n$ with respect to the absolute reference measure $\pi^n$, respectively, with respect to the local equilibrium measure $\nu_t^n$.

The precise statement of the Eulerian hydrodynamic limit is the following

**Theorem.** *Assume Conditions (A)–(E) and let* $\delta \in [0, 1)$ *be fixed. If*

$$H\left(\mu_0^n \mid \nu_0^n\right) = o(n)$$

*then*

$$H\left(\mu_t^n \mid \nu_t^n\right) = o(n)$$

*uniformly for* $t \in [0, T]$.

The Theorem follows from direct application of Yau's relative entropy method. For the proof and its direct consequences see [10, 22] or [27]. For the main consequences of this Theorem, see e.g. Corollary 1 of [22].

## 3. Low Density Asymptotics and the Main Result: Hydrodynamic Limit Under Intermediate Scaling

*3.1. General properties and low density asymptotics of the macroscopic fluxes.* The fluxes in the Euler equation (2.17) are regular smooth functions $\mathcal{D}$.

From the left-right symmetry of the microscopic models it follows that

$$\Phi(\rho, -u) = \Phi(\rho, u), \qquad \Psi(\rho, -u) = -\Psi(\rho, u). \tag{3.1}$$

It is also obvious that for $u \in [-u_*, u_*]$,

$$\Psi(0, u) = 0. \tag{3.2}$$

We make two assumptions about the low density asymptotics of the macroscopic fluxes. Here is the first one:

(G) We assume that $\Psi_{\rho u}(0, 0) \neq 0$. Actually, by possibly redefining the time scale and orientation of space, without loss of generality we assume

$$\Psi_{\rho u}(0, 0) = 1. \tag{3.3}$$

Considering the Onsager relation (2.16) with $u = 0$ and taking the Taylor expansion around $\rho = 0$ it follows that

$$\Phi_\rho(0, 0) = \Psi_{\rho u}(0, 0)\mathbf{Var}_{0,0}(\zeta) = 1, \tag{3.4}$$

where in the second equality we used the choice (2.4) of the scaling factor $v_0$ in (2.1).

We denote

$$\gamma := \frac{1}{2}\Phi_{uu}(0, 0). \tag{3.5}$$

Our results will hold for $\gamma > 1$ only.

From (3.1) and (3.3) it follows that

$$\Phi_u(0, u) - \Psi_\rho(0, u) = (2\gamma - 1)u + \mathcal{O}(|u|^3). \tag{3.6}$$

The second condition imposed on the low density asymptotics of the macroscopic fluxes is:

(H) For $u \in [-u_*, u_*]$, $u \neq 0$,

$$\Phi_u(0, u) - \Psi_\rho(0, u) \neq 0, \tag{3.7}$$

$$\Phi_\rho(0, u) \neq 0, \quad \Psi_{\rho u}(0, u) \neq 0. \tag{3.8}$$

*Remarks.*

(1) (G) is a very natural nondegeneracy condition: if $\Psi_{\rho u}(0, 0)$ vanished then in the perturbation calculus to be performed, higher order terms would be dominant and a different scaling limit should be taken.

(2) Due to (3.1), (3.3) and (3.6) conditions (3.7), (3.8) hold anyway in a neighborhood of $u = 0$, and this would suffice; we assume Condition (H) for technical convenience only. Condition (3.7) amounts to forbidding other non-hyperbolic points on $\overline{\partial\mathcal{D} \cap \{\rho = 0\}}$, beside the point $(\rho, u) = (0, 0)$. Condition (3.8) reflects the natural monotonicity requirements (i) and (ii) formulated about the microscopic models at the beginning of Sect. 2. These conditions are used in the proof of Lemma 2, for the details see [25].

We are interested in the behavior of the pde near the isolated non-hyperbolic point $(\rho, u) = (0, 0)$. The asymptotic expansion for $\rho + u^2 \ll 1$ of the macroscopic fluxes and their first partial derivatives is

$$\begin{aligned}
\Psi(\rho, u) &= \rho u\big(1 + \mathcal{O}(\rho + u^2)\big), & \Phi(\rho, u) &= (\rho + \gamma u^2)\big(1 + \mathcal{O}(\rho + u^2)\big),\\
\Psi_\rho(\rho, u) &= u\big(1 + \mathcal{O}(\rho + u^2)\big), & \Phi_\rho(\rho, u) &= 1 + \mathcal{O}(\rho + u^2),\\
\Psi_u(\rho, u) &= \rho\big(1 + \mathcal{O}(\rho + u^2)\big), & \Phi_u(\rho, u) &= 2\gamma u\big(1 + \mathcal{O}(\rho + u^2)\big).
\end{aligned} \tag{3.9}$$

We are looking for "small solutions" of the pde (2.17): Let $\rho_0(x)$ and $u_0(x)$ be given profiles and assume that $\rho^\varepsilon(t,x)$, $u^\varepsilon(t,x)$ is the solution of the pde (2.17) with initial condition

$$\rho^\varepsilon(0,x) = \varepsilon^2 \rho_0(x), \quad u^\varepsilon(0,x) = \varepsilon\, u_0(x).$$

Then, at least formally,

$$\varepsilon^{-2}\rho^\varepsilon(\varepsilon^{-1}t,x) \to \rho(t,x), \quad \varepsilon^{-1}u^\varepsilon(\varepsilon^{-1}t,x) \to u(t,x),$$

where $\rho(t,x)$, $u(t,x)$ is solution of the pde (1.1) with initial condition

$$\rho(0,x) = \rho_0(x), \quad u(0,x) = u_0(x).$$

*3.2. The main result.* The asymptotic computations of Subsect. 3.1 suggest the scaling under which we should derive the pde (1.1) as a hydrodynamic limit: fix a (small) positive $\beta$ and choose the scaling:

| | space | time | particle density | 'slope of the wall' |
|---|---|---|---|---|
| MICRO | $nx$ | $n^{1+\beta}t$ | $n^{-2\beta}\rho$ | $n^{-\beta}u$ |
| MACRO | $x$ | $t$ | $\rho$ | $u$ |

Ideally the result should be valid for $0 < \beta < 1/2$ but we are able to prove much less than that.

Choose a model satisfying Conditions (A)–(F) of Sect. 2 and Conditions (G), (H) of Subsect. 3.1, and let $\gamma$ be given by (3.5), corresponding to the microscopic system chosen. Let the microscopic system of size $n$ (defined on the discrete torus $\mathbb{T}^n$) evolve on macroscopic time scale according to the infinitesimal generator $G^n$ (see (2.6)) with speed-up factors

$$\lambda(n) = n^{1+\beta}, \qquad \kappa(n) = n^{1+\beta+\delta},$$

with $\beta > 0$ and some further conditions to be imposed on $\beta$ and $\delta$ (see (3.12)). Denote by $\mu_t^n$ the true distribution of the microscopic system at macroscopic time $t$ with $\mu_0^n$ the initial distribution (see (2.7)).

We use the translation invariant product measure

$$\pi^n := \pi^n_{n^{-2\beta},0} \tag{3.10}$$

as *absolute reference measure*. Global entropy will be considered relative to this measure, Radon-Nikodym derivatives of $\mu_t^n$ and the local equilibrium measure $\nu_t^n$ to be defined below, with respect to $\pi^n$ will be used.

Given a smooth solution $\big(\rho(t,x), u(t,x)\big)$, $(t,x) \in [0,T] \times \mathbb{T}$, of the pde (1.1), define the *local equilibrium measure* $\nu_t^n$ on $\Omega^n$ as follows:

$$\nu_t^n := \prod_{j\in\mathbb{T}^n} \pi_{n^{-2\beta}\rho(t,\frac{j}{n}),n^{-\beta}u(t,\frac{j}{n})}. \tag{3.11}$$

This time-dependent measure *mimics on a microscopic level* the macroscopic evolution governed by the pde (1.1). Our main result is the following:

**Theorem 1.** *Assume that the microscopic system of interacting particles satisfies conditions (A)–(F) of Subsects. 2.2, 2.4 and the uniform log-Sobolev condition (I) of Subsect. 6.2. Additionally, assume that the macroscopic fluxes satisfy conditions (G), (H) of Subsect. 3.1 and* $\gamma > 1$. *Choose* $\beta \in (0, 1/2)$ *and* $\delta \in (1/2, 1)$ *so that*

$$2\delta - 8\beta > 1 \quad \textit{and} \quad \delta + 3\beta < 1. \tag{3.12}$$

*Let* $\big(\rho(t,x), u(t,x)\big)$, $(t,x) \in [0,T] \times \mathbb{T}$, *be a smooth solution of the pde (1.1), such that* $\inf_{x\in\mathbb{T}} \rho(0,x) > 0$ *and let* $\nu_t^n$, $t \in [0,T]$ *be the corresponding local equilibrium measure defined in (3.11).*

*Under these conditions, if*

$$H\big(\mu_0^n \,\big|\, \nu_0^n\big) = o(n^{1-2\beta}) \tag{3.13}$$

*then*

$$H\big(\mu_t^n \,\big|\, \nu_t^n\big) = o(n^{1-2\beta}) \tag{3.14}$$

*uniformly for* $t \in [0,T]$.

*Remarks.* (i) From (3.13) via the identity (4.5) and the entropy inequality it also follows that

$$H\big(\mu_0^n \,\big|\, \pi^n\big) = \mathcal{O}(n^{1-2\beta}). \tag{3.15}$$

See the beginning of Subsect. 4.2
(ii) If $\gamma > 3/4$, in smooth solutions vacuum does not appear. That is

$$\inf_{x\in\mathbb{T}} \rho(0,x) > 0 \qquad \text{implies} \qquad \inf_{(t,x)\in[0,T]\times\mathbb{T}} \rho(t,x) > 0.$$

(iii) Although for the $\{-1, 0, +1\}$-*model* we have $\gamma = 1$, our proof can also be extended to cover this model. Actually, in that case the proof is much simpler, since the Eulerian pde is equal to the limit pde (1.1) and thus the cutoff function (see Sect. 5) can be determined explicitly.

**Corollary 1.** *Assume the conditions of Theorem 1. Let* $g, h : \mathbb{T} \to \mathbb{R}$ *be smooth test functions. Then for any* $t \in [0,T]$,
*(i)*

$$\frac{1}{n}\sum_{j\in\mathbb{T}^n}\Big\{g(\frac{j}{n})n^{2\beta}\eta_j(t) + h(\frac{j}{n})n^{\beta}\zeta_j(t)\Big\} \stackrel{\mathbf{P}}{\to} \int_{\mathbb{T}} g(x)\rho(t,x) + h(x)u(t,x)\,dx.$$

*(ii)*

$$H\big(\mu_0^n \,\big|\, \pi^n\big) - H\big(\mu_t^n \,\big|\, \pi^n\big) = o(n^{1-2\beta}).$$

Corollary 1 can be easily proved by the standard use of the entropy inequality.

## 4. Notations and General Preparatory Computations

This section is completely standard in the context of the relative entropy method, so we shall be sketchy.

*4.1. Notation.* We denote

$$h^n(t) := n^{-1+2\beta} H\big(\mu_t^n \,\big|\, \nu_t^n\big), \qquad s^n(t) := n^{-1+2\beta}\left(H\big(\mu_0^n \,\big|\, \pi^n\big) - H\big(\mu_t^n \,\big|\, \pi^n\big)\right).$$

We know *a priori* that $t \mapsto s^n(t)$ is monotone increasing and due to (3.15),

$$s^n(t) = \mathcal{O}(1), \qquad \text{uniformly for } t \in [0, \infty). \tag{4.1}$$

In fact, from Theorem 1 it follows (see Corollary 1) that as long as the solution $\rho(t, x), u(t, x)$ of the pde (1.1) is smooth

$$s^n(t) = o(1), \qquad \text{uniformly for } t \in [0, T].$$

For $(\rho, u) \in (0, \infty) \times (-\infty, \infty)$ denote

$$\tau^n(\rho, u) := \tau(n^{-2\beta}\rho, n^{-\beta}u) - \tau(n^{-2\beta}, 0), \qquad \theta^n(\rho, u) := n^\beta \theta(n^{-2\beta}\rho, n^{-\beta}u).$$

Note that for symmetry reasons $\theta(n^{-2\beta}, 0) = 0$. Recall that $\tau$ is chemical potential rather than fugacity and for small densities the fugacity $\lambda := e^\tau$ scales like $\rho$, i.e. $\tau(n^{-2\beta}, 0) \sim -2\beta \log n$. If $\rho > 0$ and $u \in \mathbb{R}$ are fixed then $\tau^n(\rho, u)$ and $\theta^n(\rho, u)$ stay of order 1, as $n \to \infty$.

Given the smooth solution $\rho(t, x), u(t, x)$, with $\rho(t, x) > 0$ we shall use the notations

$$\tau^n(t, x) := \tau^n(\rho(t, x), u(t, x)), \qquad \theta^n(t, x) := \theta^n(\rho(t, x), u(t, x)),$$
$$v(t, x) := \log \rho(t, x).$$

The following asymptotics hold uniformly in $(t, x) \in [0, T] \times \mathbb{T}$:

$$\begin{aligned} \tau^n(t, x) &= \quad v(t, x) + \mathcal{O}(n^{-2\beta}), & \theta^n(t, x) &= \quad u(t, x) + \mathcal{O}(n^{-2\beta}), \\ \partial_x \tau^n(t, x) &= \partial_x v(t, x) + \mathcal{O}(n^{-2\beta}), & \partial_x \theta^n(t, x) &= \partial_x u(t, x) + \mathcal{O}(n^{-2\beta}), \\ \partial_t \tau^n(t, x) &= \partial_t v(t, x) + \mathcal{O}(n^{-2\beta}), & \partial_t \theta^n(t, x) &= \partial_t u(t, x) + \mathcal{O}(n^{-2\beta}). \end{aligned} \tag{4.2}$$

The logarithm of the Radon-Nikodym derivative of the time dependent reference measure $\nu_t^n$ with respect to the absolute reference measure $\pi^n$ is denoted by $f_t^n$:

$$\begin{aligned} f_t^n(\underline{\omega}) &:= \log \frac{d\nu_t^n}{d\pi^n}(\underline{\omega}) \\ &= \sum_{j \in \mathbb{T}^n} \Big\{ \tau^n(t, \frac{j}{n})\eta_j + n^{-\beta}\theta^n(t, \frac{j}{n})\zeta_j \\ &\qquad - G(\tau^n(t, \frac{j}{n}) + \tau(n^{-2\beta}, 0), n^{-\beta}\theta^n(t, \frac{j}{n})) + G(\tau(n^{-2\beta}, 0), 0) \Big\}. \end{aligned} \tag{4.3}$$

*4.2. Preparatory computations.* In order to obtain the main estimate (3.14) our aim is to get a Grönwall type inequality: we will prove that for every $t \in [0, T]$,

$$h^n(t) - h^n(0) = \int_0^t \partial_t h^n(s)ds \le C \int_0^t h^n(s)ds + o(1), \tag{4.4}$$

where the error term is uniform in $t \in [0, T]$. Since $h^n(0) = o(1)$ is assumed, Theorem 1 follows.

We start with the identity

$$H(\mu_t^n|\nu_t^n) - H(\mu_t^n|\pi^n) = -\int_{\Omega^n} f_t^n \, d\mu_t^n. \tag{4.5}$$

From this identity, the explicit form of the Radon-Nikodym derivative (4.3), the asymptotics (4.2), via the entropy inequality and (3.13), the a priori entropy bound (3.15) follows indeed, as remarked after the formulation of Theorem 1.

Next we differentiate (4.5) to obtain

$$\partial_t h^n(t) = -\int_{\Omega^n} \left( n^{3\beta} L^n f_t^n + n^{3\beta+\delta} K^n f_t^n + n^{-1+2\beta} \partial_t f_t^n \right) d\mu_t^n - \partial_t s^n(t). \tag{4.6}$$

Usually, an adjoint version of (4.6) is being used in the form of an inequality. In our case this form is needed. We emphasize that the term $-\partial_t s^n(t)$ on the right-hand side will be of crucial importance.

