# Bálint Tóth:

Scaling limits for self-interacting random walks and diffusions, 6

lecture notes for Berlin-Zürich Summer School on

Stochastic Models of Complex processes

31 Aug-4 Sept 2009

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Back to the Brownian Polymer model. (See pp 68-85 for description, $(\Omega, \pi)$ and the relevant operators over $L^2(\Omega, \pi) =: \mathcal{H}$)

Goal: CLT for the displacement in $d \geq 3$

Tools: "Kipnis-Varadhan technology"

To be checked:

② $H_{-1}$ : $\| S^{-1/2} \varphi \|^2 < \infty$

③ the Relaxed Sector Condition

$\mathbb{B} := S^{-1/2} A S^{-1/2}$ well defined

④ $\lim_{t \to \infty} t^{-1} E(X(t)^2) > 0$

[ad absurdum $B(t) + M(t)$ could just cancel ...]

## Diffusive lower bound:

$$X(t)-X(s)=\sigma\left(B(t)-B(s)\right)+\int_s^t \varphi(\eta(u))\,du$$

$$\varphi:\Omega\to\mathbb{R}^d,\quad \varphi(\underline{\omega})=\omega(0).$$

We prove that

$\left(B(t)-B(s)\right)$ and $\int_s^t \varphi(\eta(u))\,du$

are <u>uncorrelated</u>

Let (for formal reasons)

$$M(s,t):=X(t)-X(s)-\int_s^t \varphi(\eta(u))\,du$$

$$\left(=\sigma\left(B(t)-B(s)\right)\right)$$

$\mathcal{F}_{(s,t)}$ := sigma-algebra generated by $\left(\eta(u):s\leq u\leq t\right)$

Lemma

(1) Let $s \in \mathbb{R}$ be fixed. Then

$[s, \infty) \ni t \mapsto M(s,t)$ is a

(forward) martingale with respect to the filtration $(\mathcal{F}_{(-\infty,t)} : t \geq s)$.

(2) Let $t \in \mathbb{R}$ be fixed. Then

$(-\infty, t] \ni s \mapsto M(s,t)$ is a

(backward) martingale with respect to the filtration $(\mathcal{F}_{(s,\infty)} : s \leq t)$.

Proof (1) nothing to prove: $\varphi(\eta(t))$ is exactly the compensator of $X(t)$, for $t$ increasing.

(2) Use Yaglom reversibility

$\hat{\eta}(t) := -\eta(-t)$; $\hat{\eta} \overset{\text{law}}{=} \eta$.

$$X(t) - X(s) = \Psi(\eta(u) : s \leq u \leq t)$$

(the displacement is a functional of the trajectory $\eta(\cdot)$ ...)

$$\tilde{X}(t) - \tilde{X}(s) = \Psi(\tilde{\eta}(u) : s \leq u \leq t)$$

the displacement along the backwards traj.

$$\tilde{X}(t) - \tilde{X}(s) = -X(-t) + X(-s)$$

Hence:

$$\lim_{h \downarrow 0} h^{-1} \mathbb{E}\left(X(s-h) - X(s) \,\middle|\, \mathcal{F}_{[s,\infty)}\right) =$$

$$\lim_{h \downarrow 0} h^{-1}\left(-\tilde{X}(-s+h) - \tilde{X}(-s) \,\middle|\, \tilde{\mathcal{F}}_{(-\infty,-s]}\right) =$$

$$-\varphi(\tilde{\eta}(-s)) = \varphi(-\tilde{\eta}(-s)) = \varphi(\eta(s))$$

$\square$

## $H_{-1}$ - bound (and diffusive upper bound)

$$\varphi_l(\omega) = \omega_l(0) \qquad l = 1, 2, \dots, d$$

$$S = -\sigma^2 \Delta$$

$$\left( S^{-1/2} \varphi_l , S^{-1/2} \varphi_k \right) = \int_{\mathbb{R}^d} p^{-2} \, \hat{C}_{lk}(p) \, dp$$