We compute the three terms under the integral using (4.3),

$$\begin{aligned} n^{3\beta} L^n f_t^n(\underline{\omega}) = {} & \frac{1}{n} \sum_{j \in \mathbb{T}^n} \partial_x v(t, \frac{j}{n}) n^{3\beta} \psi_j + \frac{1}{n} \sum_{j \in \mathbb{T}^n} \partial_x u(t, \frac{j}{n}) n^{2\beta} \phi_j \\ & + A_1^n(t, \underline{\omega}) + A_2^n(t, \underline{\omega}) + A_3^n(t, \underline{\omega}) + A_4^n(t, \underline{\omega}), \end{aligned} \tag{4.7}$$

where $A_i^n(t, \underline{\omega})$, $i = 1, \dots, 4$ are error terms which will be easy to estimate:

$$A_1^n(t, \underline{\omega}) := \frac{1}{n} \sum_{j \in \mathbb{T}^n} \left( \partial_x \tau^n(t, \frac{j}{n}) - \partial_x v(t, \frac{j}{n}) \right) n^{3\beta} \psi_j,$$

$$A_2^n(t, \underline{\omega}) := \frac{1}{n} \sum_{j \in \mathbb{T}^n} \left( \partial_x \theta^n(t, \frac{j}{n}) - \partial_x u(t, \frac{j}{n}) \right) n^{2\beta} \phi_j,$$

$$A_3^n(t, \underline{\omega}) := \frac{1}{n} \sum_{j \in \mathbb{T}^n} \left( \nabla^n \tau^n(t, \frac{j}{n}) - \partial_x \tau^n(t, \frac{j}{n}) \right) n^{3\beta} \psi_j,$$

$$A_4^n(t, \underline{\omega}) := \frac{1}{n} \sum_{j \in \mathbb{T}^n} \left( \nabla^n \theta^n(t, \frac{j}{n}) - \partial_x \theta^n(t, \frac{j}{n}) \right) n^{2\beta} \phi_j.$$

Here and in the sequel $\nabla^n$ denotes the discrete gradient:

$$\nabla^n f(x) := n\big( f(x + 1/n) - f(x) \big).$$

See Subsect. 4.4 for the estimate of the error terms $A_j^n(t, \underline{\omega})$, $j = 1, \dots, 12$.

Next, using the gradient condition (F) of the symmetric fluxes,

$$n^{3\beta+\delta} K^n f_t^n(\underline{\omega}) = n^{-1+3\beta+\delta} \frac{1}{n} \sum_{j\in\mathbb{T}^n} \left( \left(\nabla^n\right)^2 \tau^n(t, \frac{j}{n})\kappa_j + \left(\nabla^n\right)^2 \theta^n(t, \frac{j}{n})\chi_j \right)$$
$$=: A_5^n(t, \underline{\omega}) \tag{4.8}$$

is itself a numerical error term. Finally

$$\frac{n^{2\beta}}{n} \partial_t f_t^n(\underline{\omega}) = \frac{1}{n} \sum_{j\in\mathbb{T}^n} \left\{ \partial_t v(t, \frac{j}{n}) \left(n^{2\beta}\eta_j - \rho(t, \frac{j}{n})\right) + \partial_t u(t, \frac{j}{n}) \left(n^{\beta}\zeta_j - u(t, \frac{j}{n})\right) \right\}$$
$$+ A_6^n(t, \underline{\omega}) + A_7^n(t, \underline{\omega}), \tag{4.9}$$

where

$$A_6^n(t, \underline{\omega}) := \frac{1}{n} \sum_{j\in\mathbb{T}^n} \left(\partial_t \tau^n(t, \frac{j}{n}) - \partial_t v(t, \frac{j}{n})\right)\left(n^{2\beta}\eta_j - \rho(t, \frac{j}{n})\right),$$

$$A_7^n(t, \underline{\omega}) := \frac{1}{n} \sum_{j\in\mathbb{T}^n} \left(\partial_t \theta^n(t, \frac{j}{n}) - \partial_t u(t, \frac{j}{n})\right)\left(n^{\beta}\zeta_j - u(t, \frac{j}{n})\right)$$

are again easy-to-estimate error terms.

*4.3. Blocks.* We fix once and for all a weight function $a : \mathbb{R} \to \mathbb{R}$. It is assumed that:
(1) $a(x) > 0$ for $x \in (-1, 1)$ and $a(x) = 0$ otherwise,
(2) it has total weight $\int a(x)\, dx = 1$,
(3) it is even: $a(-x) = a(x)$, and
(4) it is smooth.

We choose a *mesoscopic* block size $l = l(n)$ such that

$$1 \ll n^{(1+\delta+5\beta)/3} \ll l(n) \ll n^{\delta-\beta} \ll n. \tag{4.10}$$

This can be done due to condition (3.12) imposed on $\beta$ and $\delta$.

Given a local variable (depending on $m$ consecutive spins)

$$\xi_i = \xi_i(\underline{\omega}) = \xi(\omega_i, \ldots, \omega_{i+m-1}),$$

its *block average at macroscopic space coordinate* $x$ is defined as

$$\widehat{\xi}^n(x) = \widehat{\xi}^n(\underline{\omega}, x) := \frac{1}{l} \sum_j a\left(\frac{nx - j}{l}\right) \xi_j. \tag{4.11}$$

Since $l = l(n)$, we do not denote explicitly dependence of the block average on the mesoscopic block size $l$. Note that $x \mapsto \widehat{\xi}^n(x)$ is smooth and

$$\partial_x \widehat{\xi}^n(x) = \partial_x \widehat{\xi}^n(\underline{\omega}, x) = \frac{n}{l} \frac{1}{l} \sum_j a'\left(\frac{nx - j}{l}\right) \xi_j,$$

and it is straightforward that

$$\sup_{\underline{\omega}\in\Omega^n}\sup_{x\in\mathbb{T}}\left|\partial_x\widehat{\xi}^n(\underline{\omega},x)\right|\leq C\left(\max_{\omega_1,\dots\omega_m}\xi(\omega_1,\dots,\omega_m)\right)\frac{n}{l}. \tag{4.12}$$

We shall use the handy (but slightly abused) notation

$$\widehat{\xi}^n(t,x):=\widehat{\xi}^n(\mathcal{X}_t^n,x).$$

This is the empirical block average process of the local observable $\xi_i$.

For the scaled block average of the two conserved quantities we shall also use the notation

$$\widehat{\rho}^n(t,x):=n^{2\beta}\widehat{\eta}^n(t,x),\qquad \widehat{u}^n(t,x):=n^{\beta}\widehat{\zeta}^n(t,x). \tag{4.13}$$

Note that these block averages are expected to be of order 1 in the limit.

Introducing block averages, the main terms on the right-hand side of (4.7) and (4.9) become:

$$\begin{aligned}\frac{1}{n}\sum_{j\in\mathbb{T}^n}\partial_x v(t,\frac{j}{n})n^{3\beta}\psi_j+\frac{1}{n}\sum_{j\in\mathbb{T}^n}\partial_x u(t,\frac{j}{n})n^{2\beta}\phi_j= \\ \frac{1}{n}\sum_{j\in\mathbb{T}^n}\partial_x v(t,\frac{j}{n})n^{3\beta}\widehat{\psi}^n(\frac{j}{n})+\frac{1}{n}\sum_{j\in\mathbb{T}^n}\partial_x u(t,\frac{j}{n})n^{2\beta}\widehat{\phi}^n(\frac{j}{n}) \\ +A_8^n(t,\underline{\omega})+A_9^n(t,\underline{\omega}),\end{aligned} \tag{4.14}$$

respectively

$$\begin{aligned}\frac{1}{n}\sum_{j\in\mathbb{T}^n}\partial_t v(t,\frac{j}{n})\big(n^{2\beta}\eta_j-\rho(t,\frac{j}{n})\big)+\frac{1}{n}\sum_{j\in\mathbb{T}^n}\partial_t u(t,\frac{j}{n})\big(n^{\beta}\zeta_j-u(t,\frac{j}{n})\big)= \\ \frac{1}{n}\sum_{j\in\mathbb{T}^n}\partial_t v(t,\frac{j}{n})\big(n^{2\beta}\widehat{\eta}^n(\frac{j}{n})-\rho(t,\frac{j}{n})\big)+\frac{1}{n}\sum_{j\in\mathbb{T}^n}\partial_t u(t,\frac{j}{n})\big(n^{\beta}\widehat{\zeta}^n(\frac{j}{n})-u(t,\frac{j}{n})\big) \\ +A_{10}^n(t,\underline{\omega})+A_{11}^n(t,\underline{\omega}).\end{aligned} \tag{4.15}$$

The error terms $A_i^n(t,\underline{\omega})$ $(i=8,9,10,11)$ are of the form

$$A_i^n(t,\underline{\omega}):=\frac{1}{n}\sum_{j\in\mathbb{T}^n}\Big(w\big(t,\frac{j}{n}\big)-\frac{1}{l}\sum_k a\big(\frac{j-k}{l}\big)w\big(t,\frac{k}{n}\big)\Big)\upsilon_j,$$

with $w=\partial_x v,\ \partial_x u,\ \partial_t v,\ \partial_t u$ and $\upsilon=n^{3\beta}\psi,\ n^{2\beta}\phi,\ n^{2\beta}\eta,\ n^{\beta}\zeta$ for $i=8,9,10,11$, respectively. These error terms will be estimated in Subsect. 4.4.

Since $[0,T]\times\mathbb{T}\ni(t,x)\mapsto(\rho(t,x),u(t,x))$, is a *smooth* solution of the pde (1.1), we have

$$\partial_t v=-u\partial_x v-\partial_x u,\qquad \partial_t u=-\rho\partial_x v-2\gamma u\partial_x u.$$

Inserting these expressions into the main terms of (4.15) eventually we obtain for the integrand in (4.6),

$$
\begin{aligned}
& n^{3\beta} L^n f_t^n(\underline{\omega}) + n^{3\beta+\delta} K^n f_t^n(\underline{\omega}) + n^{-1+2\beta} \partial_t f_t^n(\underline{\omega}) = \qquad (4.16) \\
& \quad \frac{1}{n} \sum_{j\in\mathbb{T}^n} \partial_x v(t, \frac{j}{n}) \Big\{ n^{3\beta} \widehat{\psi}^n(\frac{j}{n}) - \rho(t, \frac{j}{n}) u(t, \frac{j}{n}) \\
& \qquad - u(t, \frac{j}{n}) \big( n^{2\beta} \widehat{\eta}^n(\frac{j}{n}) - \rho(t, \frac{j}{n}) \big) - \rho(t, \frac{j}{n}) \big( n^{\beta} \widehat{\zeta}^n(\frac{j}{n}) - u(t, \frac{j}{n}) \big) \Big\} \\
& \quad + \frac{1}{n} \sum_{j\in\mathbb{T}^n} \partial_x u(t, \frac{j}{n}) \Big\{ n^{2\beta} \widehat{\phi}^n(\frac{j}{n}) - \big( \rho(t, \frac{j}{n}) + \gamma u(t, \frac{j}{n})^2 \big) \\
& \qquad - \big( n^{2\beta} \widehat{\eta}^n(\frac{j}{n}) - \rho(t, \frac{j}{n}) \big) - 2\gamma u(t, \frac{j}{n}) \big( n^{\beta} \widehat{\zeta}^n(\frac{j}{n}) - u(t, \frac{j}{n}) \big) \Big\} \\
& \quad + \sum_{k=1}^{12} A_k^n(t, \underline{\omega}),
\end{aligned}
$$

where

$$
\begin{aligned}
A_{12}^n(t) &:= \frac{1}{n} \sum_{j\in\mathbb{T}^n} \big( (\partial_x v)\rho u + (\partial_x u)(\rho + \gamma u^2) \big)(t, \frac{j}{n}) \\
&= \frac{1}{n} \sum_{j\in\mathbb{T}^n} \partial_x \big( \rho u + \frac{\gamma}{3} u^3 \big)(t, \frac{j}{n}).
\end{aligned}
$$

*4.4. The error terms $A_k^n$, $k = 1, \ldots, 12$.* We estimate these error terms with the help of the entropy inequality with respect to the measure $\pi^n$. Note that the variables $\eta_j$, $\zeta_j$, $\psi_j$ and $\phi_j$ are bounded and by (3.9), (3.10) we also have

$$
\begin{aligned}
&\big|\mathbf{E}_{\pi^n}(\eta_j)\big| \le C n^{-2\beta}, \quad \mathbf{Var}_{\pi^n}(\eta_j) \le C n^{-2\beta}, \quad \mathbf{E}_{\pi^n}(\zeta_j) = 0, \qquad \mathbf{Var}_{\pi^n}(\zeta_j) \le C, \\
&\mathbf{E}_{\pi^n}(\psi_j) = 0, \qquad \mathbf{Var}_{\pi^n}(\psi_j) \le C n^{-2\beta}, \quad \big|\mathbf{E}_{\pi^n}(\phi_j)\big| \le C, \quad \mathbf{Var}_{\pi^n}(\phi_j) \le C.
\end{aligned}
$$

Applying the entropy inequality in a straightforward way and using the previous bounds with the asymptotics (4.2) and uniform approximation of $\partial_x$ of smooth functions by their discrete derivative $\nabla^n$ we get that

$$
\mathbf{E}_{\mu_t^n}\big(A_k^n(t)\big) \le C\big(n^{-\beta} \vee n^{-1+2\beta+\delta} \vee n^{\beta} l^{-1} \vee n^{-1+\beta} l\big) = o(1)
$$

for $k = 1, \ldots, 11$. The computational details are obvious. Finally, $A_{12}^n(t)$ is a simple numerical error term (no probability involved):

$$
A_{12}^n(t) \le C n^{-1} = o(1).
$$

*4.5. Sumup.* Thus, integrating (4.6), using (4.16) and the bounds of Subsect. 4.4 we obtain

$$h^n(t) = \int_0^t \mathcal{A}^n(s)\,ds + \int_0^t \mathcal{B}^n(s)\,ds - s^n(t) + o(1), \tag{4.17}$$

where

$$\begin{aligned}
&\mathcal{A}^n(t) := \\
&\quad \mathbf{E}_{\mu_t^n}\Big(\tfrac{1}{n}\textstyle\sum_{j\in\mathbb{T}^n}\Big\{\big(\partial_x v\big)\big\{n^{3\beta}\widehat{\psi}^n - \rho u - u(\widehat{\rho}^n - \rho) - \rho(\widehat{u}^n - u)\big\}\Big\}(t, \tfrac{j}{n})\Big),\\
&\mathcal{B}^n(t) := \\
&\quad \mathbf{E}_{\mu_t^n}\Big(\tfrac{1}{n}\textstyle\sum_{j\in\mathbb{T}^n}\Big\{\big(\partial_x u\big)\big\{n^{2\beta}\widehat{\phi}^n - (\rho+\gamma u^2) - (\widehat{\rho}^n - \rho) - 2\gamma u(\widehat{u}^n - u)\big\}\Big\}(t, \tfrac{j}{n})\Big).
\end{aligned} \tag{4.18}$$

The main difficulty is caused by $\mathcal{A}^n(t)$. The term $\mathcal{B}^n(t)$ is estimated exactly as it is done in [21] for the one-component systems: since $\Phi(\rho, u) = \rho + \gamma u^2$ is linear in $\rho$ and quadratic in $u$ no problem is caused by the low particle density. By repeating the arguments of [21] we obtain

$$\int_0^t \mathcal{B}^n(s)ds \le C\int_0^t h^n(s)ds + o(1). \tag{4.19}$$

In the rest of the proof we concentrate on the essentially difficult term $\mathcal{A}^n(t)$.

## 5. Cutoff

We define the *rescaled macroscopic fluxes*

$$\Psi^n(\rho, u) := n^{3\beta}\Psi(n^{-2\beta}\rho, n^{-\beta}u), \quad \Phi^n(\rho, u) := n^{2\beta}\Phi(n^{-2\beta}\rho, n^{-\beta}u), \tag{5.1}$$

defined on the scaled domain

$$\mathcal{D}^n := \{(\rho, u) : (n^{-2\beta}\rho, n^{-\beta}u) \in \mathcal{D}\}. \tag{5.2}$$

The first partial derivatives of the scaled fluxes are

$$\begin{aligned}
\Psi^n_\rho(\rho, u) &= n^\beta \Psi_\rho(n^{-2\beta}\rho, n^{-\beta}u), \qquad \Phi^n_\rho(\rho, u) = \Phi_\rho(n^{-2\beta}\rho, n^{-\beta}u),\\
\Psi^n_u(\rho, u) &= n^{2\beta}\Psi_u(n^{-2\beta}\rho, n^{-\beta}u), \qquad \Phi^n_u(\rho, u) = n^\beta\Phi_u(n^{-2\beta}\rho, n^{-\beta}u).
\end{aligned} \tag{5.3}$$

For any $(\rho, u) \in \mathbb{R}_+ \times \mathbb{R}$,

$$\begin{aligned}
&\lim_{n\to\infty}\Psi^n(\rho, u) = \rho u, &&\lim_{n\to\infty}\Psi^n_\rho(\rho, u) = u, &&\lim_{n\to\infty}\Psi^n_u(\rho, u) = \rho,\\
&\lim_{n\to\infty}\Phi^n(\rho, u) = \rho + \gamma u^2, &&\lim_{n\to\infty}\Phi^n_\rho(\rho, u) = 1, &&\lim_{n\to\infty}\Phi^n_u(\rho, u) = 2\gamma u.
\end{aligned} \tag{5.4}$$

The convergence is uniform in compact subsets of $\mathbb{R}_+ \times \mathbb{R}$. Note that

$$\begin{aligned}
\Psi^n(\widehat{\rho}^n(t,x), \widehat{u}^n(t,x)) &= n^{3\beta}\Psi(\widehat{\eta}^n(t,x), \widehat{\zeta}^n(t,x)),\\
\Phi^n(\widehat{\rho}^n(t,x), \widehat{u}^n(t,x)) &= n^{2\beta}\Phi(\widehat{\eta}^n(t,x), \widehat{\zeta}^n(t,x)).
\end{aligned}$$

*5.1. The direct approach — why it fails.* The most natural thing is to write the summand in $\mathcal{A}^n(t)$ as

$$n^{3\beta}\widehat{\psi}^n - \rho u - u(\widehat{\rho}^n - \rho) - \rho(\widehat{u}^n - u) \\ = n^{3\beta}(\widehat{\psi}^n - \Psi(\widehat{\eta}^n, \widehat{\zeta}^n)) + (\Psi^n(\widehat{\rho}^n, \widehat{u}^n) - \widehat{\rho}^n\widehat{u}^n) + (\widehat{\rho}^n - \rho)(\widehat{u}^n - u). \tag{5.5}$$

By applying Varadhan's one block estimate and controlling the error terms in the Taylor expansion of $\Psi$, the first two terms on the right-hand side can be dealt with. However, the last term causes serious problems: with proper normalization, it is asymptotically distributed with respect to the local equilibrium measure $\nu_t^n$, like a product of independent Poisson and Gaussian random variables, and thus it does *not* have a finite exponential moment. Since the robust estimates heavily rely on the entropy inequality where the finite exponential moment is needed, we have to choose another approach for estimating $\mathcal{A}^n(t)$.

Instead of writing plainly (5.5), we introduce a cutoff. We let

$$M > \sup\{\rho(t,x) \vee |u(t,x)| : (t,x) \in [0,T] \times \mathbb{T}\}.$$

Let $I^n, J^n : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be bounded functions so that $I^n + J^n = 1$ and

$$I^n(\rho,u) = 1 \quad \text{for} \quad \rho \vee |u| \le M,$$

$$I^n(\rho,u) = 0 \quad \text{for 'large' } (\rho,u).$$

The last property will be specified later.

We split the right-hand side of (5.5) in a most natural way, according to this cutoff:

$$n^{3\beta}\widehat{\psi}^n - \rho u - u(\widehat{\rho}^n - \rho) - \rho(\widehat{u}^n - u) = n^{3\beta}\widehat{\psi}^n J^n(\widehat{\rho}^n, \widehat{u}^n) \\ -(\widehat{\rho}^n u + \rho\widehat{u}^n - \rho u)J^n(\widehat{\rho}^n, \widehat{u}^n) + n^{3\beta}(\widehat{\psi}^n - \Psi(\widehat{\eta}^n, \widehat{\zeta}^n))I^n(\widehat{\rho}^n, \widehat{u}^n) \\ +(\Psi^n(\widehat{\rho}^n, \widehat{u}^n) - \widehat{\rho}^n\widehat{u}^n)I^n(\widehat{\rho}^n, \widehat{u}^n) + (\widehat{\rho}^n - \rho)(\widehat{u}^n - u)I^n(\widehat{\rho}^n, \widehat{u}^n). \tag{5.6}$$

The second term on the right-hand side is linear in the block averages, so it does not cause any problem. The third term is estimated by use of Varadhan's one block estimate. The fourth term is Taylor approximation. Finally, the last term can be handled with the entropy inequality *if the cutoff $I^n(\rho,u)$ is strong enough* to tame the tail of the Gaussian×Poisson random variable.

The main difficulty is caused by the first term on the right-hand side. This term certainly can not be estimated with the robust method, i.e. with entropy inequality: we would run into the same problem we wanted to overcome by introducing the cutoff. The only way this term may be small is by some cancellation. It turns out that the desired cancellations indeed occur (in the form of a martingale appearing in the space-time average) if and only if

$$J^n(\rho,u) = S^n_\rho(\rho,u), \tag{5.7}$$

where $S^n(\rho,u)$ is a particular *Lax entropy of the scaled Euler equation*

$$\begin{cases} \partial_t\rho + \partial_x\Psi^n(\rho,u) = 0 \\ \partial_t u + \partial_x\Phi^n(\rho,u) = 0, \end{cases} \tag{5.8}$$

with $\Psi^n(\rho, u)$ and $\Phi^n(\rho, u)$ defined in (5.1). This means that there exists a flux function $F^n(\rho, u)$ with

$$F^n_\rho = \Psi^n_\rho S^n_\rho + \Phi^n_\rho S^n_u, \qquad F^n_u = \Psi^n_u S^n_\rho + \Phi^n_u S^n_u, \tag{5.9}$$

or equivalently, the following pde holds

$$\Psi^n_u S^n_{\rho\rho} + \left(\Phi^n_u - \Psi^n_\rho\right) S^n_{\rho u} - \Phi^n_\rho S^n_{uu} = 0. \tag{5.10}$$

*5.2. The cutoff function.* In the present subsection we describe the cutoff function (5.7) – or rather: the respective Lax entropies. In Lemma 2 we state some related estimates which will be of paramount importance in our further proof. The construction of the needed Lax entropies is outlined in Subsect. 10.2 of the Appendix. The proof that the Lax entropies described there indeed satisfy the conditions of Lemma 2, is pure classical pde theory. It is a straightforward, although quite lengthy (and not entirely trivial) calculation. Since the full proof would lengthen our paper considerably, we omit these computations. The interested reader can find the detailed proof in [25].

**Lemma 2.** *Let $M > 0$ and $\varepsilon > 0$ be fixed arbitrary numbers. There exist twice-differentiable Lax entropy/flux pairs $S^n(\rho, u)$, $F^n(\rho, u)$ defined on $\mathcal{D}^n$ for every (large enough) $n$ such that the following inequalities hold. The positive constants $A, B, C$ depend on $M$ and $\varepsilon$, but not on $n$,*

$$|S^n_\rho(\rho, u) - 1\!\!1_{\{\rho \geq A + B|u|\}}| \leq C\, 1\!\!1_{\{M \leq \rho < A + B|u|, |u| \geq M\}}, \tag{5.11}$$

$$|S^n_u(\rho, u)| \leq C\, 1\!\!1_{\{M \leq \rho < A + B|u|, |u| \geq M\}}, \tag{5.12}$$

$$|S^n_{\rho\rho}(\rho, u)| \leq \frac{\varepsilon}{1+\rho}\, 1\!\!1_{\{M \leq \rho < A + B|u|, |u| \geq M\}}, \tag{5.13}$$

$$|S^n_{\rho u}(\rho, u)| \leq \frac{\varepsilon}{1+\sqrt{\rho}+|u|}\, 1\!\!1_{\{M \leq \rho < A + B|u|, |u| \geq M\}}, \tag{5.14}$$

$$|S^n_{uu}(\rho, u)| \leq \varepsilon\, 1\!\!1_{\{M \leq \rho < A + B|u|, |u| \geq M\}}, \tag{5.15}$$

$$|F^n(\rho, u) - \Psi^n(\rho, u) S^n_\rho(\rho, u)| \leq C(1+u^2)\, 1\!\!1_{\{M \leq \rho < A + B|u|, |u| \geq M\}}. \tag{5.16}$$

It is easy to see that the function $I^n = 1 - S^n_\rho$ is indeed a cutoff: $I^n = 0$ if $\rho \vee |u| \leq M$ and $I^n = 1$ for 'large' values of $(\rho, u)$, namely for $\rho \geq A + B\,|u|$.

The choice of $M$ will be specified by the large deviation bounds given in Proposition 1 (via Lemma 6), the choice of $\varepsilon$ will be determined in Subsect. 7.4 (see (7.16)).

*5.3. Outline of the further steps of proof.* In Sect. 7 we give an estimate for the terms with 'large' values of $(\rho, u)$, we prove that

$$\left| \int_0^t \mathbf{E}_{\mu^n_s} \left( \frac{1}{n} \sum_{j \in \mathbb{T}^n} \left\{ \left(\partial_x v\right)\left(n^{3\beta} \widehat{\psi}^n\right) J^n(\widehat{\rho}^n, \widehat{u}^n) \right\}(s, \frac{j}{n}) \right) ds \right| \tag{5.17}$$

$$\leq \frac{1}{2} h^n(t) + \frac{1}{2} s^n(t) + C \int_0^t h^n(s)\, ds + o(1).$$

In Sect. 8 we estimate the terms with 'small' values of $(\rho, u)$, the section is divided into four subsections. In Subsect. 8.1 we prove

$$\left| \mathbf{E}_{\mu_s^n} \left( \frac{1}{n} \sum_{j \in \mathbb{T}^n} \left\{ \left(\partial_x v\right) \left(\widehat{\rho}^n u + \rho \widehat{u}^n - \rho u\right) J^n(\widehat{\rho}^n, \widehat{u}^n) \right\}(s, \frac{j}{n}) \right) \right| \tag{5.18}$$
$$\leq C\, h^n(s) + o(1).$$

In Subsect. 8.2 we prove the one block estimate

$$\left| \int_0^t \mathbf{E}_{\mu_s^n} \left( \frac{1}{n} \sum_{j \in \mathbb{T}^n} \left\{ \left(\partial_x v\right) n^{3\beta} \left(\widehat{\psi}^n - \Psi(\widehat{\eta}^n, \widehat{\zeta}^n)\right) I^n(\widehat{\rho}^n, \widehat{u}^n) \right\}(s, \frac{j}{n}) \right) ds \right| \tag{5.19}$$
$$= o(1).$$

In Subsect. 8.3 we control the Taylor approximation

$$\left| \mathbf{E}_{\mu_s^n} \left( \frac{1}{n} \sum_{j \in \mathbb{T}^n} \left\{ \left(\partial_x v\right) \left(\Psi^n(\widehat{\rho}^n, \widehat{u}^n) - \widehat{\rho}^n \widehat{u}^n\right) I^n(\widehat{\rho}^n, \widehat{u}^n) \right\}(s, \frac{j}{n}) \right) \right| \tag{5.20}$$
$$\leq C\, h^n(s) + o(1).$$

Finally, in Subsection 8.4 we control the fluctuations

$$\left| \mathbf{E}_{\mu_s^n} \left( \frac{1}{n} \sum_{j \in \mathbb{T}^n} \left\{ \left(\partial_x v\right) \left(\widehat{\rho}^n - \rho\right) \left(\widehat{u}^n - u\right) I^n(\widehat{\rho}^n, \widehat{u}^n) \right\}(s, \frac{j}{n}) \right) \right| \tag{5.21}$$
$$\leq C\, h^n(s) + o(1).$$

Having all these done, from (4.18), (5.6) and the bounds (5.17), (5.18), (5.19), (5.20), (5.21) it follows that

$$\int_0^t \mathcal{A}^n(s) ds \leq \frac{1}{2} h^n(t) + \frac{1}{2} s^n(t) + C \int_0^t h^n(s) ds + o(1). \tag{5.22}$$

Finally, from (4.17), (4.19), (5.22) and noting that $s^n(t) \geq 0$ we get the desired Grönwall inequality (4.4) and the Theorem follows. Note the importance of the term $-\partial_t s^n(t)$ on the right-hand side of (4.6).

## 6. Tools

*6.1. Fixed time estimates.* In the estimates with fixed time $s \in [0, T]$ we shall use the notation

$$L = L(n) := n^{-2\beta} l. \tag{6.1}$$

Note that $L \gg 1$ as $n \to \infty$.

The following general entropy estimate will be exploited all over:

**Lemma 3** (Fixed time entropy inequality)**.** *Let $l \le n$, $\mathcal{V} : \Omega^l \to \mathbb{R}$ and denote $\mathcal{V}_j(\underline{\omega}) := \mathcal{V}(\omega_j, \dots, \omega_{j+l-1})$. Then for any $\gamma > 0$,*

$$\mathbf{E}_{\mu_s^n}\left(\frac{1}{n}\sum_{j\in\mathbb{T}^n}\mathcal{V}_j(\mathcal{X}_s^n)\right) \le \frac{1}{\gamma}h^n(s) + \frac{1}{\gamma\, L}\,\frac{1}{n}\sum_{j\in\mathbb{T}^n}\log \mathbf{E}_{\nu_s^n}\left(\exp\left\{\gamma\, L\,\mathcal{V}_j\right\}\right). \tag{6.2}$$

This lemma is a standard tool in the context of relative entropy method. For its proof we refer the reader to the original paper [27] or the monograph [10].

**Proposition 1** (Fixed time large deviation bounds)**.**
*(i) For any $c > 0$ there exists $M < \infty$ such that for any $s \in [0, T]$*

$$\mathbf{E}_{\mu_s^n}\left(\frac{1}{n}\sum_{j\in\mathbb{T}^n}\left\{\left(1+\widehat{\rho}^n+\left|\widehat{u}^n\right|\right)1\!\!1_{\{\widehat{\rho}^n\vee|\widehat{u}^n|>M\}}\right\}(s,\frac{j}{n})\right) \le c\, h^n(s) + o(1). \tag{6.3}$$

*(ii) There exist $C < \infty$ and $M < \infty$ such that for any $s \in [0, T]$*

$$\mathbf{E}_{\mu_s^n}\left(\frac{1}{n}\sum_{j\in\mathbb{T}^n}\left\{\left|\widehat{u}^n\right|^2 1\!\!1_{\{\widehat{\rho}^n\vee|\widehat{u}^n|>M\}}\right\}(s,\frac{j}{n})\right) \le C\, h^n(s) + o(1). \tag{6.4}$$

The proof of Proposition 1 is postponed to Subsect. 9.1. It relies on the entropy inequality (6.2) of Lemma 3, the stochastic dominations formulated in Lemma 5 (see Subsect. 9.1) and standard large deviation bounds.

**Proposition 2** (Fixed time fluctuation bounds)**.** *For any $M < \infty$ there exists a $C < \infty$ such that the following bounds hold:*

$$\mathbf{E}_{\mu_s^n}\left(\frac{1}{n}\sum_{j\in\mathbb{T}^n}\left|\widehat{u}^n - u\right|^2(s,\frac{j}{n})\right) \le C\, h^n(s) + o(1), \tag{6.5}$$

$$\mathbf{E}_{\mu_s^n}\left(\frac{1}{n}\sum_{j\in\mathbb{T}^n}\left\{\left|\widehat{\rho}^n - \rho\right|^2 1\!\!1_{\{\widehat{\rho}^n\le M\}}\right\}(s,\frac{j}{n})\right) \le C\, h^n(s) + o(1). \tag{6.6}$$

The proof of Proposition 2 is postponed to Subsect. 9.2. It relies on the entropy inequality (6.2) of Lemma 3, and Gaussian fluctuation estimates.

*6.2. Convergence to local equilibrium and a priori bounds.* The hydrodynamic limit relies on macroscopically fast convergence to (local) equilibrium in blocks of mesoscopic size $l$. Fix the block size $l$, $N \in [0, l\max\eta]$, $Z \in [l\min\zeta, l\max\zeta]$ and denote

$$\Omega^l_{N,Z} := \left\{\underline{\omega}\in\Omega^l\ :\ \sum_{j=1}^{l}\eta_j = N, \sum_{j=1}^{l}\zeta_j = Z\right\},$$

$$\pi^l_{N,Z}(\underline{\omega}) := \pi^l_{\lambda,\theta}(\underline{\omega}\,|\,\sum_{j=1}^{l}\eta_j = N, \sum_{j=1}^{l}\zeta_j = Z).$$

Naturally, we are only interested in the pairs $(N, Z)$ for which $\Omega^l_{N,Z}$ is not empty. Expectation with respect to the measure $\pi^l_{N,Z}$ is denoted by $\mathbf{E}^l_{N,Z}(\,\cdot\,)$. For $f : \Omega^l_{N,Z} \to \mathbb{R}$ let

$$K^l_{N,Z} f(\underline{\omega}) := \sum_{j=1}^{l-1} \sum_{\omega',\omega''} s(\omega_j, \omega_{j+1}; \omega', \omega'')\big(f(\Theta^{\omega',\omega''}_{j,j+1}\underline{\omega}) - f(\underline{\omega})\big),$$

$$D^l_{N,Z}(f) := \frac{1}{2}\sum_{j=1}^{l-1} \mathbf{E}^l_{N,Z}\left(\sum_{\omega',\omega''} s(\omega_j, \omega_{j+1}; \omega', \omega'')\big(f(\Theta^{\omega',\omega''}_{j,j+1}\underline{\omega}) - f(\underline{\omega})\big)^2\right).$$

In plain words: $\Omega^l_{N,Z}$ is the hyperplane of configurations $\underline{\omega} \in \Omega^l$ with fixed values of the conserved quantities, $\pi^l_{N,Z}$ is the *microcanonical distribution* on this hyperplane, $K^l_{N,Z}$ is the symmetric infinitesimal generator restricted to the hyperplane $\Omega^l_{N,Z}$, and finally $D^l_{N,Z}$ is the Dirichlet form associated to $K^l_{N,Z}$. Note that $K^l_{N,Z}$ is defined with *free boundary conditions*.