$$= \int_{\mathbb{R}^d} \frac{p_k p_l}{p^4} e^{-p^2/2} \, dp$$

$$= \delta_{k,l} \cdot d^{-1} \int_{\mathbb{R}^d} \frac{e^{-p^2/2}}{p^2} \, dp < \infty$$

if $\boxed{d \geq 3}$

The Relaxed Sector Condition:

$$S = -\sigma^2 \Delta$$

$$A = A_+ + A_- = \sum_{\ell} \left( a_\ell^* \nabla_\ell + \nabla_\ell a_\ell \right)$$

$$B = B_+ + B_-$$

$$B_+ = \sigma^{-2} \sum_{\ell} |\Delta|^{-1/2} a_\ell^* |\Delta|^{-1/2} \nabla_\ell$$

$$B_- = \sigma^{-2} \sum_{\ell} |\Delta|^{-1/2} \nabla_\ell \, a_\ell |\Delta|^{-1/2}$$

these are $\boxed{\text{formal}}$ descriptions

Q: Do the operators $B_\pm$, $B$ make sense as densely defined, closable, $B^* = -B$?

Computations are handy in the Hilbert space $\widehat{\mathcal{K}} = \bigoplus_n \widehat{\mathcal{K}}_n$

The action of various operators on the spaces

$$\mathcal{K} = \bigoplus_n \mathcal{K}_n \quad \text{and} \quad \hat{\mathcal{K}} = \bigoplus_n \hat{\mathcal{K}}_n$$

see pages 76 - 83 for notation, definitions of operators, basics ...

$\nabla_\ell : \mathcal{K}_n \longrightarrow \mathcal{K}_n$ (only densely defined, of course..)

$$\left(\nabla_\ell u\right)_{\underline{k}}(\underline{x}) := \sum_{m=1}^{n} \frac{\partial}{\partial x_{m\ell}} u_{\underline{k}}(\underline{x})$$

$\nabla_\ell : \hat{\mathcal{K}}_n \longrightarrow \hat{\mathcal{K}}_n$ (only densely... )

$$\left(\nabla_\ell \hat{u}\right)_{\underline{k}}(\underline{p}) := i\left(\sum_{m=1}^{n} p_{m\ell}\right) \hat{u}_{\underline{k}}(\underline{p})$$

$\Delta : \mathcal{K}_n \longrightarrow \mathcal{K}_n$ (only densely ... )

$$(\Delta u)_{\underline{k}}(\underline{x}) := \sum_{l=1}^{d} \sum_{m,m'=1}^{n} \frac{\partial^2}{\partial x_{ml}\, \partial x_{m'l}} u_{\underline{k}}(\underline{x}) .$$

$\Delta : \hat{\mathcal{K}}_n \longrightarrow \hat{\mathcal{K}}_n$ (only densely ... )

$$(\Delta \hat{u})_{\underline{k}}(\underline{p}) := -\left| \sum_{m=1}^{d} p_m \right|^2 \hat{u}_k(\underline{p})$$

$|\Delta|^{-1/2} : \mathcal{K}_n \longrightarrow \mathcal{K}_n$ (only densely)

no explicit formula

$|\Delta|^{-1/2} : \hat{\mathcal{K}}_n \longrightarrow \hat{\mathcal{K}}_n$ (only densely ... )

$$\left(|\Delta|^{-1/2} \hat{u}\right)_{\underline{k}}(\underline{p}) := \left| \sum_{m=1}^{d} p_m \right|^{-1} \hat{u}_k(\underline{p})$$

$$|\Delta|^{-1/2}\nabla_\ell : \mathcal{K}_\ell \longrightarrow \mathcal{K}_\ell \qquad \left( \left\| |\Delta|^{-1/2}\nabla_\ell \right\| = 1 \right)$$

no explicit formula

$$|\Delta|^{-1/2}\nabla_\ell : \hat{\mathcal{K}}_\ell \longrightarrow \hat{\mathcal{K}}_\ell \qquad \left( \left\| |\Delta|^{-1/2}\nabla_\ell \right\| = 1 \right)$$