The convergence to local equilibrium is *quantitatively controlled* by the following uniform logarithmic Sobolev estimate, assumed to hold:

(I) *Logarithmic Sobolev inequality:* There exists a finite constant $\aleph$ such that for any $l \in \mathbb{N}$, $N \in [0, l\max\eta]$, $Z \in [l\min\zeta, l\max\zeta]$, and any $h : \Omega^l_{N,Z} \to \mathbb{R}_+$ with $\mathbf{E}^l_{N,Z}(h) = 1$ the following bound holds:

$$\mathbf{E}^l_{N,Z}\big(h\log h\big) \le \aleph\, l^2 D^l_{N,Z}\left(\sqrt{h}\right). \tag{6.7}$$

*Remark.* The uniform logarithmic Sobolev inequality (6.7) is expected to hold for a very wide range of locally finite interacting particle systems, though we do not know about a fully general proof. In [28] the logarithmic Sobolev inequality is proved for symmetric $K$-exclusion processes. This implies that (6.7) holds for the two lane models defined in Sect. 2. In [6] Yau's method of proving logarithmic Sobolev inequality is applied and the logarithmic Sobolev inequality is stated for random stirring models with an arbitrary number of colors. In particular, (6.7) follows for the $\{-1, 0, +1\}$-model defined in Sect. 2.

The following large deviation bound goes back to Varadhan [26]. See also the monographs [10] and [4].

**Lemma 4** (Time-averaged entropy inequality, local equilibrium)**.** *Let $l \le n$, $\mathcal{V} : \Omega^l \to \mathbb{R}_+$ and denote $\mathcal{V}_j(\underline{\omega}) := \mathcal{V}(\omega_j, \dots, \omega_{j+l-1})$. Then for any $q > 0$,*

$$\int_0^t \mathbf{E}_{\mu^n_s}\left(\frac{1}{n}\sum_{j\in\mathbb{T}^n} \mathcal{V}_j(\mathcal{X}^n_s)\right) ds \le \tag{6.8}$$

$$\frac{\aleph\, l^3}{2\, q\, n^{1+3\beta+\delta}}\left(s^n(t) + \frac{2\, n^{1+3\beta+\delta}\, t}{\aleph l^3}\max_{N,Z}\log \mathbf{E}^l_{N,Z}\big(\exp\{q\mathcal{V}\}\big)\right).$$

*Remarks.* (1) Since

$$\frac{n^{1+3\beta+\delta}}{l^3} = o(1),$$

in order to efficiently apply Lemma 4 one has to choose $q = q(n)$ so that

$$\mathbf{E}^{l}_{N,Z}\big(\exp\{q\mathcal{V}\}\big) = \mathcal{O}(1),$$

*uniformly* in the block size $l = l(n)\in\mathbb{N}$, and in $N\in[0, l\max\eta]$ and $Z\in[l\min\zeta, l\max\zeta]$.
(2) The proof of the bound (6.8) explicitly relies on the logarithmic Sobolev inequality (6.7). It appears in [29] and it is reproduced in several places, see e.g. [4, 5]. We do not repeat it here.

The main probabilistic ingredients of our proof are summarized in Proposition 3 which is a consequence of Lemma 4. These are variants of the celebrated *one block estimate*, respectively, *two blocks estimate* of Varadhan and co-authors.

**Proposition 3** (Time-averaged block replacement and gradient bounds)**.** *Given a local variable $\xi : \Omega^m \to \mathbb{R}$ there exists a constant C such that the following bounds hold:*
*(i)*

$$\int_0^t \mathbf{E}_{\mu^n_s}\left(\int_{\mathbb{T}} \big|\{\widehat{\xi}^{\,n} - \Xi(\widehat{\eta}^{\,n}, \widehat{\zeta}^{\,n})\}(s,x)\big|^2\,dx\right) ds \le C\,\frac{l^2}{n^{1+3\beta+\delta}}\big(s^n(t) + o(1)\big). \tag{6.9}$$

*(ii)*

$$\int_0^t \mathbf{E}_{\mu^n_s}\left(\int_{\mathbb{T}} \big|\partial_x\widehat{\xi}^{\,n}(s,x)\big|^2\,dx\right) ds \le Cn^{1-3\beta-\delta}\big(s^n(t) + o(1)\big). \tag{6.10}$$

*(iii) Further on, if $\xi : \Omega \to \mathbb{R}$ (that is: it depends on a single spin) and $\xi(\omega) = 0$ whenever $\eta(\omega) = 0$ then the following stronger version of the gradient bound holds:*

$$\int_0^t \mathbf{E}_{\mu^n_s}\left(\int_{\mathbb{T}} \frac{\big|\partial_x\widehat{\xi}^{\,n}(s,x)\big|^2}{\widehat{\eta}^{\,n}(s,x)}\,dx\right) ds \le Cn^{1-3\beta-\delta}\big(s^n(t) + o(1)\big). \tag{6.11}$$

The proof of Proposition 3 is postponed to Subsect. 9.3. It relies on the large deviation bound (6.8) and some elementary probability estimates stated in Lemma 9 (see Subsect. 9.3).

We shall apply (6.9) to $\xi = \phi$ and $\xi = \psi$. From (6.10) it follows that

$$\int_0^t \mathbf{E}_{\mu^n_s}\left(\int_{\mathbb{T}} \big|\partial_x\widehat{u}^{\,n}(s,x)\big|^2\,dx\right) ds \le Cn^{1-\beta-\delta}\big(s^n(t) + o(1)\big), \tag{6.12}$$

$$\int_0^t \mathbf{E}_{\mu^n_s}\left(\int_{\mathbb{T}} \big|\partial_x\widehat{\rho}^{\,n}(s,x)\big|^2\,dx\right) ds \le Cn^{1+\beta-\delta}\big(s^n(t) + o(1)\big). \tag{6.13}$$

Using (6.11) the last bound is improved to

$$\int_0^t \mathbf{E}_{\mu^n_s}\left(\int_{\mathbb{T}} \frac{\big|\partial_x\widehat{\rho}^{\,n}(s,x)\big|^2}{\widehat{\rho}^{\,n}(s,x)}\,dx\right) ds \le Cn^{1-\beta-\delta}\big(s^n(t) + o(1)\big). \tag{6.14}$$

The bound (6.11) will also be applied to $\xi = \kappa$ (see (2.13) and (2.14)) to get

$$\int_0^t \mathbf{E}_{\mu^n_s}\left(\int_{\mathbb{T}} \frac{\big|n^{2\beta}\partial_x\widehat{\kappa}^{\,n}(s,x)\big|^2}{\widehat{\rho}^{\,n}(s,x)}\,dx\right) ds \le Cn^{1-\beta-\delta}\big(s^n(t) + o(1)\big). \tag{6.15}$$

## 7. Control of the Large Values of $(\rho, u)$: Proof of (5.17)

*7.1. Preparations.* In the present section we prove (5.17). First we replace the sum $\frac{1}{n}\sum_{\mathbb{T}^n}\cdots$ by $\int_{\mathbb{T}}\cdots\,dx$. Note that given a smooth function $F:\mathbb{T}\to\mathbb{R}$,

$$\left|\frac{1}{n}\sum_{j\in\mathbb{T}^n} F(\frac{j}{n}) - \int_{\mathbb{T}} F(x)\,dx\right| \le \frac{1}{n}\left(\int_{\mathbb{T}} |\partial_x F(x)|^2\,dx\right)^{1/2}. \tag{7.1}$$

Hence it follows that

$$\begin{aligned}\mathbf{E}_{\mu^n}\left(\int_0^t \frac{1}{n}\sum_{j\in\mathbb{T}^n}\left\{\left(\partial_x v\right)\left(n^{3\beta}\widehat{\psi}^n\right) J^n(\widehat{\rho}^n,\widehat{u}^n)\right\}(s,\frac{j}{n})\,ds\right) &= \\ \mathbf{E}\left(\int_0^t\int_{\mathbb{T}}\left\{\left(\partial_x v\right)\left(n^{3\beta}\widehat{\psi}^n\right) J^n(\widehat{\rho}^n,\widehat{u}^n)\right\}(s,x)\,dx\,ds\right) &+ A^n_{13},\end{aligned} \tag{7.2}$$

where $A^n_{13}$ is again a simple numerical error term:

$$\begin{aligned}\left|A^n_{13}\right| &\le Cn^{3\beta}\left\{1+\sup_{\substack{0\le s\le t\\ x\in\mathbb{T}}}\left(\left|\partial_x\widehat{\psi}^n(s,x)\right|+\left|\partial_x\widehat{\rho}^n(s,x)\right|+\left|\partial_x\widehat{u}^n(s,x)\right|\right)\right\}\\ &= \mathcal{O}(n^{5\beta}l^{-1}) = o(1).\end{aligned}$$

In the last step we use the boundedness of the function $\partial_x v(t,x)$ and the most straightforward gradient bound (4.12).

We have to prove that the main term on the right-hand side of (7.2) is negligible. Recall that $J^n = S^n_\rho$. We start with the application of the martingale identity:

$$\begin{aligned}&\mathbf{E}_{\mu^n}\left(\int_{\mathbb{T}}\left\{\left\{vS^n(\widehat{\rho}^n,\widehat{u}^n)\right\}(t,x)-\left\{vS^n(\widehat{\rho}^n,\widehat{u}^n)\right\}(0,x)\right.\right.\\ &\qquad\left.\left.-\int_0^t\left\{(\partial_t v)S^n(\widehat{\rho}^n,\widehat{u}^n)\right\}(s,x)\,ds\right\}dx\right) =\\ &\mathbf{E}_{\mu^n}\left(\int_0^t\int_{\mathbb{T}} v(s,x)\left(n^{1+\beta}L^nS^n(\widehat{\rho}^n(x),\widehat{u}^n(x))\right)(\mathcal{X}^n_s)\,dx\,ds\right)\\ &\qquad+\mathbf{E}_{\mu^n}\left(\int_0^t\int_{\mathbb{T}} v(s,x)\left(n^{1+\beta+\delta}K^nS^n(\widehat{\rho}^n(x),\widehat{u}^n(x))\right)(\mathcal{X}^n_s)\,dx\,ds\right).\end{aligned} \tag{7.3}$$

*7.2. The left-hand side of (7.3).* From (5.11), (5.12) we conclude that

$$\left|S^n(\rho,u)\right| \le C\left(\rho+|u|\right)\mathbb{1}_{\{\rho\vee|u|>M\}}.$$

Hence, using the large deviation bound (6.3) it follows that, for any fixed small $c>0$, by choosing $M$ sufficiently large, we obtain

$$\mathbf{E}_{\mu^n_s}\left(\frac{1}{n}\sum_{j\in\mathbb{T}^n}\left|S^n(\widehat{\rho}^n,\widehat{u}^n)(s,\frac{j}{n})\right|\right) \le c\,h^n(s)+o(1).$$

Applying again (7.1), choosing an appropriately small $c$ in the previous bound we get

$$|\,\text{l.h.s. of (7.3)}\,| \le \frac{1}{2}h^n(t) + C\int_0^t h^n(s)\,ds + o(1). \tag{7.4}$$

*Remark.* Note that this is the point where $M$ and thus the lower edge of the cutoff is fixed. Also note the importance of the factor $1/2$ in front of $h^n(t)$ on the right-hand side.

*7.3. The right-hand side of (7.3): first computations.* First we compute how the infinitesimal generators $n^{1+\beta}L^n$ and $n^{1+\beta+\delta}K^n$ act on the function $\underline{\omega}\mapsto S^n(\widehat{\rho}^n(x),\widehat{u}^n(x))$:

$$\begin{aligned} &n^{1+\beta}L^n S^n(\widehat{\rho}^n(x),\widehat{u}^n(x)) = \\ &\qquad\Big\{S^n_\rho(\widehat{\rho}^n,\widehat{u}^n)\big(n^{3\beta}\partial_x\widehat{\psi}^n\big) + S^n_u(\widehat{\rho}^n,\widehat{u}^n)\big(n^{2\beta}\partial_x\widehat{\phi}^n\big)\Big\}(x) + A^n_{14}(\underline{\omega},x), \end{aligned} \tag{7.5}$$

$$\begin{aligned} &n^{1+\beta+\delta}K^n S^n(\widehat{\rho}^n(x),\widehat{u}^n(x)) = \\ &\quad n^{-1+\beta+\delta}\Big\{S^n_\rho(\widehat{\rho}^n,\widehat{u}^n)\big(n^{2\beta}\partial_x^2\widehat{\kappa}^n\big) + S^n_u(\widehat{\rho}^n,\widehat{u}^n)\big(n^{\beta}\partial_x^2\widehat{\chi}^n\big)\Big\}(x) + A^n_{15}(\underline{\omega},x), \end{aligned} \tag{7.6}$$

where $A^n_{14}(x)$ and $A^n_{15}(x)$ are *numerical error terms*. These error terms are easily estimated: using the fact that the second partial derivatives of $S^n$ are uniformly bounded, $\zeta$ and $\eta$ are bounded, by simple Taylor expansion after tedious but otherwise straightforward computations we find:

$$\sup_{\underline{\omega}\in\Omega^n}\sup_{x\in\mathbb{T}}\left(\left|A^n_{14}(\underline{\omega},x)\right| + \left|A^n_{15}(\underline{\omega},x)\right|\right) \le C\left(n^{1+3\beta}l^{-2} + n^{1+5\beta+\delta}l^{-3}\right) = o(1). \tag{7.7}$$

For similar computational details see [6] or [25].

Next we do some further transformations on the main terms coming from the right-hand sides of (7.5) and (7.6). Performing integrations by part, introducing the macroscopic fluxes and using (5.9) we obtain:

$$\begin{aligned} &-\int_{\mathbb{T}} v(x)\Big\{S^n_\rho(\widehat{\rho}^n,\widehat{u}^n)\partial_x\big(n^{3\beta}\widehat{\psi}^n\big) + S^n_u(\widehat{\rho}^n,\widehat{u}^n)\partial_x\big(n^{2\beta}\widehat{\phi}^n\big)\Big\}(x)\,dx \\ &\quad= \int_{\mathbb{T}} \partial_x v(x)\Big\{\big(n^{3\beta}\widehat{\psi}^n\big)S^n_\rho(\widehat{\rho}^n,\widehat{u}^n)\Big\}(x)\,dx \\ &\qquad\quad+ \int_{\mathbb{T}} \partial_x v(x)\Big\{F^n(\widehat{\rho}^n,\widehat{u}^n) - \Psi^n(\widehat{\rho}^n,\widehat{u}^n)S^n_\rho(\widehat{\rho}^n,\widehat{u}^n)\Big\}(x)\,dx \\ &\qquad\quad+ \int_{\mathbb{T}} \partial_x v(x)\Big\{n^{2\beta}S^n_u(\widehat{\rho}^n,\widehat{u}^n)\big(\widehat{\phi}^n - \Phi(\widehat{\eta}^n,\widehat{\zeta}^n)\big)\Big\}(x)\,dx \\ &\quad+ \int_{\mathbb{T}} v(x)\Big\{n^{3\beta}\left(S^n_{\rho\rho}(\widehat{\rho}^n,\widehat{u}^n)\big(\partial_x\widehat{\rho}^n\big) + S^n_{\rho u}(\widehat{\rho}^n,\widehat{u}^n)\big(\partial_x\widehat{u}^n\big)\right)\left(\widehat{\psi}^n - \Psi(\widehat{\eta}^n,\widehat{\zeta}^n)\right) \\ &\qquad+ n^{2\beta}\left(S^n_{u\rho}(\widehat{\rho}^n,\widehat{u}^n)\big(\partial_x\widehat{\rho}^n\big) + S^n_{uu}(\widehat{\rho}^n,\widehat{u}^n)\big(\partial_x\widehat{u}^n\big)\right)\left(\widehat{\phi}^n - \Phi(\widehat{\eta}^n,\widehat{\zeta}^n)\right)\Big\}(x)\,dx. \end{aligned} \tag{7.8}$$

Note that, since $J^n = S^n_\rho$, the first term on the right-hand side is exactly the expression in the main term on the right-hand side of (7.2). Estimating the other terms on the right-hand side of (7.8) is the object of the next subsection. Also note that here we rely heavily on the fact that $S^n$ is a Lax entropy of the pde (5.8); without this we would not be able to carry out the needed calculations.

Now we turn to the main term on the right-hand side of (7.6). Straightforward integration by parts yields

$$\begin{aligned}
-\int_{\mathbb{T}} v(x)\Big\{S^n_\rho(\widehat{\rho}^n,\widehat{u}^n)\big(n^{2\beta}\partial_x^2\widehat{\kappa}^n\big)+S^n_u(\widehat{\rho}^n,\widehat{u}^n)\big(n^{\beta}\partial_x^2\widehat{\chi}^n\big)\Big\}(x)\,dx&\\
=\int_{\mathbb{T}} \partial_x v(x)\Big\{S^n_\rho(\widehat{\rho}^n,\widehat{u}^n)\big(n^{2\beta}\partial_x\widehat{\kappa}^n\big)+S^n_u(\widehat{\rho}^n,\widehat{u}^n)\big(n^{\beta}\partial_x\widehat{\chi}^n\big)\Big\}(x)\,dx&\\
+\int_{\mathbb{T}} v(x)\Big\{\big(S^n_{\rho\rho}(\widehat{\rho}^n,\widehat{u}^n)\big(\partial_x\widehat{\rho}^n\big)+S^n_{\rho u}(\widehat{\rho}^n,\widehat{u}^n)\big(\partial_x\widehat{u}^n\big)\big)\big(n^{2\beta}\partial_x\widehat{\kappa}^n\big)&\\
+\big(S^n_{\rho u}(\widehat{\rho}^n,\widehat{u}^n)\big(\partial_x\widehat{\rho}^n\big)+S^n_{uu}(\widehat{\rho}^n,\widehat{u}^n)\big(\partial_x\widehat{u}^n\big)\big)\big(n^{\beta}\partial_x\widehat{\chi}^n\big)\Big\}(x)\,dx.&
\end{aligned}\tag{7.9}$$

We will estimate the terms emerging from the right-hand side in the next subsection.

*7.4. The right-hand side of (7.3): bounds.* By (5.16)

$$\big|F^n(\widehat{\rho}^n,\widehat{u}^n)-\Psi^n(\widehat{\rho}^n,\widehat{u}^n)S^n_\rho(\widehat{\rho}^n,\widehat{u}^n)\big|\le C\big(1+\big|\widehat{u}^n\big|^2\big)\mathbb{1}_{\{\widehat{\rho}^n\vee|\widehat{u}^n|>M\}}.$$

Hence, applying the large deviation bounds (6.3) and (6.4) we obtain

$$\mathbf{E}_{\mu^n_s}\left(\int_{\mathbb{T}}\Big|\Big\{F^n(\widehat{\rho}^n,\widehat{u}^n)-\Psi^n(\widehat{\rho}^n,\widehat{u}^n)S^n_\rho(\widehat{\rho}^n,\widehat{u}^n)\Big\}(s,x)\Big|\,dx\right)\le C\,h^n(s)+o(1).\tag{7.10}$$

Next we use the bound (5.12) on $S^n_u$ and the first block replacement bound (6.9) to obtain:

$$\mathbf{E}_{\mu^n}\left(\int_0^t\int_{\mathbb{T}}\Big|\Big\{n^{2\beta}S^n_u(\widehat{\rho}^n,\widehat{u}^n)\big(\widehat{\phi}^n-\Phi(\widehat{\eta}^n,\widehat{\zeta}^n)\big)\Big\}(s,x)\Big|\,dx\,ds\right)\le C\,l\,n^{(-1-\delta+\beta)/2}=o(1).\tag{7.11}$$

For the next terms we use the bounds on the second derivatives of $S^n$, see (5.13), (5.14), (5.15), and note that here *we do not exploit* the fact that the constant factor $\varepsilon$ on the right-hand side is actually small. Together with the block replacement bounds (6.9), the gradient bounds (6.12), (6.14) and the bound (4.1) on the relative entropy $s^n(t)$ we get the following four estimates:

$$\begin{aligned}
\mathbf{E}_{\mu^n}\left(\int_0^t\int_{\mathbb{T}}\Big|\Big\{n^{3\beta}S^n_{\rho\rho}(\widehat{\rho}^n,\widehat{u}^n)\big(\partial_x\widehat{\rho}^n\big)\big(\widehat{\psi}^n-\Psi(\widehat{\eta}^n,\widehat{\zeta}^n)\big)\Big\}\Big|\,dx\,ds\right)&\le C\,l\,n^{\beta-\delta},\\
\mathbf{E}_{\mu^n}\left(\int_0^t\int_{\mathbb{T}}\Big|\Big\{n^{3\beta}S^n_{\rho u}(\widehat{\rho}^n,\widehat{u}^n)\big(\partial_x\widehat{u}^n\big)\big(\widehat{\psi}^n-\Psi(\widehat{\eta}^n,\widehat{\zeta}^n)\big)\Big\}\Big|\,dx\,ds\right)&\le C\,l\,n^{\beta-\delta},\\
\mathbf{E}_{\mu^n}\left(\int_0^t\int_{\mathbb{T}}\Big|\Big\{n^{2\beta}S^n_{u\rho}(\widehat{\rho}^n,\widehat{u}^n)\big(\partial_x\widehat{\rho}^n\big)\big(\widehat{\phi}^n-\Phi(\widehat{\eta}^n,\widehat{\zeta}^n)\big)\Big\}\Big|\,dx\,ds\right)&\le C\,l\,n^{-\delta},\\
\mathbf{E}_{\mu^n}\left(\int_0^t\int_{\mathbb{T}}\Big|\Big\{n^{2\beta}S^n_{uu}(\widehat{\rho}^n,\widehat{u}^n)\big(\partial_x\widehat{u}^n\big)\big(\widehat{\phi}^n-\Phi(\widehat{\eta}^n,\widehat{\zeta}^n)\big)\Big\}\Big|\,dx\,ds\right)&\le C\,l\,n^{-\delta},
\end{aligned}\tag{7.12}$$

where the upper bounds on the right are all $o(1)$. Using the bounds (5.11) and (5.12) on the first partial derivatives of $S^n$, and the gradient bounds (6.12), (6.13) we obtain the following two bounds:

$$n^{-1+\beta+\delta}\mathbf{E}_{\mu^n}\left(\int_0^t\int_{\mathbb{T}}\left|\left\{S^n_\rho(\widehat{\rho}^n,\widehat{u}^n)\big(\partial_x\widehat{\rho}^n\big)\right\}(s,x)\right|\,dx\,ds\right) \tag{7.13}$$
$$\le C\,n^{(-1+\delta+3\beta)/2}=o(1),$$

$$n^{-1+\beta+\delta}\mathbf{E}_{\mu^n}\left(\int_0^t\int_{\mathbb{T}}\left|\left\{S^n_u(\widehat{\rho}^n,\widehat{u}^n)\big(\partial_x\widehat{u}^n\big)\right\}(s,x)\right|\,dx\,ds\right) \tag{7.14}$$
$$\le C\,n^{(-1+\delta+\beta)/2}=o(1).$$

The following bounds are of crucial importance and they are sharp. We use (5.13), (5.14) and (5.15) again and note that here we exploit them in their *full power*: the constant factors on the right-hand side is small. These and the gradient bounds (6.12) and (6.14) yield the following bounds:

$$n^{-1+\beta+\delta}\mathbf{E}_{\mu^n}\left(\int_0^t\int_{\mathbb{T}}\left|\left\{S^n_{\rho\rho}(\widehat{\rho}^n,\widehat{u}^n)\big(\partial_x\widehat{\rho}^n\big)\big(n^{2\beta}\partial_x\widehat{\kappa}^n\big)\right\}(s,x)\right|\,dx\,ds\right)$$
$$\le c\,s^n(t)+o(1),$$
$$n^{-1+\beta+\delta}\mathbf{E}_{\mu^n}\left(\int_0^t\int_{\mathbb{T}}\left|\left\{S^n_{\rho u}(\widehat{\rho}^n,\widehat{u}^n)\big(\partial_x\widehat{u}^n\big)\big(n^{2\beta}\partial_x\widehat{\kappa}^n\big)\right\}(s,x)\right|\,dx\,ds\right)$$
$$\le c\,s^n(t)+o(1),$$
$$n^{-1+\beta+\delta}\mathbf{E}_{\mu^n}\left(\int_0^t\int_{\mathbb{T}}\left|\left\{S^n_{\rho u}(\widehat{\rho}^n,\widehat{u}^n)\big(\partial_x\widehat{\rho}^n\big)\big(n^{\beta}\partial_x\widehat{\chi}^n\big)\right\}(s,x)\right|\,dx\,ds\right) \tag{7.15}$$
$$\le c\,s^n(t)+o(1),$$
$$n^{-1+\beta+\delta}\mathbf{E}_{\mu^n}\left(\int_0^t\int_{\mathbb{T}}\left|\left\{S^n_{uu}(\widehat{\rho}^n,\widehat{u}^n)\big(\partial_x\widehat{u}^n\big)\big(n^{\beta}\partial_x\widehat{\chi}^n\big)\right\}(s,x)\right|\,dx\,ds\right)$$
$$\le c\,s^n(t)+o(1).$$

We choose $\varepsilon$ so small in Lemma 2 that

$$c\sup_{(t,x)\in[0,T]\times\mathbb{T}}|v(t,x)|<\frac{1}{2}. \tag{7.16}$$

*7.5. Sumup.* The identities (7.5), (7.6), (7.8), (7.9) and the bounds (7.7), (7.10), (7.11), (7.12), (7.14), (7.15) yield

$$\left|\mathbf{E}_{\mu^n}\left(\int_0^t\int_{\mathbb{T}}\left\{\big(\partial_x v\big)\left(n^{3\beta}\widehat{\psi}^n\right)S^n_\rho(\widehat{\rho}^n,\widehat{u}^n)\right\}(s,x)\,dx\,ds\right)-\Big(\text{r.h.s. of (7.3)}\Big)\right|$$
$$\le\frac{1}{2}s^n(t)+C\int_0^t h^n(s)\,ds+o(1). \tag{7.17}$$

Finally, from (7.2), (7.3), (7.3), (7.4) and (7.17) we obtain (5.17).

## 8. Control of the Small Values of $(\rho, u)$: Proof of the Bounds (5.18) to (5.21)

*8.1. Proof of (5.18).* We exploit the inequality

$$\left|J^n(\widehat{\rho}^n, \widehat{u}^n)\right| = \left|S^n_\rho(\widehat{\rho}^n, \widehat{u}^n)\right| \le C\,1\!\!1_{\{\widehat{\rho}^n \vee |\widehat{u}^n| > M\}},$$

see (5.11) and boundedness of the functions $\rho(t,x)$, $u(t,x)$, $\partial_x v(t,x)$. Applying the large deviation bound (6.3) we readily obtain (5.18).

*8.2. Proof of (5.19).* This is very similar to what has been done in various parts of Subsect. 7.4. We use the block replacement bound (6.9) and the bound

$$\left|I^n(\widehat{\rho}^n, \widehat{u}^n)\right| = \left|1 - S^n_\rho(\widehat{\rho}^n, \widehat{u}^n)\right| \le C \tag{8.1}$$

which follows from (5.11). We get

$$\mathbf{E}_{\mu^n}\left(\int_0^t \int_{\mathbb{T}} \left|\left\{n^{3\beta}\left(\widehat{\psi}^n - \Psi(\widehat{\eta}^n, \widehat{\zeta}^n)\right)\left|I^n(\widehat{\rho}^n, \widehat{u}^n)\right|\right\}(s,x)\right| ds\, dx\right) \le C\, l\, n^{(-1-\delta+3\beta)/2} = o(1),$$

which proves (5.19).

*8.3. Proof of (5.20).* We write

$$I^n(\widehat{\rho}^n, \widehat{u}^n) = 1\!\!1_{\{\widehat{\rho}^n \vee |\widehat{u}^n| \le M\}} + 1\!\!1_{\{\widehat{\rho}^n \vee |\widehat{u}^n| > M\}} I^n(\widehat{\rho}^n, \widehat{u}^n), \tag{8.2}$$

and note that, by Taylor expansion of the function $(\rho, u) \mapsto \Psi(\rho, u)$,

$$\left|\Psi^n(\widehat{\rho}^n, \widehat{u}^n) - \widehat{\rho}^n \widehat{u}^n\right| 1\!\!1_{\{\widehat{\rho}^n \vee |\widehat{u}^n| \le M\}} \le C\, n^{-2\beta}.$$

On the other hand

$$\left|\Psi^n(\widehat{\rho}^n, \widehat{u}^n)\right| \le C\, \widehat{\rho}^n\, \left|\widehat{u}^n\right|$$

and

$$\widehat{\rho}^n\, \left|I^n(\widehat{\rho}^n, \widehat{u}^n)\right| \le C(1 + \left|\widehat{u}^n\right|), \tag{8.3}$$

see (5.11). Thus

$$\left|\Psi^n(\widehat{\rho}^n, \widehat{u}^n) - \widehat{\rho}^n \widehat{u}^n\right| \left|I^n(\widehat{\rho}^n, \widehat{u}^n)\right| \le C\left(n^{-2\beta} + \left(\left|\widehat{u}^n\right| + \left|\widehat{u}^n\right|^2\right) 1\!\!1_{\{\widehat{\rho}^n \vee |\widehat{u}^n| > M\}}\right).$$

From this, using the large deviation bounds (6.3) and (6.4) we obtain (5.20).

*8.4. Proof of (5.21).* We use again (8.2) and (8.3) and get

$$\begin{aligned}\left|\left(\widehat{\rho}^n - \rho\right)\left(\widehat{u}^n - u\right) I^n(\widehat{\rho}^n, \widehat{u}^n)\right| \le \quad & \left|\left(\widehat{\rho}^n - \rho\right)\left(\widehat{u}^n - u\right)\right| 1\!\!1_{\{\widehat{\rho}^n \vee |\widehat{u}^n| \le M\}} \\ & + C\left(1 + \left|\widehat{u}^n\right| + \left|\widehat{u}^n\right|^2\right) 1\!\!1_{\{\widehat{\rho}^n \vee |\widehat{u}^n| > M\}}.\end{aligned}$$

Now the fluctuation bounds (6.5), (6.6), and the large deviation bounds (6.3), (6.4) together yield (5.21).

## 9. Proof of the "Tools"

*9.1. Proof of the large deviation bounds (Proposition 1).* Recall the definition (6.1) of $L$. The following lemma follows from simple coupling arguments.

**Lemma 5.** (Stochastic dominations). *There exists a constant $C$ depending only on $\max_{(s,x)\in[0,T]\times\mathbb{T}} \rho(s,x) \vee |u(s,x)|$ such that for any fixed $(s,x) \in [0,T]\times\mathbb{T}$, the following stochastic dominations hold:*

$$\mathbf{P}_{\nu_s^n}\Big(\widehat{\rho}^n(x) > z\Big) \le \mathbf{P}\Big(\mathrm{POI}(L) > (z/C)L\Big), \tag{9.1}$$

$$\mathbf{P}_{\nu_s^n}\Big(\big|\widehat{u}^n(x)\big| > z\Big) \le \mathbf{P}\Big(|\mathrm{GAU}| > \big((z/C)-1\big)\sqrt{L}\Big), \tag{9.2}$$

*where* $\mathrm{POI}(L)$ *is a Poissonian random variable with expectation $L$, and* GAU *is a standard Gaussian random variable.*

**Lemma 6.** (Large deviation bounds).

*(i) For any $q < \infty$ there exists $M < \infty$, such that for any $n \in \mathbb{N}$ and $j \in \mathbb{T}^n$ and $s \in [0,T]$,*

$$\begin{aligned}
&\log \mathbf{E}_{\nu_s^n}\Big(\exp\big\{q\,L\,\widehat{\rho}^n(\tfrac{j}{n})1\!\!1_{\{\widehat{\rho}^n(\frac{j}{n})\vee|\widehat{u}^n(\frac{j}{n})|>M\}}\big\}\Big) \le 1,\\
&\log \mathbf{E}_{\nu_s^n}\Big(\exp\big\{q\,L\big|\widehat{u}^n(\tfrac{j}{n})\big|1\!\!1_{\{\widehat{\rho}^n(\frac{j}{n})\vee|\widehat{u}^n(\frac{j}{n})|>M\}}\big\}\Big) \le 1.
\end{aligned} \tag{9.3}$$

*(ii) Let $C$ be the same as in Lemma 5. For any $q \in (0, 1/(8C^2))$ there exists $M < \infty$, such that for any $n \in N$, $j \in \mathbb{T}^n$ and $s \in [0,T]$,*

$$\log \mathbf{E}_{\nu_s^n}\Big(\exp\big\{q\,L\big|\widehat{u}^n(\tfrac{j}{n})\big|^2 1\!\!1_{\{\widehat{\rho}^n(\frac{j}{n})\vee|\widehat{u}^n(\frac{j}{n})|>M\}}\big\}\Big) \le 1. \tag{9.4}$$

*Proof.* The bounds of (9.3) follow from standard large deviation arguments using the stochastic dominations (9.1), (9.2).