$$\left(|\Delta|^{-1/2}\nabla_\ell \hat{u}\right)_{\underline{k}}(\underline{p}) := i\,\frac{\sum_{m=1}^{n} p_{m\ell}}{\left|\sum_{m=1}^{n} p_m\right|}\,\hat{u}_{\underline{k}}(\underline{p})$$

$$a^*_\ell : \mathcal{K}_n \longrightarrow \mathcal{K}_{n+1} \qquad \left( \left\| a^*_\ell \right\| \neq \sqrt{n+1} \right)$$

$$\left(a^*_\ell u\right)_{k_1\cdots k_{n+1}}(x_1,\dots,x_{n+1}) :=$$

$$(n+1)^{-1/2}\sum_{m=1}^{n+1} u_{k_1\cdots\cancel{k_m}\cdots k_{n+1}}(x_1\cdots\cancel{x_m}\cdots x_{n+1})\,\delta_{\ell,k_m}\,\delta(x_m)$$

$$a^*_\ell : \hat{\mathcal{K}}_n \longrightarrow \hat{\mathcal{K}}_{n+1} \qquad \left( \| a^*_\ell \| = \sqrt{n+1} \right)$$

$$\left( a^*_\ell \hat{u} \right)_{k_1 \cdots k_{n+1}} (p_1 \cdots p_{n+1}) :=$$

$$(n+1)^{-1/2} \sum_{m=1}^{n+1} \hat{u}_{k_1 \cdots \cancel{k_m} \cdots k_{n+1}} (p_1 \cdots \cancel{p_m} \cdots p_{n+1}) \; \delta_{\ell, k_m}$$

$$a_\ell : \mathcal{K}_n \longrightarrow \mathcal{K}_{n-1} \qquad \left( \| a_\ell \| = \sqrt{n} \right)$$

$$(a_\ell u)_{k_1 \cdots k_{n-1}} (x_1 \cdots x_{n-1}) :=$$

$$n^{1/2} \sum_k \int_{\mathbb{R}^d} u_{k_1 \cdots k_{n-1} k} (x_1 \cdots x_{n-1}\, x)\, C_{k\ell}(x)\, dx$$

$$a_\ell : \hat{\mathcal{K}}_n \longrightarrow \hat{\mathcal{K}}_{n-1} \qquad \left( \| a_\ell \| = \sqrt{n} \right)$$

$$(a_\ell \hat{u})_{k_1 \cdots k_{n-1}} (p_1 \cdots p_{n-1}) :=$$

$$n^{1/2} \sum_k \int_{\mathbb{R}^d} \hat{u}_{k_1 \cdots k_{n-1} k} (p_1 \cdots p_{n-1}\, p)\, \hat{C}_{k\ell}(p)\, dp$$

A core for the operators: $\boxed{d \geq 3}$

$$\hat{\mathcal{C}}_n := \left\{ \hat{u} \in \mathcal{K}_n : \sup_{\underline{p}, \underline{k}} \left| \hat{u}_{\underline{k}}(\underline{p}) \right| < \infty \right\}$$

$$\hat{\mathcal{C}} = \bigoplus_n \hat{\mathcal{C}}_n$$

$$\mathcal{B}_+ = \sigma^{-2} \sum_{l=1}^{d} |\Delta|^{-1/2} a_l^* |\Delta|^{-1/2} \nabla_l$$

$$|\Delta|^{-1/2} \nabla_l : \hat{\mathcal{C}}_n \longrightarrow \hat{\mathcal{C}}_n$$

$$a_l^* : \hat{\mathcal{C}}_n \longrightarrow \hat{\mathcal{C}}_{n+1}$$

$$|\Delta|^{-1/2} : \hat{\mathcal{C}}_{n+1} \longrightarrow \mathcal{J}_{n+1} \qquad (d \geq 3).$$