For the bound (9.4) we spell out the proof with $\{\widehat{\rho}^n(\frac{j}{n}) > M\}$ instead of $\{\widehat{\rho}^n(\frac{j}{n}) \vee |\widehat{u}^n(\frac{j}{n})| > M\}$; the latter follows similarly. Let $Z_L$ be a $POI(L)$-distributed and $X$ be a standard Gaussian random variable. Using the stochastic dominations (9.1) and (9.2) we obtain

$$\begin{aligned}
&\log \mathbf{E}_{\nu_s^n}\Big(\exp\big\{q\,L\big|\widehat{u}^n(\tfrac{j}{n})\big|^2 1\!\!1_{\{\widehat{\rho}^n(\frac{j}{n})>M\}}\big\}\Big)\\
&\quad\le \log\Big(1 + \mathbf{E}_{\nu_s^n}\Big(\exp\big\{q\,L\big|\widehat{u}^n(\tfrac{j}{n})\big|^2\big\}1\!\!1_{\{\widehat{\rho}^n(\frac{j}{n})>M\}}\Big)\Big)\\
&\quad\le \sqrt{\mathbf{E}_{\nu_s^n}\Big(\exp\big\{2q\,L\big|\widehat{u}^n(\tfrac{j}{n})\big|^2\big\}\Big)}\sqrt{\mathbf{P}_{\nu_s^n}\Big(\widehat{\rho}^n(\tfrac{j}{n}) > M\Big)}\\
&\quad\le \sqrt{\mathbf{E}\Big(\exp\big\{4q\,C^2\big(X^2+L\big)\big\}\Big)}\sqrt{\mathbf{P}\Big(Z_L > (M/C)L\Big)}\\
&\quad\le \big(1-8q\,C^2\big)^{-1/4}\exp\big\{\frac{L}{2}\big(4q\,C^2 + (e^{(\alpha C)/M}-1)-\alpha\big)\big\},
\end{aligned}$$

where $\alpha$ is arbitrary positive number and in the last step we used the Markov inequality. Given $q < 1/(8C^2)$, we choose $\alpha$ sufficiently large and $M > (C\alpha)/(\ln 2)$ to obtain (9.4).

Now we turn to the proof of Proposition 1:

*Proof.* The bounds (6.3), respectively, (6.4) follow directly from the entropy inequality (6.2) of Lemma 3 and the bounds (9.3), respectively, (9.4) of Lemma 6. Recall that $L \gg 1$, as $n \to \infty$.

*9.2. Proof of the fluctuation bounds (Proposition 2).* Within this proof we need the notations

$$\widetilde{u}^n(s,x) := \frac{n^\beta}{l}\sum_k a\Big(\frac{nx-k}{l}\Big)\Big(\zeta_k - n^{-\beta}u\big(s,\frac{k}{n}\big)\Big) = \widehat{u}^n(x) - \mathbf{E}_{\nu_s^n}\big(\widehat{u}^n(x)\big),$$

$$\widetilde{\rho}^n(s,x) := \frac{n^{2\beta}}{l}\sum_k a\Big(\frac{nx-k}{l}\Big)\Big(\eta_k - n^{-2\beta}\rho\big(s,\frac{k}{n}\big)\Big) = \widehat{\rho}^n(x) - \mathbf{E}_{\nu_s^n}\big(\widehat{\rho}^n(x)\big).$$

Since we have

$$\Big|\,\big|\widehat{u}^n\big(s,\frac{j}{n}\big) - u\big(s,\frac{j}{n}\big)\big| - \big|\widetilde{u}^n\big(s,\frac{j}{n}\big)\big|\,\Big| \le C\left(\frac{1}{l}+\frac{l}{n}\right) = o(1),$$

$$\Big|\,\big|\widehat{\rho}^n\big(s,\frac{j}{n}\big) - \rho\big(s,\frac{j}{n}\big)\big| - \big|\widetilde{\rho}^n\big(s,\frac{j}{n}\big)\big|\,\Big| \le C\left(\frac{1}{l}+\frac{l}{n}\right) = o(1),$$

it is enough to prove

$$\mathbf{E}_{\mu_s^n}\Big(\frac{1}{n}\sum_{j\in\mathbb{T}^n}\big|\widetilde{u}^n(s,\frac{j}{n})\big|^2\Big) \le C\,h^n(s) + o(1), \tag{9.5}$$

respectively,

$$\mathbf{E}_{\mu_s^n}\Big(\frac{1}{n}\sum_{j\in\mathbb{T}^n}\big|\widetilde{\rho}^n(s,\frac{j}{n})\big|^2\,1\!\!1_{\{|\widetilde{\rho}^n(s,\frac{j}{n})|\le M\}}\Big) \le C\,h^n(s) + o(1). \tag{9.6}$$

**Lemma 7.** *(i) There exists $q_0 > 0$ (sufficiently small, but fixed) such that for all $n \in \mathbb{N}$, $j \in \mathbb{T}^n$ and $s \in [0,T]$,*

$$\log \mathbf{E}_{\nu_s^n}\Big(\exp\big\{q_0\,L\,\big|\widetilde{u}^n\big(s,\frac{j}{n}\big)\big|^2\big\}\Big) \le 1. \tag{9.7}$$

*(ii) For any $M < \infty$ there exists $q_0 > 0$ (sufficiently small, but fixed) such that for all $n \in \mathbb{N}$, $j \in \mathbb{T}^n$ and $s \in [0,T]$,*

$$\log \mathbf{E}_{\nu_s^n}\Big(\exp\big\{q_0\,L\,\big|\widetilde{\rho}^n\big(s,\frac{j}{n}\big)\big|^2\,1\!\!1_{\{|\widetilde{\rho}^n(s,\frac{j}{n})|\le M\}}\big\}\Big) \le 1. \tag{9.8}$$

*Proof.* (i) Let $X$ be a standard Gaussian random variable, which is independent of the other random variables in question, and denote by $\langle \dots \rangle$ expectation with respect to $X$,

$$\log \mathbf{E}_{\nu_s^n}\Big( \exp\big\{q_0\, L\, \big| \widetilde{u}^n\big(s, \frac{j}{n}\big)\big|^2\big\}\Big) \tag{9.9}$$

$$= \log \mathbf{E}_{\nu_s^n}\Big( \exp\big\{\frac{q_0}{l}\, \big| \sum_k a\big(\frac{j-k}{l}\big)\big(\zeta_k - \mathbf{E}_{\nu_s^n}(\zeta_k)\big)\big|^2\big\}\Big)$$

$$= \log \Big\langle \mathbf{E}_{\nu_s^n}\Big( \exp\big\{X\sqrt{\frac{2q_0}{l}}\, \sum_k a\big(\frac{j-k}{l}\big)\big(\zeta_k - \mathbf{E}_{\nu_s^n}(\zeta_k)\big)\big\}\Big)\Big\rangle.$$

Now, note that the random variables $\zeta_k - \mathbf{E}_{\nu_s^n}(\zeta_k)$, $k \in \mathbb{T}^n$, are uniformly bounded and under the distribution $\mathbf{P}_{\nu_s^n}$ they are independent and have zero mean. Hence there exists a finite constant $C$ such that for any collection of real numbers $\lambda_k$, $k \in \mathbb{T}^n$,

$$\mathbf{E}_{\nu_s^n}\Big( \exp\big\{ \sum_k \lambda_k\big(\zeta_k - \mathbf{E}_{\nu_s^n}(\zeta_k)\big)\big\}\Big) \le \exp\big\{C \sum_k \lambda_k^2\big\}.$$

Further on, there exists a finite constant $C$ such that for any $l$,

$$\frac{1}{l}\sum_k \big| a\big(\frac{k}{l}\big)\big|^2 \le C. \tag{9.10}$$

From these it follows that for some finite constant $C$,

$$\text{r.h.s. of (9.9)} \ \le \log \Big\langle \exp\big\{q_0\, C\, X^2\big\}\Big\rangle.$$

Choosing $q_0$ sufficiently small in this last inequality we obtain (9.7).

(ii) Note first that, given $M < \infty$ fixed, there exists a zero mean bounded random variable $Y$ such that for any $r \in \mathbb{R}$,

$$r^2 1\!\!1_{\{|r| \le M\}} \le \log \mathbf{E}\Big( \exp\big\{ r\, Y \big\}\Big).$$

Let $Y_1, Y_2, \dots$ be i.i.d. copies of $Y$ which are also independent of the other random variables in question, and denote by $\langle \dots \rangle$ expectation with respect to these. Then we have

$$\log \mathbf{E}_{\nu_s^n}\Big( \exp\big\{q_0\, L\, \big| \widetilde{\rho}^n\big(s, \frac{j}{n}\big)\big|^2 1\!\!1_{\{|\widetilde{\rho}^n(s,\frac{j}{n})| \le M\}}\big\}\Big) \tag{9.11}$$

$$\le \log \Big\langle \mathbf{E}_{\nu_s^n}\Big( \exp\big\{\frac{\sum_{p=1}^{\lceil q_0 L\rceil} Y_p}{L}\, \sum_k a\big(\frac{j-k}{l}\big)\big(\eta_k - \mathbf{E}_{\nu_s^n}(\eta_k)\big)\big\}\Big)\Big\rangle.$$

Next note that for any $\overline{\lambda} < \infty$ there exists a constant $C < \infty$ such that for any $n \in \mathbb{N}$, any $s \in [0, T]$ and any collection of real numbers $\lambda_k \in [-\overline{\lambda}, \overline{\lambda}]$, $k \in \mathbb{T}^n$,

$$\mathbf{E}_{\nu_s^n}\Big( \exp\big\{ \sum_k \lambda_k\, \big(\eta_k - \mathbf{E}_{\nu_s^n}(\eta_k)\big)\big\}\Big) \le \exp\big\{Cn^{-2\beta} \sum_k \lambda_k^2\big\}.$$

Hence, using again (9.10),

$$\text{r.h.s. of (9.11)} \;\le \log\Big\langle \exp\Big\{q_0\, C\,\Big(\big(Y_1+\ldots+Y_{\lceil q_0 L\rceil}\big)/\sqrt{\lceil q_0 L\rceil}\Big)^2\Big\}\Big\rangle.$$

Now, since the i.i.d. random variables $Y_1, Y_2, \ldots$ are bounded and have zero mean, choosing $q_0$ sufficiently small this last expression can be made arbitrarily small, uniformly in $L$. Hence (9.8).

Now back to the proof of Proposition 2.

*Proof.* From (6.2) and (9.7), respectively, from (6.2) and (9.8) we deduce (9.5), respectively, (9.6). Finally, these two bounds and the arguments at the beginning of the present subsection imply (6.5), respectively, (6.6).

### *9.3. Proof of the block replacement and gradient bounds.*

*9.3.1. An elementary probability lemma.* Let $(\Omega, \pi)$ be a finite probability space and $\omega_i$, $i \in \mathbb{Z}$ i.i.d. $\Omega$-valued random variables with distribution $\pi$. Further on let

$$\boldsymbol{\zeta} : \Omega \to \mathbb{R}^d, \quad \boldsymbol{\zeta}_i := \boldsymbol{\zeta}(\omega_i), \quad \xi : \Omega^m \to \mathbb{R}, \quad \xi_i := \xi(\omega_i \ldots, \omega_{i+m-1}).$$

For $\boldsymbol{x} \in \mathrm{co}(\mathrm{Ran}(\boldsymbol{\zeta}))$ denote

$$\Xi(\boldsymbol{x}) := \frac{\mathbf{E}_\pi\big(\xi_1 \exp\{\sum_{i=1}^m \boldsymbol{\lambda}\cdot\boldsymbol{\zeta}_i\}\big)}{\mathbf{E}_\pi\big(\exp\{\boldsymbol{\lambda}\cdot\boldsymbol{\zeta}_1\}\big)^m},$$

where $\mathrm{co}(\cdot)$ stands for 'convex hull' and $\boldsymbol{\lambda} \in \mathbb{R}^d$ is chosen so that

$$\frac{\mathbf{E}_\pi\big(\boldsymbol{\zeta}_1 \exp\{\boldsymbol{\lambda}\cdot\boldsymbol{\zeta}_1\}\big)}{\mathbf{E}_\pi\big(\exp\{\boldsymbol{\lambda}\cdot\boldsymbol{\zeta}_1\}\big)} = \boldsymbol{x}.$$

For $l \in \mathbb{N}$ we denote *plain* block averages by

$$\overline{\boldsymbol{\zeta}}_l := \frac{1}{l}\sum_{j=1}^{l} \boldsymbol{\zeta}_j.$$

Finally, let $b : [0, 1] \to \mathbb{R}$ be a fixed piecewise continuous function, we define the block averages *weighted by* $b$,

$$\langle b\,,\,\boldsymbol{\zeta}\rangle_l := \frac{1}{l}\sum_{j=0}^{l} b(j/l)\boldsymbol{\zeta}_j, \quad \langle b\,,\,\xi\rangle_l := \frac{1}{l}\sum_{j=0}^{l} b(j/l)\xi_j.$$

The following lemma relies on elementary probability arguments:

**Lemma 8** (Microcanonical exponential moments of block averages)**.** *There exists a constant $C < \infty$, depending only on $m$, on the joint distribution of $(\xi_i, \boldsymbol{\zeta}_i)$ and on the function $b$, such that the following bounds hold uniformly in $l \in \mathbb{N}$ and $\boldsymbol{x} \in (\mathrm{Ran}(\boldsymbol{\zeta}) + \cdots + \mathrm{Ran}(\boldsymbol{\zeta}))/l$:*

*(i) If $\int_0^1 b(s)\,ds = 0$, then*

$$\mathbf{E}_\pi\Big(\exp\big\{q\sqrt{l}\langle b\,,\,\xi\rangle_l\big\}\,\Big|\,\overline{\boldsymbol{\zeta}}_l = \boldsymbol{x}\Big) \le \exp\{C(q^2 + q/\sqrt{l})\}. \tag{9.12}$$

*(ii) If $\int_0^1 b(s)\,ds = 1$ then*

$$\mathbf{E}_\pi\Big(\exp\big\{q\sqrt{l}\big(\langle b\,,\,\xi\rangle_l - \Xi(\langle b\,,\,\boldsymbol{\zeta}\rangle_l)\big)\big\}\,\Big|\,\overline{\boldsymbol{\zeta}}_l = \boldsymbol{x}\Big) \le \exp\{C(q^2 + q/\sqrt{l})\}. \tag{9.13}$$

*Proof.* We prove the lemma with $m = 1$, that is with $(\xi_i)_{i=1}^l$ independent rather than $m$-dependent. The $m$-dependent case follows by applying Jensen's inequality in a rather straightforward way.

(i) In order to simplify the argument we make the assumption that the function $s \mapsto b(s)$ is odd:

$$b(1-s) = -b(s). \tag{9.14}$$

The same argument works if the function $s \mapsto b(s)$ can be rearranged (by permutation of finitely many subintervals of [0, 1]) into a piecewise continuous odd function. This case is sufficient for our purposes. The proof of the fully general case — which goes through induction on $l$ — is more tedious and it is left as an exercise for the reader. Assuming (9.14) we have

$$\sqrt{l}\langle b\,,\,\xi\rangle_l = l^{-1/2}\sum_{j=0}^{[l/2]} b(j/l)(\xi_j - \xi_{l-j}),$$

and hence

$$\begin{aligned}
&\mathbf{E}_\pi\Big(\exp\big\{q\sqrt{l}\langle b\,,\,\xi\rangle_l\big\}\,\Big|\,\overline{\boldsymbol{\zeta}}_l = \boldsymbol{x}\Big)\\
&\quad= \mathbf{E}_\pi\Big(\mathbf{E}_\pi\Big(\exp\big\{q\sqrt{l}\langle b\,,\,\xi\rangle_l\big\}\,\Big|\,\boldsymbol{\zeta}_j + \boldsymbol{\zeta}_{l-j}:\ j = 0,\dots,l\Big)\,\Big|\,\overline{\boldsymbol{\zeta}}_l = \boldsymbol{x}\Big)\\
&\quad= \mathbf{E}_\pi\Big(\prod_{j=0}^{[l/2]}\mathbf{E}_\pi\Big(\exp\big\{q\,l^{-1/2}b(j/l)(\xi_j - \xi_{l-j})\big\}\,\Big|\,\boldsymbol{\zeta}_j + \boldsymbol{\zeta}_{l-j}\Big)\,\Big|\,\overline{\boldsymbol{\zeta}}_l = \boldsymbol{x}\Big)\\
&\quad\le \exp\Big\{Cq^2\sum_{j=1}^{[l/2]} l^{-1}b(j/l)^2\Big\} = \exp\{C\,q^2\}.
\end{aligned}$$

In the second step we use the fact that the pairs $\big(\xi_j, \xi_{l-j}\big)$, $j = 0, \dots, [l/2]$ are independent, given $\boldsymbol{\zeta}_j + \boldsymbol{\zeta}_{l-j}$, $j = 0, \dots, [l/2]$. In the third step we note that the variables $\xi_j$ are bounded and $\mathbf{E}\big(\xi_j - \xi_{l-j}\big|\boldsymbol{\zeta}_j + \boldsymbol{\zeta}_{l-j}\big) = 0$.