$$B_- = \sigma^{-2} \sum_{\ell=1}^{d} |\Delta|^{-1/2} \nabla_\ell \; a_\ell \, |\Delta|^{-1/2}$$

$$|\Delta|^{-1/2} : \hat{\mathcal{C}}_n \longrightarrow \hat{\mathcal{K}}_n \qquad (d \geq 3)$$

$$a_\ell : \hat{\mathcal{K}}_n \longrightarrow \hat{\mathcal{K}}_{n-1} \qquad (\text{bdd !})$$

$$|\Delta|^{-1/2} \nabla_\ell : \hat{\mathcal{K}}_{n-1} \longrightarrow \hat{\mathcal{K}}_{n-1} \qquad (\text{bdd !})$$

So, $B_+$ & $B_-$ are both densely defined

Remark: This can't be done for $d = 1,2$ !!

Next: $B_+^* \big|_{\mathcal{C}} = -B_-$

$B_-^* \big|_{\mathcal{C}} = -B_+$ Computation!

Follows both are closable and

$$\overline{B_\pm} = B_\pm^{**} \subseteq -B_\mp^* .$$

Actually:

$$\mathcal{B}_+ : \mathcal{C}_n \to \mathcal{H}_{n+1} \quad ; \quad \mathcal{B}_- : \mathcal{C}_n \to \mathcal{H}_{n-1}$$

closed: $\overline{\mathcal{B}_+} : \mathcal{D}_{+n} \to \mathcal{H}_{n+1}$ ; $\overline{\mathcal{B}_-} : \mathcal{D}_{-n} \to \mathcal{H}_{n-1}$

$$\mathcal{D}_{+n} := \mathrm{Dom}\left(\overline{\mathcal{B}_+\big|_{\mathcal{H}_n}}\right)$$

$$\mathcal{D}_{-n} := \mathrm{Dom}\left(\overline{\mathcal{B}_-\big|_{\mathcal{H}_n}}\right)$$

and

$$\left(\overline{\mathcal{B}_+}\big|_{\mathcal{D}_{+n}}\right)^* = -\overline{\mathcal{B}_-}\big|_{\mathcal{D}_{-(n+1)}}$$

$$\left(\overline{\mathcal{B}_-}\big|_{\mathcal{D}_{-n}}\right)^* = -\overline{\mathcal{B}_+}\big|_{\mathcal{D}_{+(n-1)}}$$

...

Similarly:

$\mathcal{B} = \mathcal{B}_+ + \mathcal{B}_-$ densely defined,
skew symmetric (on $\mathcal{C}$)
closable

Next to be checked:

$$\operatorname{Ran}(I \pm B)^\perp = \{0\}.$$

Let $\Psi = (\Psi_0, \Psi_1, \dots)$ $\quad \Psi_n \in \mathcal{H}_n$

$$(\Psi, (I + B)\varphi) = 0 \quad \forall \varphi \in \operatorname{Dom}(B)$$

$$\Downarrow$$

$\forall n: \forall \varphi_n \in \mathcal{C}_n$

$$(\Psi_n, \varphi_n) + (\Psi_{n+1}, B_+ \varphi_n) + (\Psi_{n-1}, B_- \varphi_n) = 0$$

$$(\Psi_n - B_- \Psi_{n+1} - B_+ \Psi_{n-1}, \varphi_n) = 0$$

That is

$$\Psi_n - B_- \Psi_{n+1} - B_+ \Psi_{n-1} = 0$$

$$\Psi - B\Psi = 0 \quad \Big| \ (\Psi, \cdot)$$

$$\|\Psi\|^2 = (\Psi, B\Psi) = 0 \quad \checkmark$$

Thus $B$ is not only skew symmetric & closed but it is

$$\boxed{\text{skew self adjoint}}$$

$$B^* = -\overline{B}$$

The convergence:

$$\forall f \in \mathcal{C}: \lim_{\lambda \to 0} \| B_\lambda f - Bf \| = 0$$

computation. $\square$