(ii) Beside $\Xi(\boldsymbol{x})$ we also introduce the functions

$$\Xi_l : \big(\mathrm{Ran}(\boldsymbol{\zeta}) + \cdots + \mathrm{Ran}(\boldsymbol{\zeta})\big)/l \to \mathbb{R}, \qquad \Xi_l(\boldsymbol{x}) := \mathbf{E}\big(\xi_1\big|\overline{\boldsymbol{\zeta}}_l = \boldsymbol{x}\big).$$

We shall exploit the following facts:
(1) The functions $\Xi(\boldsymbol{x})$ and $\Xi_l(\boldsymbol{x})$ are uniformly bounded. This follows from the boundedness of $\xi_j$.
(2) The function $\boldsymbol{x} \mapsto \Xi(\boldsymbol{x})$ is smooth with bounded first two derivatives. This follows from direct computations.
(3) There exists a finite constant $C$, such that

$$|\Xi_l(\boldsymbol{x}) - \Xi(\boldsymbol{x})| \le C l^{-1}.$$

This follows from the so-called equivalence of ensembles (see e.g. Appendix 2 of [10]).

We write

$$\langle b\,,\,\xi\rangle_l - \Xi(\langle b\,,\,\boldsymbol{\zeta}\rangle_l) = \big(\langle b\,,\,\xi\rangle_l - \overline{\xi}_l\big) + \big(\overline{\xi}_l - \Xi_l(\overline{\boldsymbol{\zeta}}_l)\big) + \big(\Xi_l(\overline{\boldsymbol{\zeta}}_l) - \Xi(\overline{\boldsymbol{\zeta}}_l)\big) + \big(\Xi(\overline{\boldsymbol{\zeta}}_l) - \Xi(\langle b\,,\,\boldsymbol{\zeta}\rangle_l)\big). \tag{9.15}$$

By applying Jensen's inequality we conclude that it is enough to bound the exponential moments of type (9.13), separately for the four terms. Bounding the first and last terms reduces directly to (9.12), the third term is uniformly $\mathcal{O}(l^{-1})$, so we only have to bound the exponential moments of the second term in (9.15), This is done by induction on $l$. Let $C(l)$ be the best constant such that for any $q \in \mathbb{R}$,

$$\mathbf{E}_\pi\Big(\exp\big\{q\sqrt{l}\big(\overline{\xi}_l - \Xi_l(\overline{\boldsymbol{\zeta}}_l)\big)\big\}\,\Big|\,\overline{\boldsymbol{\zeta}}_l = \boldsymbol{x}\Big) \le \exp\{C(l)\,q^2\}.$$

Clearly, $C(1) < \infty$. We prove that $C(l)$ stays bounded as $l \to \infty$. The following identity holds:

$$\begin{aligned}
&\mathbf{E}_\pi\Big(\exp\big\{q\sqrt{l+1}\big(\overline{\xi}_{l+1} - \Xi_{l+1}(\overline{\boldsymbol{\zeta}}_{l+1})\big)\big\}\,\Big|\,\overline{\boldsymbol{\zeta}}_{l+1} = \boldsymbol{x}\Big) = \\
&\mathbf{E}_\pi\Big(\mathbf{E}_\pi\Big(\exp\big\{q\sqrt{l+1}\big(\overline{\xi}_{l+1} - \Xi_{l+1}(\overline{\boldsymbol{\zeta}}_{l+1})\big)\big\}\,\Big|\,\overline{\boldsymbol{\zeta}}_l\,,\,\boldsymbol{\zeta}_{l+1}\Big)\,\Big|\,\overline{\boldsymbol{\zeta}}_{l+1} = \boldsymbol{x}\Big) = \\
&\mathbf{E}_\pi\Big(\mathbf{E}_\pi\Big(\exp\Big\{\frac{q\,l}{\sqrt{l+1}}\big(\overline{\xi}_l - \Xi_l(\overline{\boldsymbol{\zeta}}_l)\big)\Big\}\,\Big|\,\overline{\boldsymbol{\zeta}}_l\Big) \times \\
&\qquad \mathbf{E}_\pi\Big(\exp\Big\{\frac{q}{\sqrt{l+1}}\big(\overline{\xi}_{l+1} - \Xi_1(\overline{\boldsymbol{\zeta}}_{l+1})\big)\Big\}\,\Big|\,\boldsymbol{\zeta}_{l+1}\Big) \times \\
&\qquad\quad \exp\Big\{\frac{q\,l}{\sqrt{l+1}}\big(\Xi_l(\overline{\boldsymbol{\zeta}}_l) - \Xi_{l+1}(\overline{\boldsymbol{\zeta}}_{l+1})\big)\Big\} \times \\
&\qquad\qquad \exp\Big\{\frac{q}{\sqrt{l+1}}\big(\Xi_1(\boldsymbol{\zeta}_{l+1}) - \Xi_{l+1}(\overline{\boldsymbol{\zeta}}_{l+1})\big)\Big\}\,\Big|\,\overline{\boldsymbol{\zeta}}_l = \boldsymbol{x}\Big).
\end{aligned}$$

The terms

$$\big(\overline{\xi}_{l+1} - \Xi_1(\overline{\boldsymbol{\zeta}}_{l+1})\big), \qquad l\big(\Xi_l(\overline{\boldsymbol{\zeta}}_l) - \Xi_{l+1}(\overline{\boldsymbol{\zeta}}_{l+1})\big), \qquad \big(\Xi_1(\boldsymbol{\zeta}_{l+1}) - \Xi_{l+1}(\overline{\boldsymbol{\zeta}}_{l+1})\big)$$

are uniformly bounded and

$$\mathbf{E}_\pi\big(\overline{\xi}_{l+1} - \Xi_1(\overline{\boldsymbol{\zeta}}_{l+1})\,\big|\,\boldsymbol{\zeta}_{l+1}\big) = 0, \qquad \mathbf{E}_\pi\big(\Xi_l(\overline{\boldsymbol{\zeta}}_l) - \Xi_{l+1}(\overline{\boldsymbol{\zeta}}_{l+1})\,\big|\,\overline{\boldsymbol{\zeta}}_{l+1}\big) = 0,$$
$$\mathbf{E}_\pi\big(\Xi_1(\boldsymbol{\zeta}_{l+1}) - \Xi_{l+1}(\overline{\boldsymbol{\zeta}}_{l+1})\,\big|\,\overline{\boldsymbol{\zeta}}_{l+1}\big) = 0.$$

Using the induction hypothesis and the previous arguments, it follows that there exists a finite constant $B$ such that

$$C(l+1) \leq \frac{l}{l+1} C(l) + \frac{1}{l+1} B,$$

for every $l \geq 1$. Hence, $\limsup_{l\to\infty} C(l) \leq B$ and the lemma follows.

**Lemma 9** (Microcanonical Gaussian bounds)**.** *There exists a $q_0 > 0$, depending only on m, on the joint distribution of $(\xi_i, \boldsymbol{\zeta}_i)$ and on the function b, such that the following bounds hold uniformly in $l \in \mathbb{N}$ and $\boldsymbol{x} \in (\mathrm{Ran}(\boldsymbol{\zeta}) + \cdots + \mathrm{Ran}(\boldsymbol{\zeta}))/l$:*

*(i) If $\int_0^1 b(s)\, ds = 0$, then*

$$\log \mathbf{E}_\pi \Big( \exp \big\{ q_0\, l \langle b\,,\ \xi \rangle_l^2 \big\} \Big| \overline{\boldsymbol{\zeta}}_l = \boldsymbol{x} \Big) \leq 1. \tag{9.16}$$

*(ii) If $\int_0^1 b(s)\, ds = 1$ then*

$$\log \mathbf{E}_\pi \Big( \exp \big\{ q_0\, l \big(\langle b\,,\ \xi \rangle_l - \Xi(\langle b\,,\ \boldsymbol{\zeta} \rangle_l)\big)^2 \big\} \Big| \overline{\boldsymbol{\zeta}}_l = \boldsymbol{x} \Big) \leq 1. \tag{9.17}$$

*Proof.* The bounds (9.16) and (9.17) follow from (9.12), respectively, (9.13) by exponential Gaussian averaging (as in the proof of Lemma 7).

*9.3.2. Proof of Proposition 3.* (i) In order to prove (6.9) first note that by simple numerical approximation (no probability bounds involved)

$$\left| \int_{\mathbb{T}} \left| \{\widehat{\xi}^n - \Xi(\widehat{\eta}^n, \widehat{\zeta}^n)\}(x) \right|^2 dx - \frac{1}{n} \sum_{j\in\mathbb{T}^n} \left| \{\widehat{\xi}^n - \Xi(\widehat{\eta}^n, \widehat{\zeta}^n)\}(\frac{j}{n}) \right|^2 \right| \leq \frac{C}{l}$$

and also that $l^{-1} = o\left(l^2\, n^{-1-3\beta-\delta}\right)$. We apply Lemma 4 with

$$\mathcal{V} = \left| \{\widehat{\xi}^n - \Xi(\widehat{\eta}^n, \widehat{\zeta}^n)\}(0) \right|^2 = \left| \langle a\,,\ \xi \rangle_l - \Xi\big(\langle a\,,\ \eta \rangle_l, \langle a\,,\ \zeta \rangle_l\big) \right|^2.$$

We use the bound (9.17) of Lemma 9 with the function $b = a$. Note that $q = q_0 l$ can be chosen in (6.8) with a small, but fixed $q_0$. This yields the bound (6.9).
(ii) In order to prove (6.10) we start again with numerical approximation:

$$\left| \int_{\mathbb{T}} \left| \partial_x \widehat{\xi}^n(x) \right|^2 dx - \frac{1}{n} \sum_{j\in\mathbb{T}^n} \left| \partial_x \widehat{\xi}^n(\frac{j}{n}) \right|^2 \right| \leq C \frac{n^2}{l^3} = o(n^{1-3\beta-\delta}).$$

We apply Lemma 4 with

$$\mathcal{V} = \left| \partial_x \widehat{\xi}^n(0) \right|^2 = \frac{n^2}{l^2} \left| \langle a'\,,\ \xi \rangle_l \right|^2.$$

We use now the bound (9.16) of Lemma 9 with the function $b = a'$. We can choose $q = q_0 l^3/n^2$ with a small, but fixed $q_0$ and this yields the bound (6.10).

(iii) Next we prove (6.11). We apply Lemma 4 with

$$\mathcal{V} = \frac{|\,\partial_x \widehat{\xi}^n(0)\,|^2}{\widehat{\eta}^n(0)} = \frac{n^2}{l^3}\frac{\left|\,\sum_k a'(k/l)\xi_k\,\right|^2}{\sum_k a(k/l)\eta_k} = \frac{n^2}{2l^3}\frac{\left|\,\sum_k a'(k/l)(\xi_k-\xi_{-k})\,\right|^2}{\sum_k a(k/l)(\eta_k+\eta_{-k})},$$

where in the last equality we use the fact that the weighting function $x \mapsto a(x)$ is *even*. We will carry out similar computations as in the proofs of Lemma 8 and 9. We compute the exponential moment $\mathbf{E}^{2l+1}_{N,Z}\big(\exp\{q\mathcal{V}\}\big)$. Let $X$ be a standard Gaussian random variable, which is independent of the other random variables in question and denote by $\langle \dots \rangle$ averaging with respect to it. We have

$$\mathbf{E}^{2l+1}_{N,Z}\Big(\exp\{q\mathcal{V}\}\Big) = \mathbf{E}^{2l+1}_{N,Z}\Big(\exp\Big\{q\frac{n^2}{2l^3}\frac{\left|\,\sum_k a'(k/l)(\xi_k-\xi_{-k})\,\right|^2}{\sum_k a(k/l)(\eta_k+\eta_{-k})}\Big\}\Big)$$

$$= \Big\langle \mathbf{E}^{2l+1}_{N,Z}\Big(\exp\Big\{X\sqrt{q}\frac{n}{l^{3/2}}\frac{\sum_k a'(k/l)(\xi_k-\xi_{-k})}{\sqrt{\sum_k a(k/l)(\eta_k+\eta_{-k})}}\Big\}\Big)\Big\rangle$$

$$= \Big\langle \mathbf{E}^{2l+1}_{N,Z}\Big(\mathbf{E}^{2l+1}_{N,Z}\Big(\exp\Big\{X\sqrt{q}\frac{n}{l^{3/2}}\frac{\sum_k a'(k/l)(\xi_k-\xi_{-k})}{\sqrt{\sum_k a(k/l)(\eta_k+\eta_{-k})}}\Big\}\,\Big|\,\{\eta_k+\eta_{-k}\}_{k=0}^{l}\Big)\Big)\Big\rangle$$

$$\le \Big\langle \mathbf{E}^{2l+1}_{N,Z}\Big(\exp\Big\{CX^2 q\frac{n^2}{l^3}\frac{\sum_k a'(k/l)^2(\eta_k+\eta_{-k})}{\sum_k a(k/l)(\eta_k+\eta_{-k})}\Big\}\Big)\Big\rangle \le \Big\langle \exp\Big\{CX^2 q\frac{n^2}{l^3}\Big\}\Big\rangle,$$

where we used the facts that the random variables $\eta_k$ are non-negative, $\Omega$ is finite and $\eta(\omega)=0$ implies $\xi(\omega)=0$. In the last step we used the inequality

$$a'(x)^2 \le Ca(x),$$

which follows from the conditions on $a(x)$, see Subsect. 4.3.

From this bound it follows that in Lemma 4 we can choose $q = q_0 l^3/n^2$, with a small but fixed $q_0$, and hence the second bound in (6.11) follows.

## 10. Appendix

*10.1. Some details about the PDE (1.1).*

*Hyperbolicity:* One has to analyze the Jacobian matrix

$$D := \begin{pmatrix} (\rho u)_\rho & (\rho u)_u \\ (\rho+\gamma u^2)_\rho & (\rho+\gamma u^2)_u \end{pmatrix} = \begin{pmatrix} u & \rho \\ 1 & 2\gamma u \end{pmatrix}.$$

The eigenvalues with the corresponding right and left eigenvectors are:

$$Dr = \lambda r, \quad Ds = \mu s, \quad l^\dagger D = \lambda l^\dagger, \quad m^\dagger D = \mu m^\dagger,$$

($v^\dagger$ stands for the transpose of the column 2-vector $v$). The eigenvalues and eigenvectors are

$$\left.\begin{matrix}\lambda\\ \mu\end{matrix}\right\} = \pm\frac{1}{2}\left\{\sqrt{(2\gamma-1)^2u^2+4\rho} \pm (2\gamma+1)u\right\}$$

and

$$\left.\begin{matrix} r^\dagger \\ s^\dagger \end{matrix}\right\} = \left(\frac{1}{2}\left\{\mp\sqrt{(2\gamma-1)^2u^2+4\rho}-(2\gamma-1)u\right\}, 1\right),$$

$$\left.\begin{matrix} l^\dagger \\ m^\dagger \end{matrix}\right\} = \left(1, -\frac{1}{2}\left\{\pm\sqrt{(2\gamma-1)^2u^2+4\rho}-(2\gamma-1)u\right\}\right).$$

We can conclude that the pde (1.1) is (strictly) hyperbolic in the domain

$$\begin{aligned} \gamma \neq 1/2: &\quad \{(\rho,u)\in\mathbb{R}_+\times\mathbb{R} : (\rho,u)\neq(0,0)\}, \\ \gamma = 1/2: &\quad \{(\rho,u)\in\mathbb{R}_+\times\mathbb{R} : \rho\neq 0\}. \end{aligned}$$

<u>Riemann invariants:</u> The Riemann invariants $w = w(\rho, u)$, $z = z(\rho, u)$ of the pde are given by the relations

$$(w_\rho, w_u)\cdot s = 0 = (z_\rho, z_u)\cdot r.$$

That is, the level lines $w =$ const., respectively $z =$ const. are determined by the ordinary differential equations

$$\frac{d\rho}{du} = \mp\frac{1}{2}\left\{\sqrt{(2\gamma-1)^2u^2+4\rho}\pm(2\gamma-1)u\right\}.$$

In our case the Riemann invariants can be found explicitly. For $\gamma \neq 3/4$ we get

$$w(\rho,u) = F\left\{\left(\sqrt{(2\gamma-1)^2u^2+4\rho}+(2\gamma-1)u\right)^{\frac{2\gamma-1}{2\gamma-2}}\left(\sqrt{(2\gamma-1)^2u^2+4\rho}-(2\gamma-2)u\right)\right\},$$

$$z(\rho,u) = F\left\{\left(\sqrt{(2\gamma-1)^2u^2+4\rho}-(2\gamma-1)u\right)^{\frac{2\gamma-1}{2\gamma-2}}\left(\sqrt{(2\gamma-1)^2u^2+4\rho}+(2\gamma-2)u\right)\right\},$$

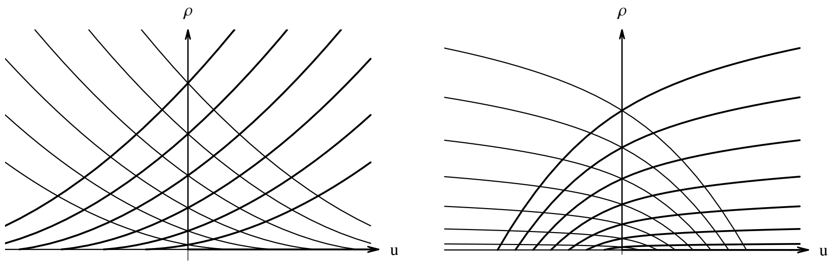
where $F : \mathbb{R} \to \mathbb{R}$ is an appropriately chosen bijection (recall that only the level sets of the Riemann invariants are determined).

Note that due to the changes of sign of $2\gamma - 1$ and $2\gamma - 2$, the above expression gives rise to *qualitatively different* behavior of the Riemann invariants. The topology of the picture changes at the critical values $\gamma = 1/2$, $\gamma = 3/4$ and $\gamma = 1$. In Fig. 1 we present the qualitative picture of the level lines of $w(\rho, u)$ and $z(\rho, u)$ for $3/4 < \gamma < 1$, and $\gamma > 1$, respectively.

In all cases the Riemann invariants satisfy the convexity conditions

$$\begin{aligned} w_{\rho\rho}w_u^2 - 2w_{\rho u}w_\rho w_u + w_{uu}w_\rho^2 \geq 0, \\ z_{\rho\rho}z_u^2 - 2z_{\rho u}z_\rho z_u + z_{uu}z_\rho^2 \geq 0, \end{aligned} \tag{10.1}$$

in $\mathbb{R}_+ \times \mathbb{R}$ for all $\gamma$. (We have to choose the sign of the function $F(\cdot)$ appropriately.) The inequalities are strict in the interior of $\mathbb{R}_+ \times \mathbb{R}$, except for the $\gamma = 1$ case, when these expressions identically vanish. These conditions are equivalent to saying that the level sets $\{(\rho, u) \in [0, \infty) \times (-\infty, \infty) : w(\rho, u) < c\}$ and $\{(\rho, u) \in [0, \infty) \times (-\infty, \infty) :$




(a) $3/4 < \gamma < 1$ (b) $1 < \gamma$

**Fig. 1.** Level lines of Riemann-invariants

$z(\rho, u) < c\}$ are convex. See [11, 12] or [19] for the importance of these convexity conditions.

It is of crucial importance for our problem that the level curves $w(\rho, u) = w =$ const. expressed as $u \mapsto \rho(u, w)$ are convex for $\gamma < 1$, linear for $\gamma = 1$ and concave for $\gamma > 1$.

*Genuine nonlinearity:* Genuine nonlinearity holds if and only if

$$(\lambda_\rho, \lambda_u) \cdot r \neq 0 \neq (\mu_\rho, \mu_u) \cdot s$$

in the interior of the domain $\mathbb{R}_+ \times \mathbb{R}$. Elementary computations show that

$$\left.\begin{matrix} (\lambda_\rho, \lambda_u) \cdot r = 0 \\ (\mu_\rho, \mu_u) \cdot s = 0 \end{matrix}\right\} \quad \Leftrightarrow \quad \rho = -\frac{4\gamma(2\gamma - 1)^2}{(\gamma + 1)^2} u^2 \text{ and } \left\{\begin{matrix} u \leq 0 \\ u \geq 0 \end{matrix}\right. . \tag{10.2}$$

Thus, for $\gamma \geq 0$, $\gamma \neq 0, 1/2$ the system is genuinely nonlinear on the closed domain $\mathbb{R}_+ \times \mathbb{R}$; for $\gamma = 0, 1/2$ it is genuinely nonlinear in the interior of $\mathbb{R}_+ \times \mathbb{R}$ (with genuine nonlinearity marginally lost on the boundary, $\rho = 0$). For $\gamma < 0$ genuine nonlinearity is lost in the interior of $\mathbb{R}_+ \times \mathbb{R}$.

*Lax entropies and entropy solutions:* Lax entropies of the pde (1.1) are solutions of the linear hyperbolic partial differential equation

$$\rho S_{\rho\rho} + (2\gamma - 1) u S_{\rho u} - S_{uu} = 0.$$

It turns out that the system is sufficiently rich in Lax entropies. In particular a globally convex Lax entropy in $\mathbb{R}_+ \times \mathbb{R}$ is

$$S(\rho, u) = \rho \log \rho + \frac{u^2}{2}. \tag{10.3}$$

*The Maximum Principle and positively invariant domains:* For $\gamma \geq 0$ our systems satisfy the conditions of Lax's Maximum Principle proved in [11], namely:

(i) they do possess a globally strictly convex Lax entropy bounded from below, see (10.3);

(ii) the Riemann invariants $w(\rho, u)$ and $z(\rho, u)$ satisfy the convexity condition (10.1);

(iii) they are genuinely nonlinear in the interior of $\mathcal{D}$, see (10.2).

Therefore, *convex domains bounded by level curves of $w(\rho, u)$ and $z(\rho, u)$ are positively invariant for entropy solutions.*

First we conclude that $\mathcal{D}$ itself is a positively invariant domain, as it should be. Second, a very essential difference between the cases $\gamma < 1$, $\gamma = 1$ and $\gamma > 1$ may be observed, which is of crucial importance for the main result of the present paper. In the case $\gamma < 1$ all convex domains bounded by level curves of the Riemann invariants are *unbounded (non-compact)* and thus there is no a priori bound on the solutions. Even starting with smooth initial data with compact support, nothing prevents the entropy solutions to blow up indefinitely after the appearance of the shocks. On the other hand, if $\gamma \geq 1$ any compact subset of $\mathcal{D}$ is contained in a compact convex domain bounded by level sets of the Riemann invariants, a fact which yields a priori bounds on the entropy solutions, given bounded initial data.

*10.2. Construction of the cutoff.* We start with the construction of some entropy/flux pairs $S(\rho, u)$, $F(\rho, u)$ for the *unscaled* Euler equation (2.17). These are the solutions of the system of pde-s

$$F_\rho = \Psi_\rho S_\rho + \Phi_\rho S_u, \qquad F_u = \Psi_u S_\rho + \Phi_u S_u, \tag{10.4}$$

defined on $\mathcal{D}$. In particular the Lax entropy $S(\rho, u)$ solves the pde:

$$\Psi_u S_{\rho\rho} + \big(\Phi_u - \Psi_\rho\big) S_{\rho u} - \Phi_\rho S_{uu} = 0. \tag{10.5}$$

The linear pde (10.5) is hyperbolic in $\mathcal{D}$. One family of its characteristic curves is solutions of the following ODE, meant in the domain $\mathcal{D}$:

$$\frac{d\rho}{du} = \frac{\sqrt{\big(\Phi_u - \Psi_\rho\big)^2 + 4\Phi_\rho\Psi_u} - \big(\Phi_u - \Psi_\rho\big)}{2\Phi_\rho}, \tag{10.6}$$

the other family is obtained by reflecting $u$ to $-u$. The characteristic curves are the same as the level lines of Riemann invariants for the pde (2.17).

First we conclude that the line segment $\mathcal{D} \cap \{u = 0\}$ is *not* characteristic for the hyperbolic pde (10.5). That is: it intersects transversally the characteristic lines defined by the differential equation (10.6). Indeed, from the Onsager relation (2.16) and obvious parity considerations it follows that the right-hand side of (10.6) restricted to $\{u = 0\}$ becomes $\big(\mathbf{Var}_{r,0}(\eta)/\mathbf{Var}_{r,0}(\zeta)\big)^{1/2}$ and this expression is obviously finite for $r \in (0, \rho^*)$. It follows that the Cauchy problem (10.5), with the following initial condition:

$$S(r, 0) = s(r), \qquad S_u(r, 0) = 0, \qquad r \in [0, \rho^*) \tag{10.7}$$

is *well posed.*

In our concrete problem the function $s(r)$ will be chosen as follows: we fix $0 < \underline{r} < \overline{r} < \rho^*$, and define

$$s(r) = \begin{cases} 0 & \text{if } r \in [0, \underline{r}), \\ \dfrac{r\log(r/\underline{r}) - (r - \underline{r})}{\log(\overline{r}/\underline{r})} & \text{if } r \in [\underline{r}, \overline{r}), \\ r - \dfrac{\overline{r} - \underline{r}}{\log(\overline{r}/\underline{r})} & \text{if } r \in [\overline{r}, \infty). \end{cases} \tag{10.8}$$

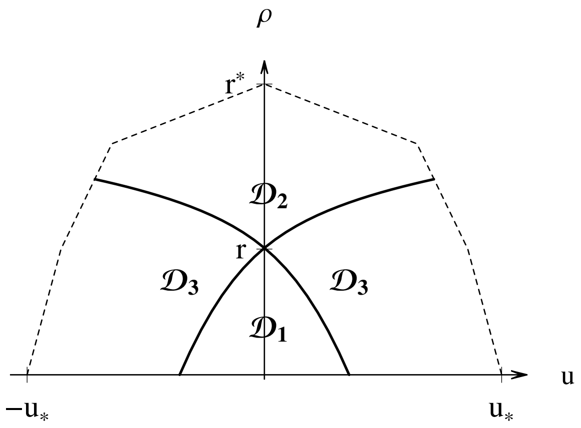



**Fig. 2.** $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$

Note that $s(r)$ and $s'(r)$ are continuous. Due to the assumption (H) imposed, and regularity of the flux functions $\Phi$ and $\Psi$, there exists some $\rho_0 > 0$ such that the ODE (10.6) is regular in $\{(\rho, u) \in \mathcal{D} : \rho < \rho_0 \text{ and } (\rho, u) \neq (0, 0)\}$. We shall not be concerned about what happens outside this strip. Denote by $\sigma(u; r)$ the solution of the ODE (10.6) with initial condition $\sigma(0; r) = r$.

For small enough $r_0 > 0$ we can partition the domain $\mathcal{D}$ in three parts for any $0 < r < r_0$ as follows:

$$\mathcal{D}_1(r) := \{(\rho, u) \in \mathcal{D} : \rho < \sigma(-|u|; r)\}, \quad \mathcal{D}_2(r) := \{(\rho, u) \in \mathcal{D} : \rho > \sigma(|u|; r)\},$$

$$\mathcal{D}_3(r) := \mathcal{D} \setminus \big(\mathcal{D}_1(r) \cup \mathcal{D}_2(r)\big) = \{(\rho, u) \in \mathcal{D} : \sigma(-|u|; r) \le \rho \le \sigma(|u|; r)\}.$$

See Fig. 2 for a sketch of the domains $\mathcal{D}_1(r), \mathcal{D}_2(r), \mathcal{D}_3(r)$.

From now on $r_0$ is *fixed forever* and we denote $\widetilde{\mathcal{D}} := \mathcal{D}_1(r_0)$. This domain is a *rectangle* in characteristic coordinates with diagonal $\widetilde{\mathcal{D}} \cap \{u = 0\}$, as opposed to $\mathcal{D}$ which may not be a full characteristic rectangle. (Actually, choosing the characteristic coordinates in a natural symmetric way, $z(\rho, u) = w(\rho, -u)$, the domain $\widetilde{\mathcal{D}}$ is a square in characteristic coordinates.)

Next we turn to the construction of a particular family of Lax entropies which will serve for obtaining the cutoff functions needed. We fix $0 < \underline{r} < \overline{r} < r_0$ and define $S : \mathcal{D} \to \mathbb{R}$ as follows:

(i) In $\widetilde{\mathcal{D}}$: $S(\rho, u)$ is a solution of the Cauchy problem (10.5)+(10.7) with $s(r)$ given in (10.8). Note that

$$S(\rho, u) = \begin{cases} 0 & \text{if } (\rho, u) \in \mathcal{D}_1(\underline{r}) \subset \widetilde{\mathcal{D}}, \\ \rho - \dfrac{\overline{r} - \underline{r}}{\log(\overline{r}/\underline{r})} & \text{if } (\rho, u) \in \mathcal{D}_2(\overline{r}) \cap \widetilde{\mathcal{D}}. \end{cases} \tag{10.9}$$

(ii) In $\mathcal{D}_2(\overline{r})$:

$$S(\rho, u) := \rho - \frac{\overline{r} - \underline{r}}{\log(\overline{r}/\underline{r})}, \qquad \text{if } (\rho, u) \in \mathcal{D}_2(\overline{r}). \tag{10.10}$$

Note that there is no contradiction: in $\widetilde{\mathcal{D}} \cap \mathcal{D}_2(\underline{r})$, (i) yields the same expression.

(iii) In $\mathcal{D}_3(\overline{r}) \setminus \widetilde{\mathcal{D}}$: $S(\rho, u)$ is defined as a solution of the Goursat problem for (10.5) with boundary conditions on the characteristic lines $\partial\widetilde{\mathcal{D}} \cap \mathcal{D}_3(\overline{r})$, respectively, $\partial\mathcal{D}_2(\overline{r}) \setminus \widetilde{\mathcal{D}}$ provided by (i), respectively, (ii).

Note that $S(\rho, u)$ is a solution of the pde (10.5), *globally* in $\mathcal{D}$, and thus there exists a flux function $F(\rho, u)$ which together with $S(\rho, u)$ satisfies (10.5).

Now we are ready to define the *scaled* functions $S^n(\rho, u)$, $F^n(\rho, u)$ on the scaled domain $\mathcal{D}^n$ given in (5.2), as follows: fix $0 < \underline{r} < \overline{r} < \infty$ and define the *unscaled* Lax entropy/flux pair as done in (i)–(iii), but with *downscaled* initial conditions

$$S(r, 0) = n^{-2\beta} s(n^{2\beta} r), \qquad S_u(r, 0) = 0. \qquad r \in [0, \rho^*), \tag{10.11}$$

with the function $r \mapsto s(r)$ given in (10.8). Now, define the pair of scaled functions $S^n, F^n : \mathcal{D}^n \to \mathbb{R}$ as

$$S^n(\rho, u) := n^{2\beta} S(n^{-2\beta}\rho, n^{-\beta}u), \qquad F^n(\rho, u) := n^{3\beta} F(n^{-2\beta}\rho, n^{-\beta}u). \tag{10.12}$$

It is straightforward to check that $S^n$, $F^n$ form a Lax entropy/flux pair of the pde (5.8):

$$F^n_\rho = \Psi^n_\rho S^n_\rho + \Phi^n_\rho S^n_u, \qquad F^n_u = \Psi^n_u S^n_\rho + \Phi^n_u S^n_u,$$

in particular $S^n$ solves the pde (5.10). If $\gamma > 1$ then for any fixed $M > 0, \varepsilon > 0$ – choosing $\underline{r}$ and $\overline{r}/\underline{r}$ large enough – this choice of $S^n$, $F^n$ will satisfy the bounds (5.11)–(5.16) of Lemma 2. The spelled out proof can be found in [25].

*Remark.* As we mentioned in the remarks after Theorem 1, our result also holds for the $\{-1, 0, +1\}$-*model*, even though we have $\gamma = 1$ in that case. For this model we have $\Psi(\rho, u) = \rho u$, $\Phi(\rho, u) = \rho + u^2$ which yields $\Psi^n(\rho, u) = \Psi(\rho, u)$, $\Phi^n(\rho, u) = \Phi(\rho, u)$. Thus our cutoff function does not depend on $n$ and it has to satisfy

$$\rho S_{\rho\rho} + u S_{\rho u} - S_{uu} = 0.$$

This pde is explicitly solvable with initial conditions (10.7), (10.8), and it is not hard to check that the bounds (5.11)–(5.16) are indeed satisfied.

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